

BMET Algebra

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Chapter 1

Equations & Functions

1.1 Variable Expressions

Learning Objectives

- Evaluate algebraic expressions.
- Evaluate algebraic expressions with exponents.

Introduction - The Language of Algebra

No one likes doing the same problem over and over again—that’s why mathematicians invented algebra. Algebra takes the basic principles of math and makes them more general, so we can solve a problem once and then use that solution to solve a group of similar problems.

In arithmetic, you’ve dealt with numbers and their arithmetical operations (such as $+$, $-$, \times , \div). In algebra, we use symbols called **variables** (which are usually letters, such as x , y , a , b , c , ...) to represent numbers and sometimes processes.

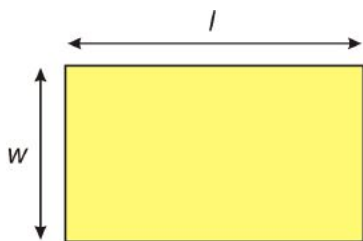
For example, we might use the letter x to represent some number we don’t know yet, which we might need to figure out in the course of a problem. Or we might use two letters, like x and y , to show a relationship between two numbers without needing to know what the actual numbers are. The same letters can represent a wide range of possible numbers, and the same letter may represent completely different numbers when used in two different problems.

Using variables offers advantages over solving each problem “from scratch.” With variables, we can:

- Formulate arithmetical laws such as $a + b = b + a$ for all real numbers a and b .
- Refer to “unknown” numbers. For instance: find a number x such that $3x + 1 = 10$.
- Write more compactly about functional relationships such as, “If you sell x tickets, then your profit will be $3x - 10$ dollars, or “ $f(x) = 3x - 10$,” where “ f ” is the profit function, and x is the input (i.e. how many tickets you sell).

Example 1

Write an algebraic expression for the perimeter and area of the rectangle below.



To find the perimeter, we add the lengths of all 4 sides. We can still do this even if we don't know the side lengths in numbers, because we can use variables like l and w to represent the unknown length and width. If we start at the top left and work clockwise, and if we use the letter P to represent the perimeter, then we can say:

$$P = l + w + l + w$$

We are adding 2 l 's and 2 w 's, so we can say that:

$$P = 2 \cdot l + 2 \cdot w$$

It's customary in algebra to omit multiplication symbols whenever possible. For example, $11x$ means the same thing as $11 \cdot x$ or $11 \times x$. We can therefore also write:

$$P = 2l + 2w$$

Area is *length multiplied by width*. In algebraic terms we get:

$$A = l \times w \rightarrow A = l \cdot w \rightarrow A = lw$$

Note: $2l + 2w$ by itself is an example of a **variable expression**; $P = 2l + 2w$ is an example of an **equation**. The main difference between expressions and **equations** is the presence of an **equals** sign ($=$).

In the above example, we found the simplest possible ways to express the perimeter and area of a rectangle when we don't yet know what its length and width actually are. Now, when we encounter a rectangle whose dimensions we do know, we can simply substitute (or **plug in**) those values in the above equations. In this chapter, we will encounter many expressions that we can evaluate by plugging in values for the variables involved.

Evaluate Algebraic Expressions

When we are given an algebraic expression, one of the most common things we might have to do with it is **evaluate** it for some given value of the variable. The following example illustrates this process.

Example 2

Let $x = 12$. Find the value of $2x - 7$.

To find the solution, we substitute 12 for x in the given expression. Every time we see x , we replace it with 12.

$$\begin{aligned} 2x - 7 &= 2(12) - 7 \\ &= 24 - 7 \\ &= 17 \end{aligned}$$

Note: At this stage of the problem, we place the substituted value in parentheses. We do this to make the written-out problem easier to follow, and to avoid mistakes. (If we didn't use parentheses and also forgot to add a multiplication sign, we would end up turning $2x$ into 212 instead of 2 times 12!)

Example 3

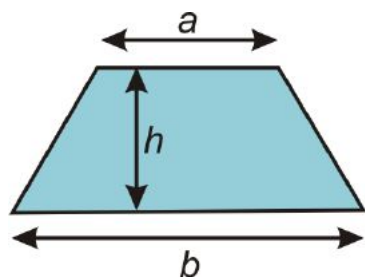
Let $y = -2$. Find the value of $\frac{7}{y} - 11y + 2$.

Solution

$$\begin{aligned}\frac{7}{(-2)} - 11(-2) + 2 &= -3\frac{1}{2} + 22 + 2 \\ &= 24 - 3\frac{1}{2} \\ &= 20\frac{1}{2}\end{aligned}$$

Many expressions have more than one variable in them. For example, the formula for the perimeter of a rectangle in the introduction has two variables: length (l) and width (w). In these cases, be careful to substitute the appropriate value in the appropriate place.

Example 5



The area of a trapezoid is given by the equation $A = \frac{h}{2}(a + b)$. Find the area of a trapezoid with bases $a = 10$ cm and $b = 15$ cm and height $h = 8$ cm.

To find the solution to this problem, we simply take the values given for the variables a , b , and h , and plug them in to the expression for A :

$$\begin{aligned}A &= \frac{h}{2}(a + b) && \text{Substitute 10 for } a, 15 \text{ for } b, \text{ and 8 for } h. \\ A &= \frac{8}{2}(10 + 15) && \text{Evaluate piece by piece. } 10 + 15 = 25; \frac{8}{2} = 4. \\ A &= 4(25) = 100\end{aligned}$$

Solution: The area of the trapezoid is 100 square centimeters.

Evaluate Algebraic Expressions with Exponents

Many formulas and equations in mathematics contain exponents. Exponents are used as a short-hand notation for repeated multiplication. For example:

$$\begin{aligned}2 \cdot 2 &= 2^2 \\ 2 \cdot 2 \cdot 2 &= 2^3\end{aligned}$$

The exponent stands for how many times the number is used as a factor (multiplied). When we deal with integers, it is usually easiest to simplify the expression. We simplify:

$$2^2 = 4$$

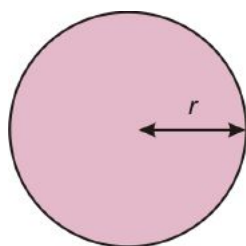
$$2^3 = 8$$

However, we need exponents when we work with variables, because it is much easier to write x^8 than $x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$.

To evaluate expressions with exponents, substitute the values you are given for each variable and simplify. It is especially important in this case to substitute using parentheses in order to make sure that the simplification is done correctly.

For a more detailed review of exponents and their properties, check out the video at <http://www.mathvids.com/lesson/r-exponents—basics>.

Example 5



The area of a circle is given by the formula $A = \pi r^2$. Find the area of a circle with radius $r = 17$ inches. Substitute values into the equation.

$$A = \pi r^2 \quad \text{Substitute 17 for } r.$$

$$A = \pi(17)^2 \quad \pi \cdot 17 \cdot 17 \approx 907.9202 \dots \text{ Round to 2 decimal places.}$$

The area is approximately 907.92 square inches.

Example 6

Find the value of $\frac{x^2y^3}{x^3+y^2}$, for $x = 2$ and $y = -4$.

Substitute the values of x and y in the following.

$$\frac{x^2y^3}{x^3+y^2} = \frac{(2)^2(-4)^3}{(2)^3+(-4)^2} \quad \text{Substitute 2 for } x \text{ and } -4 \text{ for } y.$$

$$\frac{4(-64)}{8+16} = \frac{-256}{24} = \frac{-32}{3}$$

Evaluate expressions: $(2)^2 = (2)(2) = 4$ and

$$(2)^3 = (2)(2)(2) = 8. \quad (-4)^2 = (-4)(-4) = 16 \text{ and}$$

$$(-4)^3 = (-4)(-4)(-4) = -64.$$

Example 7

The height (h) of a ball in flight is given by the formula $h = -32t^2 + 60t + 20$, where the height is given in feet and the time (t) is given in seconds. Find the height of the ball at time $t = 2$ seconds.

Solution

$$\begin{aligned}
 h &= -32t^2 + 60t + 20 \\
 &= -32(2)^2 + 60(2) + 20 && \text{Substitute 2 for } t. \\
 &= -32(4) + 60(2) + 20 \\
 &= 12
 \end{aligned}$$

The height of the ball is 12 feet.

Review Questions

- Write the following in a more condensed form by leaving out a multiplication symbol.
 - $2 \times 11x$
 - $1.35 \cdot y$
 - $3 \times \frac{1}{4}$
 - $\frac{1}{4} \cdot z$
- Evaluate the following expressions for $a = -3$, $b = 2$, $c = 5$, and $d = -4$.
 - $2a + 3b$
 - $4c + d$
 - $5ac - 2b$
 - $\frac{2a}{c-d}$
 - $\frac{3b}{d}$
 - $\frac{a-4b}{3c+2d}$
 - $\frac{1}{a+b}$
 - $\frac{ab}{cd}$
- Evaluate the following expressions for $x = -1$, $y = 2$, $z = -3$, and $w = 4$.
 - $8x^3$
 - $\frac{5x^2}{6z^3}$
 - $3z^2 - 5w^2$
 - $x^2 - y^2$
 - $\frac{z^3+w^3}{z^3-w^3}$
 - $2x^3 - 3x^2 + 5x - 4$
 - $4w^3 + 3w^2 - w + 2$
 - $3 + \frac{1}{z^2}$
- The weekly cost C of manufacturing x remote controls is given by the formula $C = 2000 + 3x$, where the cost is given in dollars.
 - What is the cost of producing 1000 remote controls?
 - What is the cost of producing 2000 remote controls?
 - What is the cost of producing 2500 remote controls?
- The volume of a box without a lid is given by the formula $V = 4x(10 - x)^2$, where x is a length in inches and V is the volume in cubic inches.
 - What is the volume when $x = 2$?
 - What is the volume when $x = 3$?

1.2 One-Step Equations

Learning Objectives

- Solve an equation using addition.
- Solve an equation using subtraction.
- Solve an equation using multiplication.
- Solve an equation using division.

Introduction

Nadia is buying a new mp3 player. Peter watches her pay for the player with a \$100 bill. She receives \$22.00 in change, and from only this information, Peter works out how much the player cost. How much was the player?

In algebra, we can solve problems like this using an **equation**. An **equation** is an algebraic expression that involves an **equals** sign. If we use the letter x to represent the cost of the mp3 player, we can write the equation $x + 22 = 100$. This tells us that the value of the player **plus** the value of the change received is **equal** to the \$100 that Nadia paid.

Another way we could write the equation would be $x = 100 - 22$. This tells us that the value of the player is **equal** to the total amount of money Nadia paid ($100 - 22$). This equation is mathematically equivalent to the first one, but it is easier to solve.

In this chapter, we will learn how to solve for the variable in a one-variable linear equation. **Linear equations** are equations in which each term is either a constant, or a constant times a single variable (raised to the first power). The term linear comes from the word line, because the graph of a linear equation is always a line.

We'll start with simple problems like the one in the last example.

Solving Equations Using Addition and Subtraction

When we work with an algebraic equation, it's important to remember that the two sides have to stay equal for the equation to stay true. We can change the equation around however we want, but whatever we do to one side of the equation, we have to do to the other side. In the introduction above, for example, we could get from the first equation to the second equation by subtracting 22 from both sides:

$$\begin{aligned}x + 22 &= 100 \\x + 22 - 22 &= 100 - 22 \\x &= 100 - 22\end{aligned}$$

Similarly, we can add numbers to each side of an equation to help solve for our unknown.

Example 1

Solve $x - 3 = 9$.

Solution

To solve an equation for x , we need to **isolate** x —that is, we need to get it by itself on one side of the equals sign. Right now our x has a 3 subtracted from it. To reverse this, we'll add 3—but we must add 3 to **both sides**.

$$\begin{aligned}
 x - 3 &= 9 \\
 x - 3 + 3 &= 9 + 3 \\
 x + 0 &= 9 + 3 \\
 x &= 12
 \end{aligned}$$

Example 2

Solve $z - 9.7 = -1.026$

Solution

It doesn't matter what the variable is—the solving process is the same.

$$\begin{aligned}
 z - 9.7 &= -1.026 \\
 z - 9.7 + 9.7 &= -1.026 + 9.7 \\
 z &= 8.674
 \end{aligned}$$

Make sure you understand the addition of decimals in this example!

Example 3

Solve $x + \frac{4}{7} = \frac{9}{5}$.

Solution

To isolate x , we need to subtract $\frac{4}{7}$ from both sides.

$$\begin{aligned}
 x + \frac{4}{7} &= \frac{9}{5} \\
 x + \frac{4}{7} - \frac{4}{7} &= \frac{9}{5} - \frac{4}{7} \\
 x &= \frac{9}{5} - \frac{4}{7}
 \end{aligned}$$

Now we have to subtract fractions, which means we need to find the LCD. Since 5 and 7 are both prime, their lowest common multiple is just their product, 35.

$$\begin{aligned}
 x &= \frac{9}{5} - \frac{4}{7} \\
 x &= \frac{7 \cdot 9}{7 \cdot 5} - \frac{4 \cdot 5}{7 \cdot 5} \\
 x &= \frac{63}{35} - \frac{20}{35} \\
 x &= \frac{63 - 20}{35} \\
 x &= \frac{43}{35}
 \end{aligned}$$

Make sure you're comfortable with decimals and fractions! To master algebra, you'll need to work with them frequently.

Solving Equations Using Multiplication and Division



Suppose you are selling pizza for \$1.50 a slice and you can get eight slices out of a single pizza. How much money do you get for a single pizza? It shouldn't take you long to figure out that you get $8 \times \$1.50 = \12.00 . You solved this problem by multiplying. Here's how to do the same thing algebraically, using x to stand for the cost in dollars of the whole pizza.

Example 4

Solve $\frac{1}{8} \cdot x = 1.5$.

Our x is being multiplied by one-eighth. To cancel that out and get x by itself, we have to multiply by the reciprocal, 8. Don't forget to multiply **both sides** of the equation.

$$\begin{aligned} 8\left(\frac{1}{8} \cdot x\right) &= 8(1.5) \\ x &= 12 \end{aligned}$$

Example 5

Solve $\frac{9x}{5} = 5$.

$\frac{9x}{5}$ is equivalent to $\frac{9}{5} \cdot x$, so to cancel out that $\frac{9}{5}$, we multiply by the reciprocal, $\frac{5}{9}$.

$$\begin{aligned} \frac{5}{9}\left(\frac{9x}{5}\right) &= \frac{5}{9}(5) \\ x &= \frac{25}{9} \end{aligned}$$

Example 6

Solve $0.25x = 5.25$.

0.25 is the decimal equivalent of one fourth, so to cancel out the 0.25 factor we would multiply by 4.

$$\begin{aligned} 4(0.25x) &= 4(5.25) \\ x &= 21 \end{aligned}$$

Solving by division is another way to isolate x . Suppose you buy five identical candy bars, and you are charged \$3.25. How much did each candy bar cost? You might just divide \$3.25 by 5, but let's see how this problem looks in algebra.

Example 7

Solve $5x = 3.25$.

To cancel the 5, we divide both sides by 5.

$$\begin{aligned} \frac{5x}{5} &= \frac{3.25}{5} \\ x &= 0.65 \end{aligned}$$

Solve $7x = \frac{5}{11}$.

$$x = \frac{5}{11.7}$$

$$x = \frac{5}{77}$$

Solve $1.375x = 1.2$.

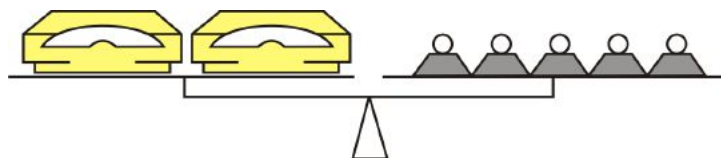
$$x = \frac{1.2}{1.375}$$
$$x = 0.\overline{872}$$

To see more examples of one - and two-step equation solving, watch the Khan Academy video series starting at <http://www.youtube.com/watch?v=bAerID24QJ0> <http://www.youtube.com/watch?v=bAerID24QJ0>.

In the year 2017, Anne will be 45years old. In what year was Anne born?

$$\begin{aligned}x + 45 &= 2017 \\x + 45 - 45 &= 2017 - 45 \\x &= 1972\end{aligned}$$

A mail order electronics company stocks a new mini DVD player and is using a balance to determine the shipping weight. Using only one-pound weights, the shipping department found that the following arrangement balances:



Since the system balances, the total weight on each side must be equal. To write our equation, we'll use x for the weight of one DVD player, which is unknown. There are two DVD players, weighing a total of $2x$

pounds, on the left side of the balance, and on the right side are 5 1-pound weights, weighing a total of 5 pounds. So our equation is $2x = 5$. Dividing both sides by 2 gives us $x = 2.5$.

Each DVD player weighs 2.5 pounds.

Example 12

In 2004, Takeru Kobayashi of Nagano, Japan, ate 53.5 hot dogs in 12 minutes. This was 3 more hot dogs than his own previous world record, set in 2002. Calculate:

- a) *How many minutes it took him to eat one hot dog.*
- b) *How many hot dogs he ate per minute.*
- c) *What his old record was.*

Solution

a) We know that the total time for 53.5 hot dogs is 12 minutes. We want to know the time for one hot dog, so that's x . Our equation is $53.5x = 12$. Then we divide both sides by 53.5 to get $x = \frac{12}{53.5}$, or $x = 0.224$ minutes.

We can also multiply by 60 to get the time in seconds; 0.224 minutes is about 13.5 seconds. So that's how long it took Takeru to eat one hot dog.

b) Now we're looking for hot dogs per minute instead of minutes per hot dog. We'll use the variable y instead of x this time so we don't get the two confused. 12 minutes, times the number of hot dogs per minute, equals the total number of hot dogs, so $12y = 53.5$. Dividing both sides by 12 gives us $y = \frac{53.5}{12}$, or $y = 4.458$ hot dogs per minute.

c) We know that his new record is 53.5, and we know that's three more than his old record. If we call his old record z , we can write the following equation: $z + 3 = 53.5$. Subtracting 3 from both sides gives us $z = 50.5$. So Takeru's old record was 50.5 hot dogs in 12 minutes.

Lesson Summary

- An equation in which each term is either a constant or the product of a constant and a single variable is a **linear equation**.
- We can add, subtract, multiply, or divide both sides of an equation by the same value and still have an equivalent equation.
- To solve an equation, **isolate** the unknown variable on one side of the equation by applying one or more arithmetic operations to both sides.

Review Questions

1. Solve the following equations for x .

- (a) $x + 11 = 7$
- (b) $x - 1.1 = 3.2$
- (c) $7x = 21$
- (d) $4x = 1$
- (e) $\frac{5x}{12} = \frac{2}{3}$
- (f) $x + \frac{5}{2} = \frac{1}{3}$
- (g) $x - \frac{5}{6} = \frac{1}{8}$
- (h) $0.01x = 11$

2. Solve the following equations for the unknown variable.

- (a) $q - 13 = -13$
 - (b) $z + 1.1 = 3.0001$
 - (c) $21s = 3$
 - (d) $t + \frac{1}{2} = \frac{1}{3}$
 - (e) $\frac{7f}{11} = \frac{7}{11}$
 - (f) $\frac{3}{4} = -\frac{1}{2} - y$
 - (g) $6r = \frac{3}{2}$
 - (h) $\frac{9b}{16} = \frac{3}{8}$
- Peter is collecting tokens on breakfast cereal packets in order to get a model boat. In eight weeks he has collected 10 tokens. He needs 25 tokens for the boat. Write an equation and determine the following information.
 - (a) How many more tokens he needs to collect, n .
 - (b) How many tokens he collects per week, w .
 - (c) How many more weeks remain until he can send off for his boat, r .
 - Juan has baked a cake and wants to sell it in his bakery. He is going to cut it into 12 slices and sell them individually. He wants to sell it for three times the cost of making it. The ingredients cost him \$8.50, and he allowed \$1.25 to cover the cost of electricity to bake it. Write equations that describe the following statements
 - (a) The amount of money that he sells the cake for (u).
 - (b) The amount of money he charges for each slice (c).
 - (c) The total profit he makes on the cake (w).
 - Jane is baking cookies for a large party. She has a recipe that will make one batch of two dozen cookies, and she decides to make five batches. To make five batches, she finds that she will need 12.5 cups of flour and 15 eggs.
 - (a) How many cookies will she make in all?
 - (b) How many cups of flour go into one batch?
 - (c) How many eggs go into one batch?
 - (d) If Jane only has a dozen eggs on hand, how many more does she need to make five batches?
 - (e) If she doesn't go out to get more eggs, how many batches can she make? How many cookies will that be?

1.3 Two-Step Equations

Learning Objectives

- Solve a two-step equation using addition, subtraction, multiplication, and division.
- Solve a two-step equation by combining like terms.
- Solve real-world problems using two-step equations.

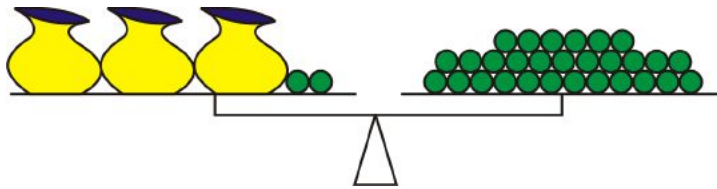
Solve a Two-Step Equation

We've seen how to solve for an unknown by isolating it on one side of an equation and then evaluating the other side. Now we'll see how to solve equations where the variable takes more than one step to isolate.

Example 1

Rebecca has three bags containing the same number of marbles, plus two marbles left over. She places them on one side of a balance. Chris, who has more marbles than Rebecca, adds marbles to the other side of the

balance. He finds that with 29 marbles, the scales balance. How many marbles are in each bag? Assume the bags weigh nothing.



Solution

We know that the system balances, so the weights on each side must be equal. If we use x to represent the number of marbles in each bag, then we can see that on the left side of the scale we have three bags (each containing x marbles) plus two extra marbles, and on the right side of the scale we have 29 marbles. The balancing of the scales is similar to the balancing of the following equation.

$$3x + 2 = 29$$

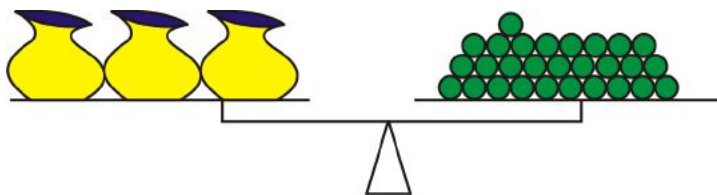
“Three bags plus two marbles **equals** 29 marbles”

To solve for x , we need to first get all the variables (terms containing an x) alone on one side of the equation. We’ve already got all the x ’s on one side; now we just need to isolate them.

$$\begin{aligned} 3x + 2 &= 29 \\ 3x + 2 - 2 &= 29 - 2 && \text{Get rid of the 2 on the left by subtracting it from both sides.} \\ 3x &= 27 \\ \frac{3x}{3} &= \frac{27}{3} && \text{Divide both sides by 3.} \\ x &= 9 \end{aligned}$$

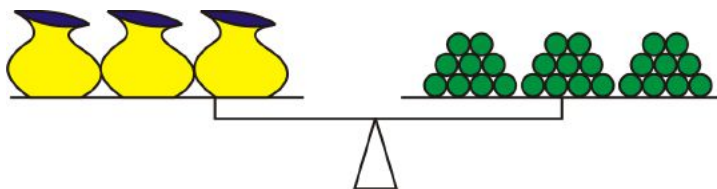
There are nine marbles in each bag.

We can do the same with the real objects as we did with the equation. Just as we subtracted 2 from both sides of the equals sign, we could remove two marbles from each side of the scale. Because we removed the same number of marbles from each side, we know the scales will still balance.



Then, because there are three bags of marbles on the left-hand side of the scale, we can divide the marbles on the right-hand side into three equal piles. You can see that there are nine marbles in each.

Three bags of marbles **balances** three piles of nine marbles.



So each bag of marbles balances nine marbles, meaning that each bag contains nine marbles.

Check out <http://www.mste.uiuc.edu/pavel/java/balance/> for more interactive balance beam activities!

Example 2

Solve $6(x + 4) = 12$.

This equation has the x buried in parentheses. To dig it out, we can proceed in one of two ways: we can either distribute the six on the left, or divide both sides by six to remove it from the left. Since the right-hand side of the equation is a multiple of six, it makes sense to divide. That gives us $x + 4 = 2$. Then we can subtract 4 from both sides to get $x = -2$.

Example 3

Solve $\frac{x-3}{5} = 7$.

It's always a good idea to get rid of fractions first. Multiplying both sides by 5 gives us $x - 3 = 35$, and then we can add 3 to both sides to get $x = 38$.

Example 4

Solve $\frac{5}{9}(x + 1) = \frac{2}{7}$.

First, we'll cancel the fraction on the left by multiplying by the reciprocal (the multiplicative inverse).

$$\begin{aligned}\frac{9}{5} \cdot \frac{5}{9}(x + 1) &= \frac{9}{5} \cdot \frac{2}{7} \\ (x + 1) &= \frac{18}{35}\end{aligned}$$

Then we subtract 1 from both sides. ($\frac{35}{35}$ is equivalent to 1.)

$$\begin{aligned}x + 1 &= \frac{18}{35} \\ x + 1 - 1 &= \frac{18}{35} - \frac{35}{35} \\ x &= \frac{18 - 35}{35} \\ x &= \frac{-17}{35}\end{aligned}$$

These examples are called **two-step equations**, because we need to perform two separate operations on the equation to isolate the variable.

Solve a Two-Step Equation by Combining Like Terms

When we look at a linear equation we see two kinds of terms: those that contain the unknown variable, and those that don't. When we look at an equation that has an x on both sides, we know that in order to solve it, we need to get all the x -terms on one side of the equation. This is called **combining like terms**. The terms with an x in them are **like terms** because they contain the same variable (or, as you will see in later chapters, the same combination of variables).

Table 1.1:

| Like Terms | Unlike Terms |
|---------------------------------------|------------------|
| $4x, 10x, -3.5x$, and $\frac{x}{12}$ | $3x$ and $3y$ |
| $3y, 0.000001y$, and y | $4xy$ and $4x$ |
| $xy, 6xy$, and $2.39xy$ | $0.5x$ and 0.5 |

To add or subtract like terms, we can use the Distributive Property of Multiplication.

$$\begin{aligned}3x + 4x &= (3 + 4)x = 7x \\0.03xy - 0.01xy &= (0.03 - 0.01)xy = 0.02xy \\-y + 16y + 5y &= (-1 + 16 + 5)y = 10y \\5z + 2z - 7z &= (5 + 2 - 7)z = 0z = 0\end{aligned}$$

To solve an equation with two or more like terms, we need to combine the terms first.

Example 5

Solve $(x + 5) - (2x - 3) = 6$.

There are two like terms: the x and the $-2x$ (don't forget that the negative sign applies to everything in the parentheses). So we need to get those terms together. The associative and distributive properties let us rewrite the equation as $x + 5 - 2x + 3 = 6$, and then the commutative property lets us switch around the terms to get $x - 2x + 5 + 3 = 6$, or $(x - 2x) + (5 + 3) = 6$.

$(x - 2x)$ is the same as $(1 - 2)x$, or $-x$, so our equation becomes $-x + 8 = 6$

Subtracting 8 from both sides gives us $-x = -2$.

And finally, multiplying both sides by -1 gives us $x = 2$.

Example 6

Solve $\frac{x}{2} - \frac{x}{3} = 6$.

This problem requires us to deal with fractions. We need to write all the terms on the left over a common denominator of six.

$$\frac{3x}{6} - \frac{2x}{6} = 6$$

Then we subtract the fractions to get $\frac{x}{6} = 6$.

Finally we multiply both sides by 6 to get $x = 36$.

Solve Real-World Problems Using Two-Step Equations

The hardest part of solving word problems is translating from words to an equation. First, you need to look to see what the equation is asking. What is the **unknown** for which you have to solve? That will be what your **variable** stands for. Then, follow what is going on with your variable all the way through the problem.

Example 7



An emergency plumber charges \$65 as a call-out fee plus an additional \$75 per hour. He arrives at a house at 9:30 and works to repair a water tank. If the total repair bill is \$196.25, at what time was the repair completed?

In order to solve this problem, we collect the information from the text and convert it to an equation.

Unknown: time taken in hours – this will be our x

The bill is made up of two parts: a call out fee and a per-hour fee. The call out is a flat fee, and independent of x —it's the same no matter how many hours the plumber works. The per-hour part depends on the number of hours (x). So the total fee is \$65 (no matter what) plus \$75 x (where x is the number of hours), or $65 + 75x$.

Looking at the problem again, we also can see that the total bill is \$196.25. So our final equation is $196.25 = 65 + 75x$.

Solving for x :

$$196.25 = 65 + 75x \quad \text{Subtract 65 from both sides.}$$

$$131.25 = 75x \quad \text{Divide both sides by 75.}$$

$$1.75 = x \quad \text{The job took 1.75 hours.}$$

Solution

The repair job was completed 1.75 hours after 9:30, so it was completed at 11:15AM.

Example 8

When Asia was young her Daddy marked her height on the door frame every month. Asia's Daddy noticed that between the ages of one and three, he could predict her height (in inches) by taking her age in months, adding 75 inches and multiplying the result by one-third. Use this information to answer the following:

- a) *Write an equation linking her predicted height, h , with her age in months, m .*
- b) *Determine her predicted height on her second birthday.*
- c) *Determine at what age she is predicted to reach three feet tall.*

Solution

a) To convert the text to an equation, first determine the type of equation we have. We are going to have an equation that links **two variables**. Our unknown will change, depending on the information we are given. For example, we could solve for height given age, or solve for age given height. However, the text gives us a way to determine **height**. Our equation will start with " $h =$ ".

The text tells us that we can predict her height by taking her age in months, adding 75, and multiplying by $\frac{1}{3}$. So our equation is $h = (m + 75) \cdot \frac{1}{3}$, or $h = \frac{1}{3}(m + 75)$.

b) To predict Asia's height on her second birthday, we substitute $m = 24$ into our equation (because 2 years is 24 months) and solve for h .

$$h = \frac{1}{3}(24 + 75)$$

$$h = \frac{1}{3}(99)$$

$$h = 33$$

Asia's height on her second birthday was predicted to be 33 inches.

c) To determine the predicted age when she reached three feet, substitute $h = 36$ into the equation and solve for m .

$$36 = \frac{1}{3}(m + 75)$$

$$108 = m + 75$$

$$33 = m$$

Asia was predicted to be 33 months old when her height was three feet.

Example 9

To convert temperatures in Fahrenheit to temperatures in Celsius, follow the following steps: Take the temperature in degrees Fahrenheit and subtract 32. Then divide the result by 1.8 and this gives the temperature in degrees Celsius.

- Write an equation that shows the conversion process.
- Convert 50 degrees Fahrenheit to degrees Celsius.
- Convert 25 degrees Celsius to degrees Fahrenheit.
- Convert -40 degrees Celsius to degrees Fahrenheit.

a) The text gives the process to convert Fahrenheit to Celsius. We can write an equation using two variables. We will use f for temperature in Fahrenheit, and c for temperature in Celsius.

First we take the temperature in Fahrenheit and subtract 32.

$$f - 32$$

Then divide by 1.8.

$$\frac{f - 32}{1.8}$$

This equals the temperature in Celsius.

$$c = \frac{f - 32}{1.8}$$

In order to convert from one temperature scale to another, simply substitute in for whichever temperature you know, and solve for the one you don't know.

- To convert 50 degrees Fahrenheit to degrees Celsius, substitute $f = 50$ into the equation.

$$c = \frac{50 - 32}{1.8}$$

$$c = \frac{18}{1.8}$$

$$c = 10$$

50 degrees Fahrenheit is equal to 10 degrees Celsius.

- To convert 25 degrees Celsius to degrees Fahrenheit, substitute $c = 25$ into the equation:

$$25 = \frac{f - 32}{1.8}$$

$$45 = f - 32$$

$$77 = f$$

25 degrees Celsius is equal to 77 degrees Fahrenheit.

- To convert -40 degrees Celsius to degrees Fahrenheit, substitute $c = -40$ into the equation.

$$-40 = \frac{f - 32}{1.8}$$

$$-72 = f - 32$$

$$-40 = f$$

-40 degrees Celsius is equal to -40 degrees Fahrenheit. (No, that's not a mistake! This is the one temperature where they are equal.)

Lesson Summary

- Some equations require more than one operation to solve. Generally it, is good to go from the outside in. If there are parentheses around an expression with a variable in it, cancel what is outside the parentheses first.
- Terms with the same variable in them (or no variable in them) are **like terms**. **Combine like terms** (adding or subtracting them from each other) to simplify the expression and solve for the unknown.

Review Questions

1. Solve the following equations for the unknown variable.
 - (a) $1.3x - 0.7x = 12$
 - (b) $6x - 1.3 = 3.2$
 - (c) $5x - (3x + 2) = 1$
 - (d) $4(x + 3) = 1$
 - (e) $5q - 7 = \frac{2}{3}$
 - (f) $\frac{3}{5}x + \frac{5}{2} = \frac{2}{3}$
 - (g) $s - \frac{3s}{8} = \frac{5}{6}$
 - (h) $0.1y + 11 = 0$
 - (i) $\frac{5q-7}{12} = \frac{2}{3}$
 - (j) $\frac{5(q-7)}{12} = \frac{2}{3}$
 - (k) $33t - 99 = 0$
 - (l) $5p - 2 = 32$
 - (m) $10y + 5 = 10$
 - (n) $10(y + 5) = 10$
 - (o) $10y + 5y = 10$
 - (p) $10(y + 5y) = 10$
2. Jade is stranded downtown with only \$10 to get home. Taxis cost \$0.75 per mile, but there is an additional \$2.35 hire charge. Write a formula and use it to calculate how many miles she can travel with her money.
3. Jasmin's Dad is planning a surprise birthday party for her. He will hire a bouncy castle, and will provide party food for all the guests. The bouncy castle costs \$150 for the afternoon, and the food will cost \$3 per person. Andrew, Jasmin's Dad, has a budget of \$300. Write an equation and use it to determine the maximum number of guests he can invite.
4. The local amusement park sells summer memberships for \$50 each. Normal admission to the park costs \$25; admission for members costs \$15.
 - (a) If Darren wants to spend no more than \$100 on trips to the amusement park this summer, how many visits can he make if he buys a membership with part of that money?
 - (b) How many visits can he make if he does not?
 - (c) If he increases his budget to \$160, how many visits can he make as a member?
 - (d) And how many as a non-member?
5. For an upcoming school field trip, there must be one adult supervisor for every five children.
 - (a) If the bus seats 40 people, how many children can go on the trip?

- (b) How many children can go if a second 40-person bus is added?
- (c) Four of the adult chaperones decide to arrive separately by car. Now how many children can go in the two buses?

1.4 Multi-Step Equations

Learning Objectives

- Solve a multi-step equation by combining like terms.
- Solve a multi-step equation using the distributive property.
- Solve real-world problems using multi-step equations.
- Check solutions to equations

Solving Multi-Step Equations by Combining Like Terms

We've seen that when we solve for an unknown variable, it can take just one or two steps to get the terms in the right places. Now we'll look at solving equations that take several steps to isolate the unknown variable. Such equations are referred to as **multi-step equations**.

In this section, we'll simply be combining the steps we already know how to do. Our goal is to end up with all the constants on one side of the equation and all the variables on the other side. We'll do this by collecting like terms. Don't forget, like terms have the same combination of variables in them.

Example 1

Solve $\frac{3x+4}{3} - 5x = 6$.

Before we can combine the variable terms, we need to get rid of that fraction.

First let's put all the terms on the left over a common denominator of three: $\frac{3x+4}{3} - \frac{15x}{3} = 6$.

Combining the fractions then gives us $\frac{3x+4-15x}{3} = 6$.

Combining like terms in the numerator gives us $\frac{4-12x}{3} = 6$.

Multiplying both sides by 3 gives us $4 - 12x = 18$.

Subtracting 4 from both sides gives us $-12x = 14$.

And finally, dividing both sides by -12 gives us $x = -\frac{14}{12}$, which reduces to $x = -\frac{7}{6}$.

Solving Multi-Step Equations Using the Distributive Property

You may have noticed that when one side of the equation is multiplied by a constant term, we can either distribute it or just divide it out. If we can divide it out without getting awkward fractions as a result, then that's usually the better choice, because it gives us smaller numbers to work with. But if dividing would result in messy fractions, then it's usually better to distribute the constant and go from there.

Example 2

Solve $7(2x - 5) = 21$.

The first thing we want to do here is get rid of the parentheses. We could use the Distributive Property, but it just so happens that 7 divides evenly into 21. That suggests that dividing both sides by 7 is the easiest way to solve this problem.

If we do that, we get $2x - 5 = \frac{21}{7}$ or just $2x - 5 = 3$. Then all we need to do is add 5 to both sides to get $2x = 8$, and then divide by 2 to get $x = 4$.

Example 3

Solve $17(3x + 4) = 7$.

Once again, we want to get rid of those parentheses. We could divide both sides by 17, but that would give us an inconvenient fraction on the right-hand side. In this case, distributing is the easier way to go.

Distributing the 17 gives us $51x + 68 = 7$. Then we subtract 68 from both sides to get $51x = -61$, and then we divide by 51 to get $x = \frac{-61}{51}$. (Yes, that's a messy fraction too, but since it's our final answer and we don't have to do anything else with it, we don't really care how messy it is.)

Example 4

Solve $4(3x - 4) - 7(2x + 3) = 3$.

Before we can collect like terms, we need to get rid of the parentheses using the Distributive Property. That gives us $12x - 16 - 14x - 21 = 3$, which we can rewrite as $(12x - 14x) + (-16 - 21) = 3$. This in turn simplifies to $-2x - 37 = 3$.

Next we add 37 to both sides to get $-2x = 40$.

And finally, we divide both sides by -2 to get $x = -20$.

Example 5

Solve the following equation for x : $0.1(3.2 + 2x) + \frac{1}{2}\left(3 - \frac{x}{5}\right) = 0$

This function contains both fractions and decimals. We should convert all terms to one or the other. It's often easier to convert decimals to fractions, but in this equation the fractions are easy to convert to decimals—and with decimals we don't need to find a common denominator!

In decimal form, our equation becomes $0.1(3.2 + 2x) + 0.5(3 - 0.2x) = 0$.

Distributing to get rid of the parentheses, we get $0.32 + 0.2x + 1.5 - 0.1x = 0$.

Collecting and combining like terms gives us $0.1x + 1.82 = 0$.

Then we can subtract 1.82 from both sides to get $0.1x = -1.82$, and finally divide by 0.1 (or multiply by 10) to get $x = -18.2$.

Solving Real-World Problems Using Multi-Step Equations

Example 6

A growers' cooperative has a farmer's market in the town center every Saturday. They sell what they have grown and split the money into several categories. 8.5% of all the money taken in is set aside for sales tax. \$150 goes to pay the rent on the space they occupy. What remains is split evenly between the seven growers. How much total money is taken in if each grower receives a \$175 share?

Let's translate the text above into an equation. The unknown is going to be the total money taken in dollars. We'll call this x .

"8.5% of all the money taken in is set aside for sales tax." This means that 91.5% of the money remains. This is $0.915x$.

"\$150 goes to pay the rent on the space they occupy." This means that what's left is $0.915x - 150$.

"What remains is split evenly between the 7 growers." That means each grower gets $\frac{0.915x - 150}{7}$.

If each grower's share is \$175, then our equation to find x is $\frac{0.915x - 150}{7} = 175$.

First we multiply both sides by 7 to get $0.915x - 150 = 1225$.

Then add 150 to both sides to get $0.915x = 1375$.

Finally divide by 0.915 to get $x \approx 1502.7322$. Since we want our answer in dollars and cents, we round to two decimal places, or \$1502.73.

The workers take in a total of \$1502.73.

Example 7

A factory manager is packing engine components into wooden crates to be shipped on a small truck. The truck is designed to hold sixteen crates, and will safely carry a 1200 lb cargo. Each crate weighs 12 lbs empty. How much weight should the manager instruct the workers to put in each crate in order to get the shipment weight as close as possible to 1200 lbs?

The unknown quantity is the weight to put in each box, so we'll call that x .

Each crate when full will weigh $x + 12$ lbs, so all 16 crates together will weigh $16(x + 12)$ lbs.

We also know that all 16 crates together should weigh 1200 lbs, so we can say that $16(x + 12) = 1200$.

To solve this equation, we can start by dividing both sides by 16: $x + 12 = \frac{1200}{16} = 75$.

Then subtract 12 from both sides: $x = 63$.

The manager should tell the workers to put 63 lbs of components in each crate.

Ohm's Law

The electrical current, I (amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship $V = I \cdot R$ where R is the resistance measured in Ohms (Ω).

Example 8

A scientist is trying to deduce the resistance of an unknown component. He labels the resistance of the unknown component $x \Omega$. The resistance of a circuit containing a number of these components is $(5x + 20)\Omega$. If a 120 volt potential difference across the circuit produces a current of 2.5 amps, calculate the resistance of the unknown component.

Solution

To solve this, we need to start with the equation $V = I \cdot R$ and substitute in $V = 120$, $I = 2.5$, and $R = 5x + 20$. That gives us $120 = 2.5(5x + 20)$.

Distribute the 2.5 to get $120 = 12.5x + 50$.

Subtract 50 from both sides to get $70 = 12.5x$.

Finally, divide by 12.5 to get $5.6 = x$.

The unknown components have a resistance of 5.6 Ω .

Distance, Speed and Time

The speed of a body is the distance it travels per unit of time. That means that we can also find out how far an object moves in a certain amount of time if we know its speed: we use the equation "distance = speed \times time."

Example 8

Shanice's car is traveling 10 miles per hour slower than twice the speed of Brandon's car. She covers 93

miles in 1 hour 30 minutes. How fast is Brandon driving?

Solution

Here, we don't know either Brandon's speed or Shanice's, but since the question asks for Brandon's speed, that's what we'll use as our variable x .

The distance Shanice covers in miles is 93, and the time in hours is 1.5. Her speed is 10 less than twice Brandon's speed, or $2x - 10$ miles per hour. Putting those numbers into the equation gives us $93 = 1.5(2x - 10)$.

First we distribute, to get $93 = 3x - 15$.

Then we add 15 to both sides to get $108 = 3x$.

Finally we divide by 3 to get $36 = x$.

Brandon is driving at 36 miles per hour.

We can check this answer by considering the situation another way: we can solve for Shanice's speed instead of Brandon's and then check that against Brandon's speed. We'll use y for Shanice's speed since we already used x for Brandon's.

The equation for Shanice's speed is simply $93 = 1.5y$. We can divide both sides by 1.5 to get $62 = y$, so Shanice is traveling at 62 miles per hour.

The problem tells us that Shanice is traveling 10 mph slower than twice Brandon's speed; that would mean that 62 is equal to 2 times 36 minus 10. Is that true? Well, 2 times 36 is 72, minus 10 is 62. The answer checks out.

In algebra, there's almost always more than one method of solving a problem. If time allows, it's always a good idea to try to solve the problem using two different methods just to confirm that you've got the answer right.

Speed of Sound

The speed of sound in dry air, v , is given by the equation $v = 331 + 0.6T$, where T is the temperature in Celsius and v is the speed of sound in meters per second.

Example 9

Tashi hits a drainpipe with a hammer and 250 meters away Minh hears the sound and hits his own drainpipe. Unfortunately, there is a one second delay between him hearing the sound and hitting his own pipe. Tashi accurately measures the time between her hitting the pipe and hearing Minh's pipe at 2.46 seconds. What is the temperature of the air?

This is a complex problem and we need to be careful in writing our equations. First of all, the distance the sound travels is equal to the speed of sound multiplied by the time, and the speed is given by the equation above. So the distance equals $(331 + 0.6T) \times \text{time}$, and the time is $2.46 - 1$ (because for 1 second out of the 2.46 seconds measured, there was no sound actually traveling). We also know that the distance is 250×2 (because the sound traveled from Tashi to Minh and back again), so our equation is $250 \times 2 = (331 + 0.6T)(2.46 - 1)$, which simplifies to $500 = 1.46(331 + 0.6T)$.

Distributing gives us $500 = 483.26 + 0.876T$, and subtracting 483.26 from both sides gives us $16.74 = 0.876T$. Then we divide by 0.876 to get $T \approx 19.1$.

The temperature is about 19.1 degrees Celsius.

Lesson Summary

- Multi-step equations are slightly more complex than one - and two-step equations, but use the same basic techniques.
- If dividing a number outside of parentheses will produce fractions, it is often better to use the **Distributive Property** to expand the terms and then combine like terms to solve the equation.

Review Questions

1. Solve the following equations for the unknown variable.

(a) $3(x - 1) - 2(x + 3) = 0$

(b) $3(x + 3) - 2(x - 1) = 0$

(c) $7(w + 20) - w = 5$

(d) $5(w + 20) - 10w = 5$

(e) $9(x - 2) - 3x = 3$

(f) $12(t - 5) + 5 = 0$

(g) $2(2d + 1) = \frac{2}{3}$

(h) $2\left(5a - \frac{1}{3}\right) = \frac{2}{7}$

(i) $\frac{2}{9}\left(i + \frac{2}{3}\right) = \frac{2}{5}$

(j) $4\left(v + \frac{1}{4}\right) = \frac{35}{2}$

(k) $\frac{g}{10} = \frac{6}{3}$

(l) $\frac{s-4}{11} = \frac{2}{5}$

(m) $\frac{2k}{7} = \frac{3}{8}$

(n) $\frac{7x+4}{3} = \frac{9}{2}$

(o) $\frac{9y-3}{6} = \frac{5}{2}$

(p) $\frac{r}{3} + \frac{r}{2} = 7$

(q) $\frac{p}{16} - \frac{2p}{3} = \frac{1}{9}$

(r) $\frac{m+3}{2} - \frac{m}{4} = \frac{1}{3}$

(s) $5\left(\frac{k}{3} + 2\right) = \frac{32}{3}$

(t) $\frac{3}{z} = \frac{2}{5}$

(u) $\frac{2}{r} + 2 = \frac{10}{3}$

(v) $\frac{t-12}{5} = \frac{3+z}{z}$

2. An engineer is building a suspended platform to raise bags of cement. The platform has a mass of 200 kg, and each bag of cement is 40 kg. He is using two steel cables, each capable of holding 250 kg. Write an equation for the number of bags he can put on the platform at once, and solve it.
3. A scientist is testing a number of identical components of unknown resistance which he labels $x\Omega$. He connects a circuit with resistance $(3x + 4)\Omega$ to a steady 12 volt supply and finds that this produces a current of 1.2 amps. What is the value of the unknown resistance?
4. Lydia inherited a sum of money. She split it into five equal parts. She invested three parts of the money in a high-interest bank account which added 10% to the value. She placed the rest of her inheritance plus \$500 in the stock market but lost 20% on that money. If the two accounts end up with exactly the same amount of money in them, how much did she inherit?
5. Pang drove to his mother's house to drop off her new TV. He drove at 50 miles per hour there and back, and spent 10 minutes dropping off the TV. The entire journey took him 94 minutes. How far away does his mother live?

1.5 Equations with Variables on Both Sides

Learning Objectives

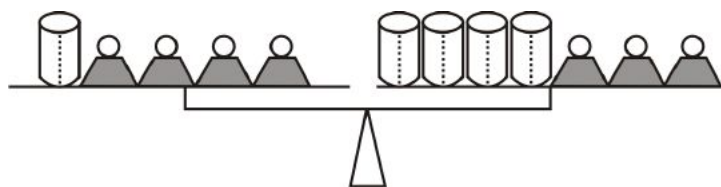
- Solve an equation with variables on both sides.
- Solve an equation with grouping symbols.
- Solve real-world problems using equations with variables on both sides.

Solve an Equation with Variables on Both Sides

When a variable appears on both sides of the equation, we need to manipulate the equation so that all variable terms appear on one side, and only constants are left on the other.

Example 1

Dwayne was told by his chemistry teacher to measure the weight of an empty beaker using a balance. Dwayne found only one lb weights, and so devised the following way of balancing the scales.



Knowing that each weight is one lb, calculate the weight of one beaker.

Solution

We know that the system balances, so the weights on each side must be equal. We can write an algebraic expression based on this fact. The unknown quantity, the weight of the beaker, will be our x . We can see that on the left hand scale we have one beaker and four weights. On the right scale, we have four beakers and three weights. The balancing of the scales is analogous to the balancing of the following equation:

$$x + 4 = 4x + 3$$

“One beaker plus 4 lbs **equals** 4 beakers plus 3 lbs”

To solve for the weight of the beaker, we want all the constants (numbers) on one side and all the variables (terms with x in them) on the other side. Since there are more beakers on the right and more weights on the left, we’ll try to move all the x terms (beakers) to the right, and the constants (weights) to the left.

First we subtract 3 from both sides to get $x + 1 = 4x$.

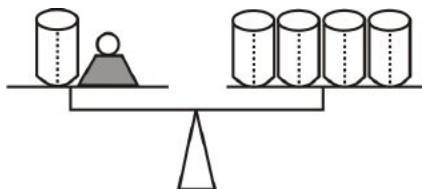
Then we subtract x from both sides to get $1 = 3x$.

Finally we divide by 3 to get $\frac{1}{3} = x$.

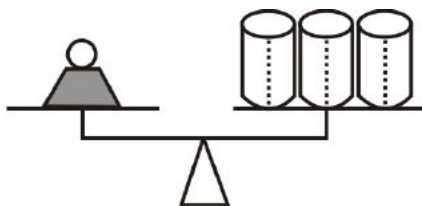
The weight of the beaker is one-third of a pound.

We can do the same with the real objects as we did with the equation. Just as we subtracted amounts from each side of the equation, we could remove a certain number of weights or beakers from each scale. Because we remove the same number of objects from each side, we know the scales will still balance.

First, we could remove three weights from each scale. This would leave one beaker and one weight on the left and four beakers on the right (in other words $x + 1 = 4x$):



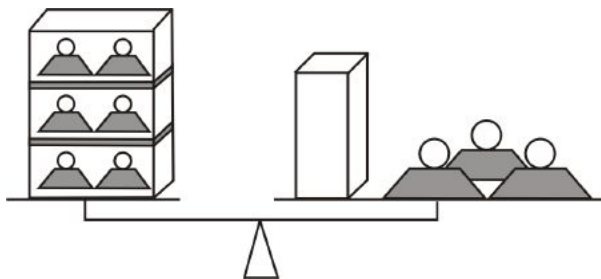
Then we could remove one beaker from each scale, leaving only one weight on the left and three beakers on the right, to get $1 = 3x$:



Looking at the balance, it is clear that the weight of one beaker is one-third of a pound.

Example 2

Sven was told to find the weight of an empty box with a balance. Sven found some one lb weights and five lb weights. He placed two one lb weights in three of the boxes and with a fourth empty box found the following way of balancing the scales:



Knowing that small weights are one lb and big weights are five lbs, calculate the weight of one box.

We know that the system balances, so the weights on each side must be equal. We can write an algebraic expression based on this equality. The unknown quantity—the weight of each empty box, in pounds—will be our x . A box with two 1 lb weights in it weighs $(x + 2)$ pounds. Our equation, based on the picture, is $3(x + 2) = x + 3(5)$.

Distributing the 3 and simplifying, we get $3x + 6 = x + 15$.

Subtracting x from both sides, we get $2x + 6 = 15$.

Subtracting 6 from both sides, we get $2x = 9$.

And finally we can divide by 2 to get $x = \frac{9}{2}$, or $x = 4.5$.

Each box weighs 4.5 lbs.

To see more examples of solving equations with variables on both sides of the equation, see the Khan Academy video at <http://www.youtube.com/watch?v=Zn-GbH2S0Dk> <http://www.youtube.com/watch?v=Zn-GbH2S0Dk>.

Solve an Equation with Grouping Symbols

As you've seen, we can solve equations with variables on both sides even when some of the variables are in parentheses; we just have to get rid of the parentheses, and then we can start combining like terms. We use the same technique when dealing with fractions: first we multiply to get rid of the fractions, and then we can shuffle the terms around by adding and subtracting.

Example 3

Solve $3x + 2 = \frac{5x}{3}$.

Solution

The first thing we'll do is get rid of the fraction. We can do this by multiplying both sides by 3, leaving $3(3x + 2) = 5x$.

Then we distribute to get rid of the parentheses, leaving $9x + 6 = 5x$.

We've already got all the constants on the left side, so we'll move the variables to the right side by subtracting $9x$ from both sides. That leaves us with $6 = -4x$.

And finally, we divide by -4 to get $-\frac{3}{2} = x$, or $x = -1.5$.

Example 4

Solve $7x + 2 = \frac{5x-3}{6}$.

Solution

Again we start by eliminating the fraction. Multiplying both sides by 6 gives us $6(7x + 2) = 5x - 3$, and distributing gives us $42x + 12 = 5x - 3$.

Subtracting $5x$ from both sides gives us $37x + 12 = -3$.

Subtracting 12 from both sides gives us $37x = -15$.

Finally, dividing by 37 gives us $x = -\frac{15}{37}$.

Example 5

Solve the following equation for x : $\frac{14x}{(x+3)} = 7$

Solution

The form of the left hand side of this equation is known as a **rational function** because it is the ratio of two other functions: $14x$ and $(x + 3)$. But we can solve it just like any other equation involving fractions.

First we multiply both sides by $(x + 3)$ to get rid of the fraction. Now our equation is $14x = 7(x + 3)$.

Then we distribute: $14x = 7x + 21$.

Then subtract $7x$ from both sides: $7x = 21$.

And divide by 7: $x = 3$.

Solve Real-World Problems Using Equations with Variables on Both Sides

Here's another chance to practice translating problems from words to equations. What is the equation asking? What is the **unknown** variable? What quantity will we use for our variable?

The text explains what's happening. Break it down into small, manageable chunks, and follow what's going on with our variable all the way through the problem.

More on Ohm's Law

Recall that the electrical current, I (amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship $V = I \cdot R$ where R is the resistance measured in Ohms (Ω).

The resistance R of a number of components wired in a **series** (one after the other) is simply the sum of all the resistances of the individual components.

Example 6

In an attempt to find the resistance of a new component, a scientist tests it in series with standard resistors. A fixed voltage causes a 4.8 amp current in a circuit made up from the new component plus a 15Ω resistor in series. When the component is placed in a series circuit with a 50Ω resistor, the same voltage causes a 2.0 amp current to flow. Calculate the resistance of the new component.

This is a complex problem to translate, but once we convert the information into equations it's relatively straightforward to solve. First, we are trying to find the resistance of the new component (in Ohms, Ω). This is our x . We don't know the voltage that is being used, but we can leave that as a variable, V . Our first situation has a total resistance that equals the unknown resistance plus 15Ω . The current is 4.8 amps. Substituting into the formula $V = I \cdot R$, we get $V = 4.8(x + 15)$.

Our second situation has a total resistance that equals the unknown resistance plus 50Ω . The current is 2.0 amps. Substituting into the same equation, this time we get $V = 2(x + 50)$.

We know the voltage is fixed, so the V in the first equation must equal the V in the second. That means we can set the right-hand sides of the two equations equal to each other: $4.8(x + 15) = 2(x + 50)$. Then we can solve for x .

Distribute the constants first: $4.8x + 72 = 2x + 100$.

Subtract $2x$ from both sides: $2.8x + 72 = 100$.

Subtract 72 from both sides: $2.8x = 28$.

Divide by 2.8: $x = 10$.

The resistance of the component is 10Ω .

Lesson Summary

If an unknown variable appears on both sides of an equation, distribute as necessary. Then simplify the equation to have the unknown on only one side.

Review Questions

1. Solve the following equations for the unknown variable.

(a) $3(x - 1) = 2(x + 3)$

(b) $7(x + 20) = x + 5$

(c) $9(x - 2) = 3x + 3$

(d) $2\left(a - \frac{1}{3}\right) = \frac{2}{5}\left(a + \frac{2}{3}\right)$

(e) $\frac{2}{7}\left(t + \frac{2}{3}\right) = \frac{1}{5}\left(t - \frac{2}{3}\right)$

(f) $\frac{1}{7}\left(v + \frac{1}{4}\right) = 2\left(\frac{3v}{2} - \frac{5}{2}\right)$

(g) $\frac{y-4}{11} = \frac{2}{5} \cdot \frac{2y+1}{3}$

(h) $\frac{z}{16} = \frac{2(3z+1)}{9}$

- (i) $\frac{q}{\frac{1}{6}} + \frac{q}{\frac{1}{6}} = \frac{(3q+1)}{9} + \frac{3}{2}$
 (j) $\frac{\frac{3}{x}}{\frac{2}{x+1}} = \frac{2}{x+1}$
 (k) $\frac{5}{2+p} = \frac{3}{p-8}$
- Manoj and Tamar are arguing about a number trick they heard. Tamar tells Andrew to think of a number, multiply it by five and subtract three from the result. Then Manoj tells Andrew to think of a number, add five and multiply the result by three. Andrew says that whichever way he does the trick he gets the same answer.
 - What was the number Andrew started with?
 - What was the result Andrew got both times?
 - Name another set of steps that would have resulted in the same answer if Andrew started with the same number.
 - Manoj and Tamar try to come up with a harder trick. Manoj tells Andrew to think of a number, double it, add six, and then divide the result by two. Tamar tells Andrew to think of a number, add five, triple the result, subtract six, and then divide the result by three.
 - Andrew tries the trick both ways and gets an answer of 10 each time. What number did he start out with?
 - He tries again and gets 2 both times. What number did he start out with?
 - Is there a number Andrew can start with that will *not* give him the same answer both ways?
 - Bonus:** Name another set of steps that would give Andrew the same answer every time as he would get from Manoj's and Tamar's steps.
 - I have enough money to buy five regular priced CDs and have \$6 left over. However, all CDs are on sale today, for \$4 less than usual. If I borrow \$2, I can afford nine of them.
 - How much are CDs on sale for today?
 - How much would I have to borrow to afford nine of them if they weren't on sale?
 - Five identical electronics components were connected in series. A fixed but unknown voltage placed across them caused a 2.3 amp current to flow. When two of the components were replaced with standard 10Ω resistors, the current dropped to 1.9 amps. What is the resistance of each component?
 - Solve the following resistance problems. Assume the same voltage is applied to all circuits.
 - Three unknown resistors plus 20Ω give the same current as one unknown resistor plus 70Ω .
 - One unknown resistor gives a current of 1.5 amps and a 15Ω resistor gives a current of 3.0 amps.
 - Seven unknown resistors plus 18Ω gives twice the current of two unknown resistors plus 150Ω .
 - Three unknown resistors plus 1.5Ω gives a current of 3.6 amps and seven unknown resistors plus seven 12Ω resistors gives a current of 0.2 amps.

Check Solutions to Equations

You will often need to check solutions to equations in order to check your work. In a math class, checking that you arrived at the correct solution is very good practice. We check the solution to an equation by replacing the variable in an equation with the value of the solution. A solution should result in a true statement when plugged into the equation.

Example 4

Check that the given number is a solution to the corresponding equation.

- $y = -1$; $3y + 5 = -2y$
- $z = 3$; $z^2 + 2z = 8$
- $x = -\frac{1}{2}$; $3x + 1 = x$

Solution

Replace the variable in each equation with the given value.

a)

$$\begin{aligned}3(-1) + 5 &= -2(-1) \\-3 + 5 &= 2 \\2 &= 2\end{aligned}$$

This is a true statement. This means that $y = -1$ is a solution to $3y + 5 = -2y$.

b)

$$\begin{aligned}3^2 + 2(3) &= 8 \\9 + 6 &= 8 \\15 &= 8\end{aligned}$$

This is not a true statement. This means that $z = 3$ is **not a solution** to $z^2 + 2z = 8$.

c)

$$\begin{aligned}3\left(-\frac{1}{2}\right) + 1 &= -\frac{1}{2} \\ \left(-\frac{3}{2}\right) + 1 &= -\frac{1}{2} \\ -\frac{3}{2} + \frac{2}{2} &= -\frac{1}{2}\end{aligned}$$

This is a true statement. This means that $x = -\frac{1}{2}$ is a solution to $3x + 1 = x$.

Solve Real-World Problems Using an Equation

Let's use what we have learned about defining variables and writing equations to solve some real-world problems.

Example 6

Tomatoes cost \$0.50 each and avocados cost \$2.00 each. Anne buys six more tomatoes than avocados. Her total bill is \$8. How many tomatoes and how many avocados did Anne buy?

Solution

Define

Let a = the number of avocados Anne buys.

Translate

Anne buys six more tomatoes than avocados. This means that $a + 6$ = the number of tomatoes.

Tomatoes cost \$0.50 each and avocados cost \$2.00 each. Her total bill is \$8. This means that .50 times the number of tomatoes plus 2 times the number of avocados equals 8.

$$\begin{aligned}0.5(a + 6) + 2a &= 8 \\0.5a + 0.5 \cdot 6 + 2a &= 8 \\2.5a + 3 &= 8 \\2.5a &= 5 \\a &= 2\end{aligned}$$

Remember that a = the number of avocados, so Anne buys two avocados. The number of tomatoes is $a + 6 = 2 + 6 = 8$.

Answer

Anne bought 2 avocados and 8 tomatoes.

Check

If Anne bought two avocados and eight tomatoes, the total cost is: $(2 \times 2) + (8 \times 0.5) = 4 + 4 = 8$. **The answer checks out.**

Example 7

To organize a picnic Peter needs at least two times as many hamburgers as hot dogs. He has 24 hot dogs. What is the possible number of hamburgers Peter has?

Solution

Define

Let x = number of hamburgers

Translate

Peter needs at least two times as many hamburgers as hot dogs. He has 24 hot dogs.

This means that twice the number of hot dogs is less than or equal to the number of hamburgers.

$$2 \times 24 \leq x, \text{ or } 48 \leq x$$

Answer

Peter needs at least 48 hamburgers.

Check

48 hamburgers is twice the number of hot dogs. So more than 48 hamburgers is more than twice the number of hot dogs. **The answer checks out.**

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Review Questions

- Define the variables and translate the following expressions into equations.
 - Peter's Lawn Mowing Service charges \$10 per job and \$0.20 per square yard. Peter earns \$25 for a job.
 - Renting the ice-skating rink for a birthday party costs \$200 plus \$4 per person. The rental costs \$324 in total.
 - Renting a car costs \$55 per day plus \$0.45 per mile. The cost of the rental is \$100.
 - Nadia gave Peter 4 more blocks than he already had. He already had 7 blocks.
- Check whether the given number is a solution to the corresponding equation.
 - $a = -3$; $4a + 3 = -9$
 - $x = \frac{4}{3}$; $\frac{3}{4}x + \frac{1}{2} = \frac{3}{2}$
 - $y = 2$; $2.5y - 10.0 = -5.0$
 - $z = -5$; $2(5 - 2z) = 20 - 2(z - 1)$
- The cost of a Ford Focus is 27% of the price of a Lexus GS 450h. If the price of the Ford is \$15000, what is the price of the Lexus?

4. On your new job you can be paid in one of two ways. You can either be paid \$1000 per month plus 6% commission of total sales or be paid \$1200 per month plus 5% commission on sales over \$2000. For what amount of sales is the first option better than the second option? Assume there are always sales over \$2000.
5. A phone company offers a choice of three text-messaging plans. Plan A gives you unlimited text messages for \$10 a month; Plan B gives you 60 text messages for \$5 a month and then charges you \$0.05 for each additional message; and Plan C has no monthly fee but charges you \$0.10 per message.
 - (a) If m is the number of messages you send per month, write an expression for the monthly cost of each of the three plans.
 - (b) For what values of m is Plan A cheaper than Plan B?
 - (c) For what values of m is Plan A cheaper than Plan C?
 - (d) For what values of m is Plan B cheaper than Plan C?
 - (e) For what values of m is Plan A the cheapest of all? (Hint: for what values is A both cheaper than B and cheaper than C?)
 - (f) For what values of m is Plan B the cheapest of all? (Careful—for what values is B cheaper than A?)
 - (g) For what values of m is Plan C the cheapest of all?
 - (h) If you send 30 messages per month, which plan is cheapest?
 - (i) What is the cost of each of the three plans if you send 30 messages per month?

1.6 Patterns and Equations

Learning Objectives

- Write an equation.
- Use a verbal model to write an equation.
- Solve problems using equations.

Introduction

In mathematics, and especially in algebra, we look for patterns in the numbers we see. The tools of algebra help us describe these patterns with words and with **equations**(formulas or functions). An equation is a mathematical recipe that gives the value of one variable in terms of another.

For example, if a theme park charges \$12 admission, then the number of people who enter the park every day and the amount of money taken in by the ticket office are related mathematically, and we can write a rule to find the amount of money taken in by the ticket office.

In words, we might say “The amount of money taken in is equal to twelve times the number of people who enter the park.”

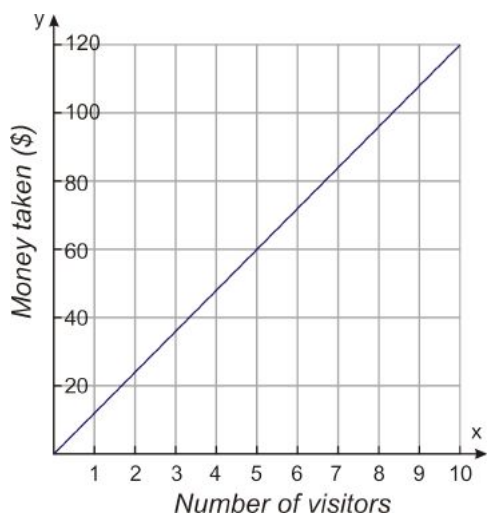
We could also make a table. The following table relates the number of people who visit the park and the total money taken in by the ticket office.

| | | | | | | | |
|---------------------|----|----|----|----|----|----|----|
| Number of visitors | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| Money taken in (\$) | 12 | 24 | 36 | 48 | 60 | 72 | 84 |

Clearly, we would need a **big** table to cope with a busy day in the middle of a school vacation!

A third way we might relate the two quantities (visitors and money) is with a graph. If we plot the money taken in on the **vertical axis** and the number of visitors on the **horizontal axis**, then we would have

a graph that looks like the one shown below. Note that this graph shows a smooth line that includes non-whole number values of x (e.g. $x = 2.5$). In real life this would not make sense, because fractions of people can't visit a park. This is an issue of domain and range, something we will talk about later.



The method we will examine in detail in this lesson is closer to the first way we chose to describe the relationship. In words we said that “*The amount of money taken in is twelve times the number of people who enter the park.*” In mathematical terms we can describe this sort of relationship with **variables**. A variable is a letter used to represent an unknown quantity. We can see the beginning of a mathematical formula in the words:

The amount of money taken in **is** twelve **times** the number of people who enter the park.

This can be translated to:

$$\text{the amount of money taken in} = 12 \times (\text{the number of people who enter the park})$$

We can now see which quantities can be assigned to **letters**. First we must state which letters (or **variables**) relate to which quantities. We call this **defining the variables**:

Let x = the number of people who enter the theme park.

Let y = the total amount of money taken in at the ticket office.

We now have a fourth way to describe the relationship: with an algebraic equation.

$$y = 12x$$

Writing a mathematical equation using variables is very convenient. You can perform all of the operations necessary to solve this problem without having to write out the known and unknown quantities over and over again. At the end of the problem, you just need to remember which quantities x and y represent.

Write an Equation

An equation is a term used to describe a collection of **numbers** and **variables** related through mathematical **operators**. An **algebraic equation** will contain letters that represent real quantities. For example, if we wanted to use the algebraic equation in the example above to find the money taken in for a certain number of visitors, we would substitute that number for x and then solve the resulting equation for y .

Example 1

A theme park charges \$12 entry to visitors. Find the money taken in if 1296 people visit the park.

Let's break the solution to this problem down into steps. This will be a useful strategy for all the problems in this lesson.

Step 1: Extract the important information.

$$\begin{aligned}(\text{number of dollars taken in}) &= 12 \times (\text{number of visitors}) \\ (\text{number of visitors}) &= 1296\end{aligned}$$

Step 2: Translate into a mathematical equation. To do this, we pick variables to stand for the numbers.

$$\begin{aligned}\text{Let } y &= (\text{number of dollars taken in}). \\ \text{Let } x &= (\text{number of visitors}).\end{aligned}$$

$$\begin{aligned}(\text{number of dollars taken in}) &= 12 \times (\text{number of visitors}) \\ y &= 12 \times x\end{aligned}$$

Step 3: Substitute in any known values for the variables.

$$\begin{aligned}y &= 12 \times x \\ x &= 1296 \\ \therefore \\ y &= 12 \times 1296\end{aligned}$$

Step 4: Solve the equation.

$$y = 12 \times 1296 = 15552$$

The amount of money taken in is \$15552.

Step 5: Check the result.

If \$15552 is taken at the ticket office and tickets are \$12, then we can divide the total amount of money collected by the price per individual ticket.

$$(\text{number of people}) = \frac{15552}{12} = 1296$$

1296 is indeed the number of people who entered the park. **The answer checks out.**

Example 2

The following table shows the relationship between two quantities. First, write an equation that describes the relationship. Then, find out the value of b when a is 750.

| | | | | | | |
|-----|----|----|----|----|-----|-----|
| a | 0 | 10 | 20 | 30 | 40 | 50 |
| b | 20 | 40 | 60 | 80 | 100 | 120 |

Step 1: Extract the important information.

We can see from the table that every time a increases by 10, b increases by 20. However, b is not simply twice the value of a . We can see that when $a = 0$, $b = 20$, and this gives a clue as to what rule the pattern follows. The rule linking a and b is:

“To find b , double the value of a and add 20.”

Step 2: Translate into a mathematical equation:

Table 1.2:

| Text | Translates to | Mathematical Expression |
|----------------------------|---------------|-------------------------|
| “To find b ” | \rightarrow | $b =$ |
| “double the value of a ” | \rightarrow | $2a$ |
| “add 20” | \rightarrow | $+ 20$ |

Our equation is $b = 2a + 20$.

Step 3: Solve the equation.

The original problem asks for the value of b when a is 750. When a is 750, $b = 2a + 20$ becomes $b = 2(750) + 20$. Following the order of operations, we get:

$$\begin{aligned}
 b &= 2(750) + 20 \\
 &= 1500 + 20 \\
 &= 1520
 \end{aligned}$$

Step 4: Check the result.

In some cases you can check the result by plugging it back into the original equation. Other times you must simply double-check your math. In either case, checking your answer is *always* a good idea. In this case, we can plug our answer for b into the equation, along with the value for a , and see what comes out. $1520 = 2(750) + 20$ is TRUE because both sides of the equation are equal. A true statement means that **the answer checks out**.

Use a Verbal Model to Write an Equation

In the last example we developed a **rule**, written in words, as a way to develop an algebraic **equation**. We will develop this further in the next few examples.

Example 3

The following table shows the values of two related quantities. Write an equation that describes the relationship mathematically.

Table 1.3:

| x -value | y -value |
|------------|------------|
| -2 | 10 |
| 0 | 0 |
| 2 | -10 |
| 4 | -20 |
| 6 | -30 |

Step 1: Extract the important information.

We can see from the table that y is five times bigger than x . The value for y is negative when x is positive, and it is positive when x is negative. Here is the rule that links x and y :

“ y is the negative of five times the value of x ”

Step 2: Translate this statement into a mathematical equation.

Table 1.4:

| Text | Translates to | Mathematical Expression |
|--------------------------------------|---------------|-------------------------|
| “y is” | → | $y =$ |
| “negative 5 times the value of x ” | → | $-5x$ |

Our equation is $y = -5x$.

Step 3: There is nothing in this problem to **solve** for. We can move to Step 4.

Step 4: Check the result.

In this case, the way we would check our answer is to use the equation to generate our own xy pairs. If they match the values in the table, then we know our equation is correct. We will plug in -2, 0, 2, 4, and 6 for x and solve for y :

Table 1.5:

| x | y |
|-----|---------------|
| -2 | $-5(-2) = 10$ |
| 0 | $-5(0) = 0$ |
| 2 | $-5(2) = -10$ |
| 4 | $-5(4) = -20$ |
| 6 | $-5(6) = -30$ |

The y -values in this table match the ones in the earlier table. **The answer checks out.**

Example 4

Zarina has a \$100 gift card, and she has been spending money on the card in small regular amounts. She checks the balance on the card weekly and records it in the following table.

Table 1.6:

| Week Number | Balance (\$) |
|-------------|--------------|
| 1 | 100 |
| 2 | 78 |
| 3 | 56 |
| 4 | 34 |

Write an equation for the money remaining on the card in any given week.

Step 1: Extract the important information.

The balance remaining on the card is not just a constant multiple of the week number; 100 is 100 times 1, but 78 is not 100 times 2. But there is still a pattern: the balance decreases by 22 whenever the week number increases by 1. This suggests that the balance is somehow related to the amount “-22 times the week number.”

In fact, the balance equals “-22 times the week number, plus *something*.” To determine what that *something* is, we can look at the values in one row on the table—for example, the first row, where we have a balance

of \$100 for week number 1.

Step 2: Translate into a mathematical equation.

First, we define our variables. Let n stand for the week number and b for the balance.

Then we can translate our verbal expression as follows:

Table 1.7:

| Text | Translates to | Mathematical Expression |
|---|---------------|-------------------------|
| Balance equals -22 times the week number, plus <i>something</i> . | \rightarrow | $b = -22n + ?$ |

To find out what that ? represents, we can plug in the values from that first row of the table, where $b = 100$ and $n = 1$. This gives us $100 = -22(1) + ?$.

So what number gives 100 when you add -22 to it? The answer is 122, so that is the number the ? stands for. Now our final equation is:

$$b = -22n + 122$$

Step 3: All we were asked to **find** was the expression. We weren't asked to solve it, so we can move to Step 4.

Step 4: Check the result.

To check that this equation is correct, we see if it really reproduces the data in the table. To do that we plug in values for n :

$$n = 1 \rightarrow b = -22(1) + 122 = 122 - 22 = 100$$

$$n = 2 \rightarrow b = -22(2) + 122 = 122 - 44 = 78$$

$$n = 3 \rightarrow b = -22(3) + 122 = 122 - 66 = 56$$

$$n = 4 \rightarrow b = -22(4) + 122 = 122 - 88 = 34$$

The equation perfectly reproduces the data in the table. **The answer checks out.**

Solve Problems Using Equations

Let's solve the following real-world problem by using the given information to write a mathematical equation that can be solved for a solution.

Example 5

A group of students are in a room. After 25 students leave, it is found that $\frac{2}{3}$ of the original group is left in the room. How many students were in the room at the start?

Step 1: Extract the important information

We know that 25 students leave the room.

We know that $\frac{2}{3}$ of the original number of students are left in the room.

We need to find how many students were in the room at the start.

Step 2: Translate into a mathematical equation. Initially we have an unknown number of students in the room. We can refer to this as the original number.

Let's define the variable x = the original number of students in the room. After 25 students leave the room, the number of students in the room is $x - 25$. We also know that the number of students left is $\frac{2}{3}$ of x . So we have two expressions for the number of students left, and those two expressions are equal because they represent the same number. That means our equation is:

$$\frac{2}{3}x = x - 25$$

Step 3: Solve the equation.

Add 25 to both sides.

$$\begin{aligned} x - 25 &= \frac{2}{3}x \\ x - 25 + 25 &= \frac{2}{3}x + 25 \\ x &= \frac{2}{3}x + 25 \end{aligned}$$

Subtract $\frac{2}{3}x$ from both sides.

$$\begin{aligned} x - \frac{2}{3}x &= \frac{2}{3}x - \frac{2}{3}x + 25 \\ \frac{1}{3}x &= 25 \end{aligned}$$

Multiply both sides by 3.

$$\begin{aligned} 3 \cdot \frac{1}{3}x &= 3 \cdot 25 \\ x &= 75 \end{aligned}$$

Remember that x represents the original number of students in the room. So, there were 75 students in the room to start with.

Step 4: Check the answer:

If we start with 75 students in the room and 25 of them leave, then there are $75 - 25 = 50$ students left in the room.

$\frac{2}{3}$ of the original number is $\frac{2}{3} \cdot 75 = 50$.

This means that the number of students who are left over equals $\frac{2}{3}$ of the original number. **The answer checks out.**

The method of defining variables and writing a mathematical equation is the method you will use the most in an algebra course. This method is often used together with other techniques such as making a table of values, creating a graph, drawing a diagram and looking for a pattern.

Review Questions

Table 1.8:

| Day | Profit |
|-----|--------|
| 1 | 20 |
| 2 | 40 |
| 3 | 60 |
| 4 | 80 |
| 5 | 100 |

- The above table depicts the profit in dollars taken in by a store each day.
 - Write a mathematical equation that describes the relationship between the variables in the table.
 - What is the profit on day 10?
 - If the profit on a certain day is \$200, what is the profit on the next day?
- Write a mathematical equation that describes the situation: *A full cookie jar has 24 cookies. How many cookies are left in the jar after you have eaten some?*
 - How many cookies are in the jar after you have eaten 9 cookies?
 - How many cookies are in the jar after you have eaten 9 cookies and then eaten 3 more?
- Write a mathematical equation for the following situations and solve.
 - Seven times a number is 35. What is the number?
 - Three times a number, plus 15, is 24. What is the number?
 - Twice a number is three less than five times another number. Three times the second number is 15. What are the numbers?
 - One number is 25 more than 2 times another number. If each number were multiplied by five, their sum would be 350. What are the numbers?
 - The sum of two consecutive integers is 35. What are the numbers?
 - Peter is three times as old as he was six years ago. How old is Peter?
- How much water should be added to one liter of pure alcohol to make a mixture of 25% alcohol?
- A mixture of 50% alcohol and 50% water has 4 liters of water added to it. It is now 25% alcohol. What was the total volume of the original mixture?
- In Crystal's silverware drawer there are twice as many spoons as forks. If Crystal adds nine forks to the drawer, there will be twice as many forks as spoons. How many forks and how many spoons are in the drawer right now?
- Mia drove to Javier's house at 40 miles per hour. Javier's house is 20 miles away. Mia arrived at Javier's house at 2:00 pm. What time did she leave?
 - Mia left Javier's house at 6:00 pm to drive home. This time she drove 25% faster. What time did she arrive home?
 - The next day, Mia took the expressway to Javier's house. This route was 24 miles long, but she was able to drive at 60 miles per hour. How long did the trip take?
 - When Mia took the same route back, traffic on the expressway was 20% slower. How long did the return trip take?
- The price of an mp3 player decreased by 20% from last year to this year. This year the price of the player is \$120. What was the price last year?
- SmartCo sells deluxe widgets for \$60 each, which includes the cost of manufacture plus a 20% markup. What does it cost SmartCo to manufacture each widget?
- Jae just took a math test with 20 questions, each worth an equal number of points. The test is worth 100 points total.
 - Write an equation relating the number of questions Jae got right to the total score he will get on the test.

- (b) If a score of 70 points earns a grade of $C-$, how many questions would Jae need to get right to get a $C-$ on the test?
- (c) If a score of 83 points earns a grade of B , how many questions would Jae need to get right to get a B on the test?
- (d) Suppose Jae got a score of 60% and then was allowed to retake the test. On the retake, he got all the questions right that he got right the first time, and also got half the questions right that he got wrong the first time. What is his new score?

1.7 Functions as Rules and Tables

Learning Objectives

- Identify the domain and range of a function.
- Make a table for a function.
- Write a function rule.
- Represent a real-world situation with a function.

Introduction

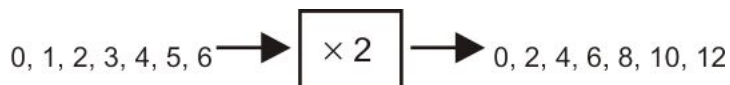
A **function** is a rule for relating two or more variables. For example, the price you pay for phone service may depend on the number of minutes you talk on the phone. We would say that the cost of phone service is a *function* of the number of minutes you talk. Consider the following situation.

Josh goes to an amusement park where he pays \$2 per ride.

There is a relationship between the number of rides Josh goes on and the total amount he spends that day: To figure out the amount he spends, we multiply the number of rides by two. This rule is an example of a **function**. Functions usually—but *not always*—are rules based on mathematical operations. You can think of a function as a box or a machine that contains a mathematical operation.



Whatever number we feed into the function box is changed by the given operation, and a new number comes out the other side of the box. When we input different values for the number of rides Josh goes on, we get different values for the amount of money he spends.

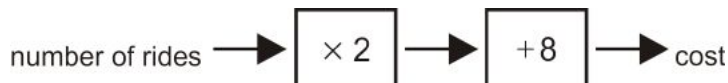


The input is called the **independent variable** because its value can be any number. The output is called the **dependent variable** because its value depends on the input value.

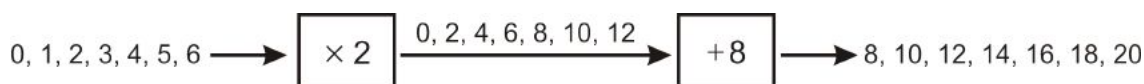
Functions usually contain more than one mathematical operation. Here is a situation that is slightly more complicated than the example above.

Jason goes to an amusement park where he pays \$8 admission and \$2 per ride.

The following function represents the total amount Jason pays. The rule for this function is "multiply the number of rides by 2 and add 8."



When we input different values for the number of rides, we arrive at different outputs (costs).



These flow diagrams are useful in visualizing what a function is. However, they are cumbersome to use in practice. In algebra, we use the following short-hand notation instead:

$$\begin{array}{c}
 \text{input} \\
 \downarrow \\
 \underbrace{f(x)}_{\text{function box}} = y \leftarrow \text{output}
 \end{array}$$

First, we define the variables:

x = the number of rides Jason goes on

y = the total amount of money Jason spends at the amusement park.

So, x represents the input and y represents the output. The notation $f()$ represents the function or the mathematical operations we use on the input to get the output. In the last example, the cost is 2 times the number of rides plus 8. This can be written as a function:

$$f(x) = 2x + 8$$

In algebra, the notations y and $f(x)$ are typically used interchangeably. Technically, though, $f(x)$ represents the function itself and y represents the output of the function.

Identify the Domain and Range of a Function

In the last example, we saw that we can input the number of rides into the function to give us the total cost for going to the amusement park. The set of all values that we can use for the input is called the **domain** of the function, and the set of all values that the output could turn out to be is called the **range** of the function. In many situations the **domain** and **range** of a function are both simply the set of all real numbers, but this isn't always the case. Let's look at our amusement park example.

Example 1

Find the domain and range of the function that describes the situation:

Jason goes to an amusement park where he pays \$8 admission and \$2 per ride.

Solution

Here is the function that describes this situation:

$$f(x) = 2x + 8 = y$$

In this function, x is the number of rides and y is the total cost. To find the domain of the function, we need to determine which numbers make sense to use as the input (x).

- The values have to be zero or positive, because Jason can't go on a negative number of rides.

- The values have to be integers because, for example, Jason could not go on 2.25 rides.
- Realistically, there must be a maximum number of rides that Jason can go on because the park closes, he runs out of money, etc. However, since we aren't given any information about what that maximum might be, we must consider that all non-negative integers are possible values regardless of how big they are.

Answer For this function, the domain is the set of all non-negative integers.

To find the range of the function we must determine what the values of y will be when we apply the function to the input values. The domain is the set of all non-negative integers: $\{0, 1, 2, 3, 4, 5, 6, \dots\}$. Next we plug these values into the function for x . If we plug in 0, we get 8; if we plug in 1, we get 10; if we plug in 2, we get 12, and so on, counting by 2s each time. Possible values of y are therefore 8, 10, 12, 14, 16, 18, 20... or in other words all even integers greater than or equal to 8.

Answer The range of this function is the set of all even integers greater than or equal to 8.

Example 2

Find the domain and range of the following functions.

a) A ball is dropped from a height and it bounces up to 75% of its original height.

b) $y = x^2$

Solution

a) Let's define the variables:

x = original height

y = bounce height

A function that describes the situation is $y = f(x) = 0.75x$. x can represent any real value greater than zero, since you can drop a ball from any height greater than zero. A little thought tells us that y can also represent any real value greater than zero.

Answer

The domain is the set of all real numbers greater than zero. The range is also the set of all real numbers greater than zero.

b) Since there is no word problem attached to this equation, we can assume that we can use any real number as a value of x . When we square a real number, we always get a non-negative answer, so y can be any non-negative real number.

Answer

The domain of this function is all real numbers. The range of this function is all non-negative real numbers.

In the functions we've looked at so far, x is called the **independent variable** because it can be any of the values from the domain, and y is called the **dependent variable** because its value depends on x . However, any letters or symbols can be used to represent the dependent and independent variables. Here are three different examples:

$$y = f(x) = 3x$$

$$R = f(w) = 3w$$

$$v = f(t) = 3t$$

These expressions all represent the same function: a function where the dependent variable is three times the independent variable. Only the symbols are different. In practice, we usually pick symbols for the

dependent and independent variables based on what they represent in the real world—like t for time, d for distance, v for velocity, and so on. But when the variables don't represent anything in the real world—or even sometimes when they do—we traditionally use y for the dependent variable and x for the independent variable.

For another look at the domain of a function, see the following video, where the narrator solves a sample problem from the California Standards Test about finding the domain of an unusual function: <http://www.youtube.com/watch?v=NRB6s77nx2gI>.

Make a Table For a Function

A table is a very useful way of arranging the data represented by a function. We can match the input and output values and arrange them as a table. For example, the values from Example 1 above can be arranged in a table as follows:

| | | | | | | | |
|-----|---|----|----|----|----|----|----|
| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| y | 8 | 10 | 12 | 14 | 16 | 18 | 20 |

A table lets us organize our data in a compact manner. It also provides an easy reference for looking up data, and it gives us a set of coordinate points that we can plot to create a graph of the function.

Example 3

Make a table of values for the function $f(x) = \frac{1}{x}$. Use the following numbers for input values: -1, -0.5, -0.2, -0.1, -0.01, 0.01, 0.1, 0.2, 0.5, 1.

Solution

Make a table of values by filling the first row with the input values and the next row with the output values calculated using the given function.

| | | | | | | | | | | |
|----------------------|----------------|------------------|------------------|------------------|-------------------|------------------|-----------------|-----------------|-----------------|---------------|
| x | -1 | -0.5 | -0.2 | -0.1 | -0.01 | 0.01 | 0.1 | 0.2 | 0.5 | 1 |
| $f(x) = \frac{1}{x}$ | $\frac{1}{-1}$ | $\frac{1}{-0.5}$ | $\frac{1}{-0.2}$ | $\frac{1}{-0.1}$ | $\frac{1}{-0.01}$ | $\frac{1}{0.01}$ | $\frac{1}{0.1}$ | $\frac{1}{0.2}$ | $\frac{1}{0.5}$ | $\frac{1}{1}$ |
| y | -1 | -2 | -5 | -10 | -100 | 100 | 10 | 5 | 2 | 1 |

When you're given a function, you won't usually be told what input values to use; you'll need to decide for yourself what values to pick based on what kind of function you're dealing with. We will discuss how to pick input values throughout this book.

Write a Function Rule

In many situations, we collect data by conducting a survey or an experiment, and then organize the data in a table of values. Most often, we want to find the function rule or formula that fits the set of values in the table, so we can use the rule to predict what could happen for values that are not in the table.

Example 4

Write a function rule for the following table:

| | | | | | |
|---------------|----|----|----|----|-----|
| Number of CDs | 2 | 4 | 6 | 8 | 10 |
| Cost in \$ | 24 | 48 | 72 | 86 | 120 |

Solution

You pay \$24 for 2 CDs, \$48 for 4 CDs, \$120 for 10 CDs. That means that each CD costs \$12.

We can write a function rule:

Cost = \$12 \times (number of CDs) or $f(x) = 12x$

Example 5

Write a function rule for the following table:

| | | | | | | | |
|-----|----|----|----|---|---|---|---|
| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
| y | 3 | 2 | 1 | 0 | 1 | 2 | 3 |

Solution

You can see that a negative number turns into the same number, only positive, while a non-negative number stays the same. This means that the function being used here is the absolute value function: $f(x) = |x|$.

Coming up with a function based on a set of values really is as tricky as it looks. There's no rule that will tell you the function every time, so you just have to think of all the types of functions you know and guess which one might be a good fit, and then check if your guess is right. In this book, though, we'll stick to writing functions for linear relationships, which are the simplest type of function.

Represent a Real-World Situation with a Function

Let's look at a few real-world situations that can be represented by a function.

Example 5

Maya has an internet service that currently has a monthly access fee of \$11.95 and a connection fee of \$0.50 per hour. Represent her monthly cost as a function of connection time.

Solution**Define**

Let x = the number of hours Maya spends on the internet in one month

Let y = Maya's monthly cost

Translate

The cost has two parts: the one-time fee of \$11.95 and the per-hour charge of \$0.50. So the total cost is the flat fee + the charge per hour \times the number of hours.

Answer

The function is $y = f(x) = 11.95 + 0.50x$.

Example 6

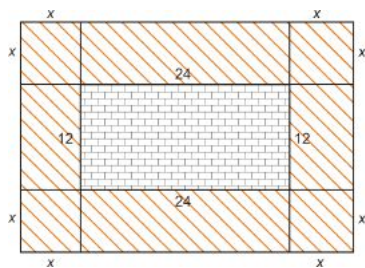
Alfredo wants a deck build around his pool. The dimensions of the pool are 12 feet \times 24 feet and the decking costs \$3 per square foot. Write the cost of the deck as a function of the width of the deck.

Solution**Define**

Let x = width of the deck

Let y = cost of the deck

Make a sketch and label it



Translate

You can look at the decking as being formed by several rectangles and squares. We can find the areas of all the separate pieces and add them together:

$$\text{Area} = 12x + 12x + 24x + 24x + x^2 + x^2 + x^2 + x^2 = 72x + 4x^2$$

To find the total cost, we then multiply the area by the cost per square foot (\$3).

Answer

$$f(x) = 3(72x + 4x^2) = 216x + 12x^2$$

Example 7

A cell phone company sells two million phones in their first year of business. The number of phones they sell doubles each year. Write a function that gives the number of phones that are sold per year as a function of how old the company is.

Solution

Define

Let x = age of company in years

Let y = number of phones that are sold per year

Make a table

| | | | | | | | |
|--------------------|---|---|---|----|----|----|-----|
| Age (years) | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| Millions of phones | 2 | 4 | 8 | 16 | 32 | 64 | 128 |

Write a function rule

The number of phones sold per year doubles every year, so the first year the company sells 2 million phones, the next year it sells 2×2 million, the next year it sells $2 \times 2 \times 2$ million, and so on. You might remember that when we multiply a number by itself several times we can use exponential notation: $2 = 2^1$, $2 \times 2 = 2^2$, $2 \times 2 \times 2 = 2^3$, and so on. In this problem, the exponent just happens to match the company's age in years, which makes our function easy to describe.

Answer

$$y = f(x) = 2^x$$

Review Questions

1. Identify the domain and range of the following functions.

- (a) Dustin charges \$10 per hour for mowing lawns.
 - (b) Maria charges \$25 per hour for tutoring math, with a minimum charge of \$15.
 - (c) $f(x) = 15x - 12$
 - (d) $f(x) = 2x^2 + 5$
 - (e) $f(x) = \frac{1}{x}$
 - (f) $f(x) = \sqrt{x}$
2. What is the range of the function $y = x^2 - 5$ when the domain is -2, -1, 0, 1, 2?
 3. What is the range of the function $y = 2x - \frac{3}{4}$ when the domain is -2.5, -1.5, 5?
 4. What is the domain of the function $y = 3x$ when the range is 9, 12, 15?
 5. What is the range of the function $y = 3x$ when the domain is 9, 12, 15?
 6. Angie makes \$6.50 per hour working as a cashier at the grocery store. Make a table that shows how much she earns if she works 5, 10, 15, 20, 25, or 30 hours.
 7. The area of a triangle is given by the formula $A = \frac{1}{2}bh$. If the base of the triangle measures 8 centimeters, make a table that shows the area of the triangle for heights 1, 2, 3, 4, 5, and 6 centimeters.
 8. Make a table of values for the function $f(x) = \sqrt{2}x + 3$ for input values -1, 0, 1, 2, 3, 4, 5.
 9. Write a function rule for the following table:

| | | | | |
|-----|---|----|----|----|
| x | 3 | 4 | 5 | 6 |
| y | 9 | 16 | 15 | 36 |

10. Write a function rule for the following table:

| | | | | |
|-------|----|----|----|----|
| Hours | 0 | 1 | 2 | 3 |
| Cost | 15 | 20 | 25 | 30 |

11. Write a function rule for the following table:

| | | | | |
|-----|----|----|---|---|
| x | 0 | 1 | 2 | 3 |
| y | 24 | 12 | 6 | 3 |

12. Write a function that represents the number of cuts you need to cut a ribbon into x pieces.
13. Write a function that represents the number of cuts you need to divide a pizza into x slices.
14. Solomon charges a \$40 flat rate plus \$25 per hour to repair a leaky pipe.
 - (a) Write a function that represents the total fee charged as a function of hours worked.
 - (b) How much does Solomon earn for a 3-hour job?
 - (c) How much does he earn for three separate 1-hour jobs?
15. Rochelle has invested \$2500 in a jewelry making kit. She makes bracelets that she can sell for \$12.50 each.
 - (a) Write a function that shows how much money Rochelle makes from selling b bracelets.
 - (b) Write a function that shows how much money Rochelle has after selling b bracelets, minus her investment in the kit.
 - (c) How many bracelets does Rochelle need to make before she breaks even?
 - (d) If she buys a \$50 display case for her bracelets, how many bracelets does she now need to sell to break even?

1.8 Problem-Solving Strategies: Make a Table and Look for a Pattern

Learning Objectives

- Read and understand given problem situations.
- Develop and use the strategy “make a table.”
- Develop and use the strategy “look for a pattern.”
- Plan and compare alternative approaches to solving a problem.
- Solve real-world problems using the above strategies as part of a plan.

Introduction

In this section, we will apply the problem-solving plan you learned about in the last section to solve several real-world problems. You will learn how to develop and use the methods **make a table** and **look for a pattern**.

Read and Understand Given Problem Situations

The most difficult parts of problem-solving are most often the first two steps in our problem-solving plan. You need to read the problem and make sure you understand what you are being asked. Once you understand the problem, you can devise a strategy to solve it.

Let’s apply the first two steps to the following problem.

Example 1:

Six friends are buying pizza together and they are planning to split the check equally. After the pizza was ordered, one of the friends had to leave suddenly, before the pizza arrived. Everyone left had to pay \$1 extra as a result. How much was the total bill?

Solution

Understand

We want to find how much the pizza cost.

We know that five people had to pay an extra \$1 each when one of the original six friends had to leave.

Strategy

We can start by making a list of possible amounts for the total bill.

We divide the amount by six and then by five. The total divided by five should equal \$1 more than the total divided by six.

Look for any patterns in the numbers that might lead you to the correct answer.

In the rest of this section you will learn how to make a table or look for a pattern to figure out a solution for this type of problem. After you finish reading the rest of the section, you can finish solving this problem for homework.

Develop and Use the Strategy: Make a Table

The method “Make a Table” is helpful when solving problems involving numerical relationships. When data is organized in a table, it is easier to recognize patterns and relationships between numbers. Let’s apply this strategy to the following example.

Example 2

Josie takes up jogging. On the first week she jogs for 10 minutes per day, on the second week she jogs for 12 minutes per day. Each week, she wants to increase her jogging time by 2 minutes per day. If she jogs six days each week, what will be her total jogging time on the sixth week?

Solution

Understand

We know in the first week Josie jogs 10 minutes per day for six days.

We know in the second week Josie jogs 12 minutes per day for six days.

Each week, she increases her jogging time by 2 minutes per day and she jogs 6 days per week.

We want to find her total jogging time in week six.

Strategy

A good strategy is to list the data we have been given in a table and use the information we have been given to find new information.

We are told that Josie jogs 10 minutes per day for six days in the first week and 12 minutes per day for six days in the second week. We can enter this information in a table:

Table 1.9:

| Week | Minutes per Day | Minutes per Week |
|------|-----------------|------------------|
| 1 | 10 | 60 |
| 2 | 12 | 72 |

You are told that each week Josie increases her jogging time by 2 minutes per day and jogs 6 times per week. We can use this information to continue filling in the table until we get to week six.

Table 1.10:

| Week | Minutes per Day | Minutes per Week |
|------|-----------------|------------------|
| 1 | 10 | 60 |
| 2 | 12 | 72 |
| 3 | 14 | 84 |
| 4 | 16 | 96 |
| 5 | 18 | 108 |
| 6 | 20 | 120 |

Apply strategy/solve

To get the answer we read the entry for week six.

Answer: In week six Josie jogs a total of 120 minutes.

Check

Josie increases her jogging time by two minutes per day. She jogs six days per week. This means that she increases her jogging time by 12 minutes per week.

Josie starts at 60 minutes per week and she increases by 12 minutes per week for five weeks.

That means the total jogging time is $60 + 12 \times 5 = 120$ *minutes*.

The answer checks out.

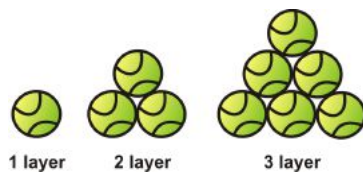
You can see that making a table helped us organize and clarify the information we were given, and helped guide us in the next steps of the problem. We solved this problem solely by making a table; in many situations, we would combine this strategy with others to get a solution.

Develop and Use the Strategy: Look for a Pattern

Looking for a pattern is another strategy that you can use to solve problems. The goal is to look for items or numbers that are repeated or a series of events that repeat. The following problem can be solved by finding a pattern.

Example 3

You arrange tennis balls in triangular shapes as shown. How many balls will there be in a triangle that has 8 rows?



Solution

Understand

We know that we arrange tennis balls in triangles as shown.

We want to know how many balls there are in a triangle that has 8 rows.

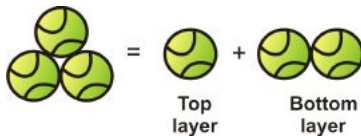
Strategy

A good strategy is to make a table and list how many balls are in triangles of different rows.

One row: It is simple to see that a triangle with one row has only one ball.



Two rows: For a triangle with two rows, we add the balls from the top row to the balls from the bottom row. It is useful to make a sketch of the separate rows in the triangle.



$$3 = 1 + 2$$

Three rows: We add the balls from the top triangle to the balls from the bottom row.



$$6 = 3 + 3$$

Now we can fill in the first three rows of a table.

Table 1.11:

| Number of Rows | Number of Balls |
|----------------|-----------------|
| 1 | 1 |
| 2 | 3 |
| 3 | 6 |

We can see a pattern.

To create the next triangle, we add a new bottom row to the existing triangle.

The new bottom row has the same number of balls as there are rows. (For example, a triangle with 3 rows has 3 balls in the bottom row.)

To get the total number of balls for the new triangle, we add the number of balls in the old triangle to the number of balls in the new bottom row.

Apply strategy/solve:

We can complete the table by following the pattern we discovered.

Number of balls = number of balls in previous triangle + number of rows in the new triangle

Table 1.12:

| Number of Rows | Number of Balls |
|----------------|-----------------|
| 1 | 1 |
| 2 | 3 |
| 3 | 6 |
| 4 | $6 + 4 = 10$ |
| 5 | $10 + 5 = 15$ |
| 6 | $15 + 6 = 21$ |
| 7 | $21 + 7 = 28$ |
| 8 | $28 + 8 = 36$ |

Answer There are 36 balls in a triangle arrangement with 8 rows.

Check

Each row of the triangle has one more ball than the previous one. In a triangle with 8 rows, row 1 has 1 ball, row 2 has 2 balls, row 3 has 3 balls, row 4 has 4 balls, row 5 has 5 balls, row 6 has 6 balls, row 7 has 7 balls, row 8 has 8 balls.

When we add these we get: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$ balls

The answer checks out.

Notice that in this example we made tables and drew diagrams to help us organize our information and find a pattern. Using several methods together is a very common practice and is very useful in solving word problems.

Plan and Compare Alternative Approaches to Solving Problems

In this section, we will compare the methods of “Making a Table” and “Looking for a Pattern” by using each method in turn to solve a problem.

Example 4

Andrew cashes a \$180 check and wants the money in \$10 and \$20 bills. The bank teller gives him 12 bills. How many of each kind of bill does he receive?

Solution

Method 1: Making a Table

Understand

Andrew gives the bank teller a \$180 check.

The bank teller gives Andrew 12 bills. These bills are a mix of \$10 bills and \$20 bills.

We want to know how many of each kind of bill Andrew receives.

Strategy

Let’s start by making a table of the different ways Andrew can have twelve bills in tens and twenties.

Andrew could have twelve \$10 bills and zero \$20 bills, or eleven \$10 bills and one \$20 bill, and so on.

We can calculate the total amount of money for each case.

Apply strategy/solve

Table 1.13:

| \$10 bills | \$ 20 bills | Total amount |
|------------|-------------|------------------------------|
| 12 | 0 | $\$10(12) + \$20(0) = \$120$ |
| 11 | 1 | $\$10(11) + \$20(1) = \$130$ |
| 10 | 2 | $\$10(10) + \$20(2) = \$140$ |
| 9 | 3 | $\$10(9) + \$20(3) = \$150$ |
| 8 | 4 | $\$10(8) + \$20(4) = \$160$ |
| 7 | 5 | $\$10(7) + \$20(5) = \$170$ |
| 6 | 6 | $\$10(6) + \$20(6) = \$180$ |
| 5 | 7 | $\$10(5) + \$20(7) = \$190$ |
| 4 | 8 | $\$10(4) + \$20(8) = \$200$ |
| 3 | 9 | $\$10(3) + \$20(9) = \$210$ |
| 2 | 10 | $\$10(2) + \$20(10) = \$220$ |
| 1 | 11 | $\$10(1) + \$20(11) = \$230$ |
| 0 | 12 | $\$10(0) + \$20(12) = \$240$ |

In the table we listed all the possible ways you can get twelve \$10 bills and \$20 bills and the total amount of money for each possibility. The correct amount is given when Andrew has six \$10 bills and six \$20 bills.

Answer: Andrew gets six \$10 bills and six \$20 bills.

Check

Six \$10 bills and six \$20 bills $\rightarrow 6(\$10) + 6(\$20) = \$60 + \$120 = \$180$

The answer checks out.

Let's solve the same problem using the method "Look for a Pattern."

Method 2: Looking for a Pattern

Understand

Andrew gives the bank teller a \$180 check.

The bank teller gives Andrew 12 bills. These bills are a mix of \$10 bills and \$20 bills.

We want to know how many of each kind of bill Andrew receives.

Strategy

Let's start by making a table just as we did above. However, this time we will look for patterns in the table that can be used to find the solution.

Apply strategy/solve

Let's fill in the rows of the table until we see a pattern.

Table 1.14:

| \$10 bills | \$20 bills | Total amount |
|------------|------------|------------------------------|
| 12 | 0 | $\$10(12) + \$20(0) = \$120$ |
| 11 | 1 | $\$10(11) + \$20(1) = \$130$ |
| 10 | 2 | $\$10(10) + \$20(2) = \$140$ |

We see that every time we reduce the number of \$10 bills by one and increase the number of \$20 bills by one, the total amount increases by \$10. The last entry in the table gives a total amount of \$140, so we have \$40 to go until we reach our goal. This means that we should reduce the number of \$10 bills by four and increase the number of \$20 bills by four. That would give us six \$10 bills and six \$20 bills.

$$6(\$10) + 6(\$20) = \$60 + 120 = \$180$$

Answer: Andrew gets six \$10 bills and six \$20 bills.

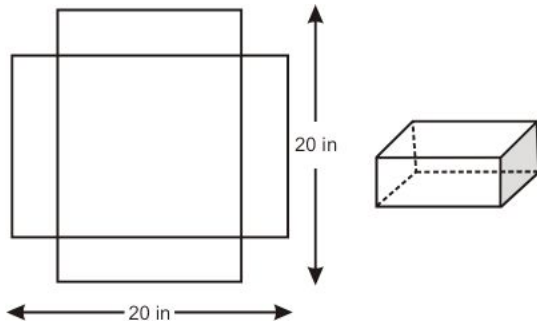
Check

Six \$10 bills and six \$20 bills $\rightarrow 6(\$10) + 6(\$20) = \$60 + 120 = \180

The answer checks out.

You can see that the second method we used for solving the problem was less tedious. In the first method, we listed all the possible options and found the answer we were seeking. In the second method, we started by listing the options, but we found a pattern that helped us find the solution faster. The methods of "Making a Table" and "Looking for a Pattern" are both more powerful if used alongside other problem-solving methods.

Solve Real-World Problems Using Selected Strategies as Part of a Plan



Example 5

Anne is making a box without a lid. She starts with a 20 in. square piece of cardboard and cuts out four equal squares from each corner of the cardboard as shown. She then folds the sides of the box and glues the edges together. How big does she need to cut the corner squares in order to make the box with the biggest volume?

Solution

Step 1:

Understand

Anne makes a box out of a 20 in \times 20 in piece of cardboard.

She cuts out four equal squares from the corners of the cardboard.

She folds the sides and glues them to make a box.

How big should the cut out squares be to make the box with the biggest volume?

Step 2:

Strategy

We need to remember the formula for the volume of a box.

Volume = Area of base \times height

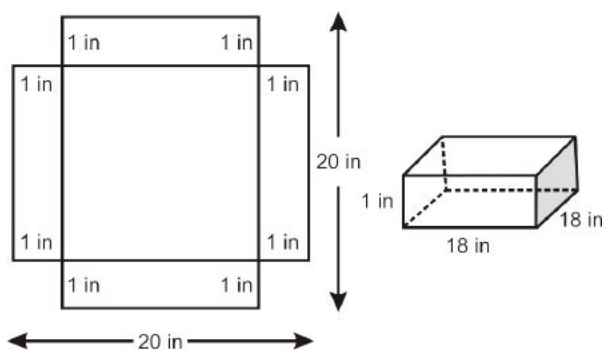
Volume = width \times length \times height

Make a table of values by picking different values for the side of the squares that we are cutting out and calculate the volume.

Step 3:

Apply strategy/solve

Let's "make" a box by cutting out four corner squares with sides equal to 1 inch. The diagram will look like this:



You see that when we fold the sides over to make the box, the height becomes 1 inch, the width becomes 18 inches and the length becomes 18 inches.

Volume = width \times length \times height

Volume = $18 \times 18 \times 1 = 324 \text{ in}^3$

Let's make a table that shows the value of the box for different square sizes:

Table 1.15:

| Side of Square | Box Height | Box Width | Box Length | Volume |
|----------------|------------|-----------|------------|-------------------------------|
| 1 | 1 | 18 | 18 | $18 \times 18 \times 1 = 324$ |
| 2 | 2 | 16 | 16 | $16 \times 16 \times 2 = 512$ |
| 3 | 3 | 14 | 14 | $14 \times 14 \times 3 = 588$ |
| 4 | 4 | 12 | 12 | $12 \times 12 \times 4 = 576$ |
| 5 | 5 | 10 | 10 | $10 \times 10 \times 5 = 500$ |
| 6 | 6 | 8 | 8 | $8 \times 8 \times 6 = 384$ |
| 7 | 7 | 6 | 6 | $6 \times 6 \times 7 = 252$ |
| 8 | 8 | 4 | 4 | $4 \times 4 \times 8 = 128$ |
| 9 | 9 | 2 | 2 | $2 \times 2 \times 9 = 36$ |
| 10 | 10 | 0 | 0 | $0 \times 0 \times 10 = 0$ |

We stop at a square of 10 inches because at this point we have cut out all of the cardboard and we can't make a box any more. From the table we see that we can make the biggest box if we cut out squares with a side length of three inches. This gives us a volume of 588 in^3 .

Answer The box of greatest volume is made if we cut out squares with a side length of three inches.

Step 4:

Check

We see that 588 in^3 is the largest volume appearing in the table. We picked integer values for the sides of the squares that we are cut out. Is it possible to get a larger value for the volume if we pick non-integer values? Since we get the largest volume for the side length equal to three inches, let's make another table with values close to three inches that is split into smaller increments:

Table 1.16:

| Side of Square | Box Height | Box Width | Box Length | Volume |
|----------------|------------|-----------|------------|-----------------------------------|
| 2.5 | 2.5 | 15 | 15 | $15 \times 15 \times 2.5 = 562.5$ |

Table 1.16: (continued)

| Side of Square | Box Height | Box Width | Box Length | Volume |
|----------------|------------|-----------|------------|---------------------------------------|
| 2.6 | 2.6 | 14.8 | 14.8 | $14.8 \times 14.8 \times 2.6 = 569.5$ |
| 2.7 | 2.7 | 14.6 | 14.6 | $14.6 \times 14.6 \times 2.7 = 575.5$ |
| 2.8 | 2.8 | 14.4 | 14.4 | $14.4 \times 14.4 \times 2.8 = 580.6$ |
| 2.9 | 2.9 | 14.2 | 14.2 | $14.2 \times 14.2 \times 2.9 = 584.8$ |
| 3 | 3 | 14 | 14 | $14 \times 14 \times 3 = 588$ |
| 3.1 | 3.1 | 13.8 | 13.8 | $13.8 \times 13.8 \times 3.1 = 590.4$ |
| 3.2 | 3.2 | 13.6 | 13.6 | $13.6 \times 13.6 \times 3.2 = 591.9$ |
| 3.3 | 3.3 | 13.4 | 13.4 | $13.4 \times 13.4 \times 3.3 = 592.5$ |
| 3.4 | 3.4 | 13.2 | 13.2 | $13.2 \times 13.2 \times 3.4 = 592.4$ |
| 3.5 | 3.5 | 13 | 13 | $13 \times 13 \times 3.5 = 591.5$ |

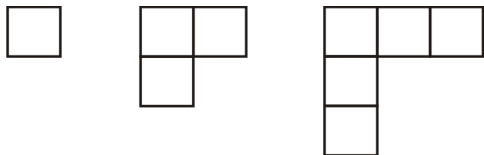
Notice that the largest volume is not when the side of the square is three inches, but rather when the side of the square is 3.3 inches.

Our original answer was not incorrect, but it was not as accurate as it could be. We can get an even more accurate answer if we take even smaller increments of the side length of the square. To do that, we would choose smaller measurements that are in the neighborhood of 3.3 inches.

Meanwhile, our first answer checks out if we want it rounded to zero decimal places, but **a more accurate answer is 3.3 inches**.

Review Questions

1. Go back and find the solution to the problem in Example 1.
2. Britt has \$2.25 in nickels and dimes. If she has 40 coins in total, how many of each coin does she have?
3. Jeremy divides a 160-square-foot garden into plots that are either 10 or 12 square feet each. If there are 14 plots in all, how many plots are there of each size?
4. A pattern of squares is put together as shown. How many squares are in the 12th diagram?



5. In Harrisville, local housing laws specify how many people can live in a house or apartment: the maximum number of people allowed is twice the number of bedrooms, plus one. If Jan, Pat, and their four children want to rent a house, how many bedrooms must it have?
6. A restaurant hosts children's birthday parties for a cost of \$120 for the first six children (including

- the birthday child) and \$30 for each additional child. If Jaden's parents have a budget of \$200 to spend on his birthday party, how many guests can Jaden invite?
7. A movie theater with 200 seats charges \$8 general admission and \$5 for students. If the 5:00 showing is sold out and the theater took in \$1468 for that showing, how many of the seats are occupied by students?
 8. Oswald is trying to cut down on drinking coffee. His goal is to cut down to 6 cups per week. If he starts with 24 cups the first week, then cuts down to 21 cups the second week and 18 cups the third week, how many weeks will it take him to reach his goal?
 9. Taylor checked out a book from the library and it is now 5 days late. The late fee is 10 cents per day. How much is the fine?
 10. Mikhail is filling a sack with oranges.
 - (a) If each orange weighs 5 ounces and the sack will hold 2 pounds, how many oranges will the sack hold before it bursts?
 - (b) Mikhail plans to use these oranges to make breakfast smoothies. If each smoothie requires $\frac{3}{4}$ cup of orange juice, and each orange will yield half a cup, how many smoothies can he make?
 11. Jessamyn takes out a \$150 loan from an agency that charges 12% of the original loan amount in interest each week. If she takes five weeks to pay off the loan, what is the total amount (loan plus interest) she will need to pay back?
 12. How many hours will a car traveling at 75 miles per hour take to catch up to a car traveling at 55 miles per hour if the slower car starts two hours before the faster car?
 13. Grace starts biking at 12 miles per hour. One hour later, Dan starts biking at 15 miles per hour, following the same route. How long will it take him to catch up with Grace?
 14. A new theme park opens in Milford. On opening day, the park has 120 visitors; on each of the next three days, the park has 10 more visitors than the day before; and on each of the three days after that, the park has 20 more visitors than the day before.
 - (a) How many visitors does the park have on the seventh day?
 - (b) How many total visitors does the park have all week?
 15. Lemuel wants to enclose a rectangular plot of land with a fence. He has 24 feet of fencing. What is the largest possible area that he could enclose with the fence?
 16. Quizzes in Keiko's history class are worth 20 points each. Keiko scored 15 and 18 points on her last two quizzes. What score does she need on her third quiz to get an average score of 17 on all three?
 17. Tickets to an event go on sale for \$20 six weeks before the event, and go up in price by \$5 each week. What is the price of tickets one week before the event?
 18. Mark is three years older than Janet, and the sum of their ages is 15. How old are Mark and Janet?
 19. In a one-on-one basketball game, Jane scored $1\frac{1}{2}$ times as many points as Russell. If the two of them together scored 10 points, how many points did Jane score?
 20. Scientists are tracking two pods of whales during their migratory season. On the first day of June, one pod is 120 miles north of a certain group of islands, and every day thereafter it gets 15 miles closer to the islands. The second pod starts out 160 miles east of the islands on June 3, and heads toward the islands at a rate of 20 miles a day.
 - (a) Which pod will arrive at the islands first, and on what day?
 - (b) How long after that will it take the other pod to reach the islands?
 - (c) Suppose the pod that reaches the islands first immediately heads south from the islands at a rate of 15 miles a day, and the pod that gets there second also heads south from there at a rate of 25 miles a day. On what day will the second pod catch up with the first?
 - (d) How far will both pods be from the islands on that day?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9611>.

1.9 Ratios and Proportions

Learning Objectives

- Write and understand a ratio.
- Write and solve a proportion.
- Solve proportions using cross products.
- Solve problems using scale drawings.

Introduction

Nadia is counting out money with her little brother. She gives her brother all the nickels and pennies. She keeps the quarters and dimes for herself. Nadia has four quarters and six dimes. Her brother has fifteen nickels and five pennies and is happy because he has more coins than his big sister. How would you explain to him that he is actually getting a bad deal?

Write a ratio

A **ratio** is a way to compare two numbers, measurements or quantities. When we write a ratio, we divide one number by another and express the answer as a fraction. There are two distinct ratios in the problem above. For example, the ratio of the **number** of Nadia's coins to her brother's is $\frac{4+6}{15+5}$, or $\frac{10}{20} = \frac{1}{2}$. (Ratios should always be simplified.) In other words, Nadia has half as many coins as her brother.

Another ratio we could look at is the **value** of the coins. The value of Nadia's coins is $(4 \times 25) + (6 \times 10) = 160$ cents. The value of her brother's coins is $(15 \times 5) + (5 \times 1) = 80$ cents. The ratio of the **value** of Nadia's coins to her brother's is $\frac{160}{80} = \frac{2}{1}$. So the value of Nadia's money is twice the value of her brother's.

Notice that even though the denominator is one, we still write it out and leave the ratio as a fraction instead of a whole number. A ratio with a denominator of one is called a **unit rate**.

Example 1

The price of a Harry Potter Book on Amazon.com is \$10.00. The same book is also available used for \$6.50. Find two ways to compare these prices.

Solution

We could compare the numbers by expressing the difference between them: $\$10.00 - \$6.50 = \$3.50$.

We can also use a ratio to compare them: $\frac{10.00}{6.50} = \frac{100}{65} = \frac{20}{13}$ (after multiplying by 10 to remove the decimals, and then simplifying).

So we can say that **the new book is \$3.50 more than the used book**, or we can say that **the new book costs $\frac{20}{13}$ times as much as the used book**.

Example 2

A tournament size shuffleboard table measures 30 inches wide by 14 feet long. Compare the length of the

table to its width and express the answer as a ratio.

Solution

We could just write the ratio as $\frac{14 \text{ feet}}{30 \text{ inches}}$. But since we're comparing two lengths, it makes more sense to convert all the measurements to the same units. 14 feet is $14 \times 12 = 168$ inches, so our new ratio is $\frac{168}{30} = \frac{28}{5}$.

Example 3

*A family car is being tested for fuel efficiency. It drives non-stop for 100 miles and uses 3.2 gallons of gasoline. Write the ratio of distance traveled to fuel used as a **unit rate**.*

Solution

The ratio of distance to fuel is $\frac{100 \text{ miles}}{3.2 \text{ gallons}}$. But a unit rate has to have a denominator of one, so to make this ratio a unit rate we need to divide both numerator and denominator by 3.2. $\frac{\frac{100}{3.2} \text{ miles}}{\frac{3.2}{3.2} \text{ gallons}} = \frac{31.25 \text{ miles}}{1 \text{ gallon}}$ or **31.25 miles per gallon**.

Write and Solve a Proportion

When two ratios are equal to each other, we call it a proportion. For example, the equation $\frac{10}{5} = \frac{6}{9}$ is a proportion. We know it's true because we can reduce both fractions to $\frac{2}{3}$.

(Check this yourself to make sure!)

We often use proportions in science and business—for example, when scaling up the size of something. We generally use them to solve for an unknown, so we use algebra and label the unknown variable x .

Example 4

A small fast food chain operates 60 stores and makes \$1.2 million profit every year. How much profit would the chain make if it operated 250 stores?

Solution

First, we need to write a **ratio**: the ratio of profit to number of stores. That would be $\frac{\$1,200,000}{60}$.

Now we want to know how much profit 250 stores would make. If we label that profit x , then the ratio of profit to stores in that case is $\frac{x}{250}$.

Since we're assuming the profit is proportional to the number of stores, the ratios are equal and our proportion is $\frac{1,200,000}{60} = \frac{x}{250}$.

(Note that we can drop the units – not because they are the same in the numerator and denominator, but because they are the same on both sides of the equation.)

To solve this equation, first we simplify the left-hand fraction to get $20,000 = \frac{x}{250}$. Then we multiply both sides by 250 to get $5,000,000 = x$.

If the chain operated 250 stores, the annual profit would be 5 million dollars.

Example 5

A chemical company makes up batches of copper sulfate solution by adding 250 kg of copper sulfate powder to 1000 liters of water. A laboratory chemist wants to make a solution of identical concentration, but only needs 350 mL (0.35 liters) of solution. How much copper sulfate powder should the chemist add to the water?

Solution

The ratio of powder to water in the first case, in kilograms per liter, is $\frac{250}{1000}$, which reduces to $\frac{1}{4}$. In the

second case, the unknown amount is how much powder to add. If we label that amount x , the ratio is $\frac{x}{0.35}$. So our proportion is $\frac{1}{4} = \frac{x}{0.35}$.

To solve for x , first we multiply both sides by 0.35 to get $\frac{0.35}{4} = x$, or $x = 0.0875$.

The mass of copper sulfate that the chemist should add is 0.0875 kg, or 87.5 grams.

Solve Proportions Using Cross Products

One neat way to simplify proportions is to cross multiply. Consider the following proportion:

$$\frac{16}{4} = \frac{20}{5}$$

If we want to eliminate the fractions, we could multiply both sides by 4 and then multiply both sides by 5. But suppose we just do both at once?

$$\begin{aligned} 4 \times 5 \times \frac{16}{4} &= 4 \times 5 \times \frac{20}{5} \\ 5 \times 16 &= 4 \times 20 \end{aligned}$$

Now comparing this to the proportion we started with, we see that the denominator from the left hand side ends up being multiplied by the numerator on the right hand side. You can also see that the denominator from the *right* hand side ends up multiplying the numerator on the *left* hand side.

In effect the two denominators have multiplied across the equal sign:


$$\frac{16}{4} = \frac{20}{5}$$

becomes $5 \times 16 = 4 \times 20$.

This movement of denominators is known as **cross multiplying**. It is extremely useful in solving proportions, especially when the unknown variable is in the denominator.

Example 6

Solve this proportion for x : $\frac{4}{3} = \frac{9}{x}$

Solution

Cross multiply to get $4x = 9 \times 3$, or $4x = 27$. Then divide both sides by 4 to get $x = \frac{27}{4}$, or $x = 6.75$.

Example 7

Solve the following proportion for x : $\frac{0.5}{3} = \frac{56}{x}$

Solution

Cross multiply to get $0.5x = 56 \times 3$, or $0.5x = 168$. Then divide both sides by 0.5 to get $x = 336$.

Solve Real-World Problems Using Proportions

Example 8

A cross-country train travels at a steady speed. It covers 15 miles in 20 minutes. How far will it travel in 7 hours assuming it continues at the same speed?

Solution

We've done speed problems before; remember that speed is just the ratio $\frac{\text{distance}}{\text{time}}$, so that ratio is the one we'll use for our proportion. We can see that the speed is $\frac{15 \text{ miles}}{20 \text{ minutes}}$, and that speed is also equal to $\frac{x \text{ miles}}{7 \text{ hours}}$. To set up a proportion, we first have to get the units the same. 20 minutes is $\frac{1}{3}$ of an hour, so our proportion will be $\frac{15}{\frac{1}{3}} = \frac{x}{7}$. This is a very awkward looking ratio, but since we'll be cross multiplying, we can leave it as it is.

Cross multiplying gives us $7 \times 15 = \frac{1}{3}x$. Multiplying both sides by 3 then gives us $3 \times 7 \times 15 = x$, or $x = 315$.

The train will travel 315 miles in 7 hours.

Example 9

In the United Kingdom, Alzheimer's disease is said to affect one in fifty people over 65 years of age. If approximately 250000 people over 65 are affected in the UK, how many people over 65 are there in total?

Solution

The fixed ratio in this case is the 1 person in 50. The unknown quantity x is the total number of people over 65. Note that in this case we don't need to include the units, as they will cancel between the numerator and denominator.

Our proportion is $\frac{1}{50} = \frac{250000}{x}$. Each ratio represents $\frac{\text{people with Alzheimer's}}{\text{total people}}$.

Cross multiplying, we get $1 \cdot x = 250000 \cdot 50$, or $x = 12,500,000$.

There are approximately 12.5 million people over the age of 65 in the UK.

For some more advanced ratio problems and applications, watch the Khan Academy video at <http://www.youtube.com/>

Scale and Indirect Measurement

One place where ratios are often used is in making maps. The **scale** of a map describes the relationship between distances on a map and the corresponding distances on the earth's surface. These measurements are expressed as a fraction or a ratio.

So far we have only written ratios as fractions, but outside of mathematics books, ratios are often written as two numbers separated by a colon (:). For example, instead of $\frac{2}{3}$, we would write 2:3.

Ratios written this way are used to express the relationship between a map and the area it represents. For example, a map with a scale of 1:1000 would be a map where one unit of measurement (such as a centimeter) on the map would represent 1000 of the same unit (1000 centimeters, or 10 meters) in real life.

Example 10

Anne is visiting a friend in London, and is using the map below to navigate from Fleet Street to Borough Road. She is using a 1:100,000 scale map, where 1 cm on the map represents 1 km in real life. Using a ruler, she measures the distance on the map as 8.8 cm. How far is the real distance from the start of her journey to the end?



Solution

The scale is the ratio of distance on the map to the corresponding distance in real life. Written as a fraction, it is $\frac{1}{100000}$. We can also write an equivalent ratio for the distance Anne measures on the map and the distance in real life that she is trying to find: $\frac{8.8}{x}$. Setting these two ratios equal gives us our proportion: $\frac{1}{100000} = \frac{8.8}{x}$. Then we can cross multiply to get $x = 880000$.

That's how many *centimeters* it is from Fleet Street to Borough Road; now we need to convert to kilometers. There are 100000 cm in a km, so we have to divide our answer by 100000.

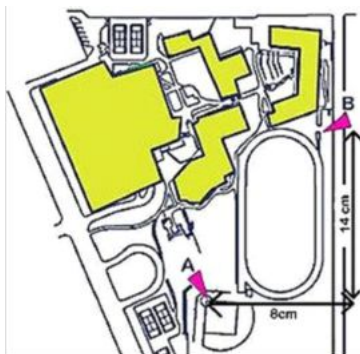
$$\frac{880000}{100000} = 8.8.$$

The distance from Fleet Street to Borough Road is 8.8 km.

In this problem, we could have just used our intuition: the 1 *cm* = 1 *km* scale tells us that any number of cm on the map is equal to the same number of km in real life. But not all maps have a scale this simple. You'll usually need to refer to the map scale to convert between measurements on the map and distances in real life!

Example 11

Antonio is drawing a map of his school for a project in math. He has drawn out the following map of the school buildings and the surrounding area



He is trying to determine the scale of his figure. He knows that the distance from the point marked A on the baseball diamond to the point marked B on the athletics track is 183 meters. Use the dimensions marked on the drawing to determine the scale of his map.

Solution

We know that the real-life distance is 183 m, and the scale is the ratio $\frac{\text{distance on map}}{\text{distance in real life}}$.

To find the distance on the map, we use Pythagoras' Theorem: $a^2 + b^2 = c^2$, where a and b are the horizontal and vertical lengths and c is the diagonal between points A and B.

$$\begin{aligned}
 8^2 + 14^2 &= c^2 \\
 64 + 196 &= c^2 \\
 260 &= c^2 \\
 \sqrt{260} &= c \\
 16.12 &\approx c
 \end{aligned}$$

So the distance on the map is about 16.12 cm. The distance in real life is 183 m, which is 18300 cm. Now we can divide:

$$\text{Scale} = \frac{16.12}{18300} \approx \frac{1}{1135.23}$$

The scale of Antonio's map is approximately 1:1100.

Another visual use of ratio and proportion is in **scale drawings**. Scale drawings (often called **plans**) are used extensively by architects. The equations governing scale are the same as for maps; the scale of a drawing is the ratio $\frac{\text{distance on diagram}}{\text{distance in real life}}$.

Example 12

Oscar is trying to make a scale drawing of the Titanic, which he knows was 883 ft long. He would like his drawing to be at a 1:500 scale. How many inches long does his sheet of paper need to be?

Solution

We can reason intuitively that since the scale is 1:500, the paper must be $\frac{883}{500} = 1.766$ feet long. Converting to inches means the length is $12(1.766) = 21.192$ inches.

Oscar's paper should be at least 22 inches long.

Example 13

The Rose Bowl stadium in Pasadena, California measures 880 feet from north to south and 695 feet from east to west. A scale diagram of the stadium is to be made. If 1 inch represents 100 feet, what would be the dimensions of the stadium drawn on a sheet of paper? Will it fit on a standard 8.5×11 inch sheet of paper?

Solution

Instead of using a proportion, we can simply use the following equation: (distance on diagram) = (distance in real life) \times (scale). (We can derive this from the fact that $\text{scale} = \frac{\text{distance on diagram}}{\text{distance in real life}}$.)

Plugging in, we get

$$\text{height on paper} = 880 \text{ feet} \times \frac{1 \text{ inch}}{100 \text{ feet}} = 8.8 \text{ inches}$$

$$\text{width on paper} = 695 \text{ feet} \times \frac{1 \text{ inch}}{100 \text{ feet}} = 6.95 \text{ inches}$$

The scale diagram will be $8.8 \text{ in} \times 6.95 \text{ in}$. It will fit on a standard sheet of paper.

Lesson Summary

- A **ratio** is a way to compare two numbers, measurements or quantities by dividing one number by the other and expressing the answer as a fraction.
- A **proportion** is formed when two ratios are set equal to each other.

- **Cross multiplication** is useful for solving equations in the form of proportions. To cross multiply, multiply the bottom of each ratio by the top of the other ratio and set them equal. For instance, cross multiplying $\frac{11}{5} = \frac{x}{3}$ results in $11 \times 3 = 5x$.
- **Scale** is a proportion that relates map distance to real life distance.

Review Questions

- Write the following comparisons as ratios. Simplify fractions where possible.
 - \$150 to \$3
 - 150 boys to 175 girls
 - 200 minutes to 1 hour
 - 10 days to 2 weeks
- Write the following ratios as a unit rate.
 - 54 hotdogs to 12 minutes
 - 5000 lbs to 250 square inches
 - 20 computers to 80 students
 - 180 students to 6 teachers
 - 12 meters to 4 floors
 - 18 minutes to 15 appointments
- Solve the following proportions.
 - $\frac{13}{6} = \frac{5}{x}$
 - $\frac{1.25}{7} = \frac{3.6}{x}$
 - $\frac{6}{19} = \frac{x}{11}$
 - $\frac{1}{x} = \frac{0.01}{5}$
 - $\frac{300}{4} = \frac{x}{99}$
 - $\frac{2.75}{9} = \frac{x}{(\frac{2}{9})}$
 - $\frac{1.3}{4} = \frac{x}{1.3}$
 - $\frac{0.1}{1.01} = \frac{1.9}{x}$
- A restaurant serves 100 people per day and takes in \$908. If the restaurant were to serve 250 people per day, how much money would it take in?
- The highest mountain in Canada is Mount Yukon. It is $\frac{298}{67}$ the size of Ben Nevis, the highest peak in Scotland. Mount Elbert in Colorado is the highest peak in the Rocky Mountains. Mount Elbert is $\frac{220}{67}$ the height of Ben Nevis and $\frac{11}{12}$ the size of Mont Blanc in France. Mont Blanc is 4800 meters high. How high is Mount Yukon?
- At a large high school it is estimated that two out of every three students have a cell phone, and one in five of all students have a cell phone that is one year old or less. Out of the students who own a cell phone, what proportion owns a phone that is more than one year old?
- Use the map in Example 10. Using the scale printed on the map, determine the distances (rounded to the nearest half km) between:
 - Points 1 and 4
 - Points 22 and 25
 - Points 18 and 13
 - Tower Bridge and London Bridge

1.10 Percent Problems

Learning Objectives

- Find a percent of a number.
- Use the percent equation.
- Find the percent of change.

Introduction

A **percent** is simply a ratio with a base unit of 100. When we write a ratio as a fraction, the percentage we want to represent is the numerator, and the denominator is 100. For example, 43% is another way of writing $\frac{43}{100}$. $\frac{43}{1000}$, on the other hand, is equal to $\frac{4.3}{100}$, so it would be equivalent to 4.3%. $\frac{2}{5}$ is equal to $\frac{40}{100}$, or 40%. To convert any fraction to a percent, just convert it to an equivalent fraction with a denominator of 100, and then take the numerator as your percent value.

To convert a percent to a decimal, just move the decimal point two spaces to the right:

$$\begin{aligned}67\% &= 0.67 \\0.2\% &= 0.002 \\150\% &= 1.5\end{aligned}$$

And to convert a decimal to a percent, just move the decimal point two spaces to the left:

$$\begin{aligned}2.3 &= 230\% \\0.97 &= 97\% \\0.00002 &= 0.002\%\end{aligned}$$

Finding and Converting Percentages

Before we work with percentages, we need to know how to convert between percentages, decimals and fractions.

Converting percentages to fractions is the easiest. The word “percent” simply means “per 100”—so, for example, 55% means 55 per 100, or $\frac{55}{100}$. This fraction can then be simplified to $\frac{11}{20}$.

Example 1

Convert 32.5% to a fraction.

Solution

32.5% is equal to 32.5 per 100, or $\frac{32.5}{100}$. To reduce this fraction, we first need to multiply it by $\frac{10}{10}$ to get rid of the decimal point. $\frac{325}{1000}$ then reduces to $\frac{13}{40}$.

Converting fractions to percentages can be a little harder. To convert a fraction directly to a percentage, you need to express it as an equivalent fraction with a denominator of 100.

Example 2

Convert $\frac{7}{8}$ to a percent.

Solution

To get the denominator of this fraction equal to 100, we have to multiply it by 12.5. Multiplying the numerator by 12.5 also, we get $\frac{87.5}{100}$, which is equivalent to 87.5%.

But what about a fraction like $\frac{1}{6}$, where there's no convenient number to multiply the denominator by to get 100? In a case like this, it's easier to do the division problem suggested by the fraction in order to convert the fraction to a decimal, and *then* convert the decimal to a percent. 1 divided by 6 works out to 0.166666.... Moving the decimal two spaces to the right tells us that this is equivalent to about 16.7%.

Why can we convert from decimals to percents just by moving the decimal point? Because of what decimal places represent. 0.1 is another way of representing one tenth, and 0.01 is equal to one hundredth—and one hundredth is one percent. By the same logic, 0.02 is 2 percent, 0.35 is 35 percent, and so on.

Example 3

Convert 2.64 to a percent.

Solution

To convert to a percent, simply move the decimal two places to the right. $2.64 = 264\%$.

Does a percentage greater than 100 even make sense? Sure it does—percentages greater than 100 come up in real life all the time. For example, a business that made 10 million dollars last year and 13 million dollars this year would have made 130% as much money this year as it did last year.

The only situation where a percentage greater than 100 doesn't make sense is when you're talking about dividing up something that you only have a fixed amount of—for example, if you took a survey and found that 56% of the respondents gave one answer and 72% gave another answer (for a total of 128%), you'd know something went wrong with your math somewhere, because there's no way you could have gotten answers from more than 100% of the people you surveyed.

Converting percentages to decimals is just as easy as converting decimals to percentages—simply move the decimal to the left instead of to the right.

Example 4

Convert 58% to a decimal.

Solution

The decimal point here is invisible—it's right after the 8. So moving it to the left two places gives us 0.58.

It can be hard to remember which way to move the decimal point when converting from decimals to percents or vice versa. One way to check if you're moving it the right way is to check whether your answer is a bigger or smaller number than you started out with. If you're converting from percents to decimals, you should end up with a smaller number—just think of how a number like 50 percent, where 50 is **greater** than 1, represents a fraction like $\frac{1}{2}$ (or 0.50 in decimal form), where $\frac{1}{2}$ is **less** than 1. Conversely, if you're converting from decimals to percents, you should end up with a bigger number.

One way you might remember this is by remembering that a percent sign is bigger than a decimal point—so percents should be bigger numbers than decimals.

Example 5

Convert 3.4 to a percent.

Solution

If you move the decimal point to the left, you get 0.034%. That's a smaller number than you started out with, but you're moving from decimals to percents, so you want the number to get bigger, not smaller. Move it to the right instead to get 340%.

Now let's try another fraction.

Example 6

Convert $\frac{2}{7}$ to a percent.

Solution

$\frac{2}{7}$ doesn't convert easily unless you change it to a decimal first. 2 divided by 7 is approximately 0.285714..., and moving the decimal and rounding gives us 28.6%.

The following Khan Academy video shows several more examples of finding percents and might be useful for further practice: http://www.youtube.com/watch?v=_SpE4hQ8D_o http://www.youtube.com/watch?v=_SpE4hQ8D_o.

Use the Percent Equation

The percent equation is often used to solve problems. It goes like this:

$$\text{Rate} \times \text{Total} = \text{Part}$$

or

$$R\% \text{ of Total is Part}$$

Rate is the ratio that the percent represents ($R\%$ in the second version).

Total is often called the **base unit**.

Part is the amount we are comparing with the base unit.

Example 7

Find 25% of \$80.

Solution

We are looking for the **part**. The **total** is \$80. 'of' means multiply. $R\%$ is 25%, so we can use the second form of the equation: 25% of \$80 is Part, or $0.25 \times 80 = \text{Part}$.

$0.25 \times 80 = 20$, so the Part we are looking for is **\$20**.

Example 8

Express \$90 as a percentage of \$160.

Solution

This time we are looking for the **rate**. We are given the **part** (\$90) and the **total** (\$160). Using the rate equation, we get $\text{Rate} \times 160 = 90$. Dividing both sides by 160 tells us that the rate is 0.5625, or 56.25%.

Example 9

\$50 is 15% of what total sum?

This time we are looking for the **total**. We are given the **part** (\$50) and the **rate** (15%, or 0.15). Using the rate equation, we get $0.15 \times \text{Total} = 50$. Dividing both sides by 0.15, we get $\text{Total} = \frac{50}{0.15} \approx 333.33$. So **\$50 is 15% of \$333.33**.

Find Percent of Change

A useful way to express changes in quantities is through percents. You've probably seen signs such as "20% extra free," or "save 35% today." When we use percents to represent a change, we generally use the formula

$$\text{Percent change} = \frac{\text{final amount} - \text{original amount}}{\text{original amount}} \times 100\%$$

or

$$\frac{\text{percent change}}{100} = \frac{\text{actual change}}{\text{original amount}}$$

This means that a **positive** percent change is an **increase**, while a **negative** change is a **decrease**.

Example 10

A school of 500 students is expecting a 20% increase in students next year. How many students will the school have?

Solution

First let's solve this using the first formula. Since the 20% change is an increase, we represent it in the formula as 20 (if it were a decrease, it would be -20.) Plugging in all the numbers, we get

$$20\% = \frac{\text{final amount} - 500}{500} \times 100\%$$

Dividing both sides by 100%, we get $0.2 = \frac{\text{final amount} - 500}{500}$.

Multiplying both sides by 500 gives us $100 = \text{final amount} - 500$.

Then adding 500 to both sides gives us 600 as the final number of students.

How about if we use the second formula? Then we get $\frac{20}{100} = \frac{\text{actual change}}{500}$. (Reducing the first fraction to $\frac{1}{5}$ will make the problem easier, so let's rewrite the equation as $\frac{1}{5} = \frac{\text{actual change}}{500}$)

Cross multiplying is our next step; that gives us $500 = 5 \times (\text{actual change})$. Dividing by 5 tells us the change is equal to 100. We were told this was an increase, so if we start out with 500 students, after an increase of 100 we know there will be a total of 600.

Markup

A **markup** is an increase from the price a store pays for an item from its supplier to the retail price it charges to the public. For example, a 100% mark-up (commonly known in business as *keystone*) means that the price is doubled. Half of the retail price covers the cost of the item from the supplier, half is profit.

Example 11

A furniture store places a 30% markup on everything it sells. It offers its employees a 20% discount from the sales price. The employees are demanding a 25% discount, saying that the store would still make a profit. The manager says that at a 25% discount from the sales price would cause the store to lose money. Who is right?

Solution

We'll consider this problem two ways. First, let's consider an item that the store buys from its supplier for a certain price, say \$1000. The markup would be 30% of 1000, or \$300, so the item would sell for \$1300 and the store would make a \$300 profit.

And what if an employee buys the product? With a discount of 20%, the employee would pay 80% of the \$1300 retail price, or $0.8 \times \$1300 = \1040 .

But with a 25% discount, the employee would pay 75% of the retail price, or $0.75 \times \$1300 = \975 .

So with a 20% employee discount, the store still makes a \$40 profit on the item they bought for \$1000—but with a 25% employee discount, the store loses \$25 on the item.

Now let's use algebra to see how this works for an item of any price. If x is the price of an item, then the store's markup is 30% of x , or $0.3x$, and the retail price of the item is $x + 0.3x$, or $1.3x$. An employee buying the item at a 20% discount would pay $0.8 \times 1.3x = 1.04x$, while an employee buying it at a 25% discount would pay $0.75 \times 1.3x = 0.975x$.

So the manager is right: a 20% employee discount still allows the store to make a profit, while a 25% employee discount would cause the store to lose money.

It may not seem to make sense that the store would lose money after applying a 30% markup and only a 25% discount. The reason it does work out that way is that the discount is bigger in absolute dollars after the markup is factored in. That is, an employee getting 25% off an item is getting 25% off the original price *plus* 25% off the 30% markup, and those two numbers together add up to more than 30% of the original price.

Solve Real-World Problems Using Percents

Example 12

*In 2004 the US Department of Agriculture had 112071 employees, of which 87846 were Caucasian. Of the remaining minorities, African-American and Hispanic employees had the two largest demographic groups, with 11754 and 6899 employees respectively.**

- a) Calculate the total percentage of minority (non-Caucasian) employees at the USDA.
- b) Calculate the percentage of African-American employees at the USDA.
- c) Calculate the percentage of minority employees who were neither African-American nor Hispanic.

Solution

- a) Use the percent equation $\text{Rate} \times \text{Total} = \text{Part}$.

The *total* number of employees is 112071. We know that the number of Caucasian employees is 87846, which means that there must be $112071 - 87846 = 24225$ non-Caucasian employees. This is the *part*. Plugging in the total and the part, we get $\text{Rate} \times 112071 = 24225$.

Divide both sides by 112071 to get $\text{Rate} = \frac{24225}{112071} \approx 0.216$. Multiply by 100 to get this as a percent: 21.6%.

21.6% of USDA employees in 2004 were from minority groups.

- b) Here, the total is still 112071 and the part is 11754, so we have $\text{Rate} \times 112071 = 11754$. Dividing, we get $\text{Rate} = \frac{11754}{112071} \approx 0.105$, or 10.5%.

10.5% of USDA employees in 2004 were African-American.

- c) Here, our total is just the number of non-Caucasian employees, which we found out is 24225. Subtracting the African-American and Hispanic employees leaves $24225 - 11754 - 6899 = 5572$ employees in the group we're looking at.

So with 24225 for the whole and 5572 for the part, our equation is $\text{Rate} \times 24225 = 5572$, or $\text{Rate} = \frac{5572}{24225} \approx 0.230$, or 23%.

23% of USDA minority employees in 2004 were neither African-American nor Hispanic.

Example 13

In 1995 New York had 18136000 residents. There were 827025 reported crimes, of which 152683 were violent. By 2005 the population was 19254630 and there were 85839 violent crimes out of a total of 491829

reported crimes. (Source: New York Law Enforcement Agency Uniform Crime Reports.) Calculate the percentage change from 1995 to 2005 in:

- a) Population of New York
- b) Total reported crimes
- c) violent crimes

Solution

This is a percentage change problem. Remember the formula for percentage change:

$$\text{Percent change} = \frac{\text{final amount} - \text{original amount}}{\text{original amount}} \times 100\%$$

In these problems, the final amount is the 2005 statistic, and the initial amount is the 1995 statistic.

a) Population:

$$\begin{aligned} \text{Percent change} &= \frac{19254630 - 18136000}{18136000} \times 100\% \\ &= \frac{1118630}{18136000} \times 100\% \\ &\approx 0.0617 \times 100\% \\ &= 6.17\% \end{aligned}$$

The population grew by 6.17%.

b) Total reported crimes:

$$\begin{aligned} \text{Percent change} &= \frac{491829 - 827025}{827025} \times 100\% \\ &= \frac{-335196}{827025} \times 100\% \\ &\approx -0.4053 \times 100\% \\ &= -40.53\% \end{aligned}$$

The total number of reported crimes fell by 40.53%.

c) Violent crimes:

$$\begin{aligned} \text{Percent change} &= \frac{85839 - 152683}{152683} \times 100\% \\ &= \frac{-66844}{152683} \times 100\% \\ &\approx -0.4377 \times 100\% \\ &= -43.77\% \end{aligned}$$

The total number of violent crimes fell by 43.77%.

Lesson Summary

- A **percent** is simply a ratio with a base unit of 100—for example, $13\% = \frac{13}{100}$.
- The **percent equation** is $\text{Rate} \times \text{Total} = \text{Part}$, or $R\% \text{ of Total is Part}$.
- The percent change equation is $\text{Percent change} = \frac{\text{final amount} - \text{original amount}}{\text{original amount}} \times 100\%$. A **positive** percent change means the value **increased**, while a **negative** percent change means the value **decreased**.

Review Questions

- Express the following decimals as a percent.
 - 0.011
 - 0.001
 - 0.91
 - 1.75
 - 20
- Express the following percentages in decimal form.
 - 15%
 - 0.08%
 - 222%
 - 3.5%
 - 341.9%
- Express the following fractions as a percent (round to two decimal places when necessary).
 - $\frac{1}{6}$
 - $\frac{5}{24}$
 - $\frac{6}{7}$
 - $\frac{11}{7}$
 - $\frac{13}{97}$
- Express the following percentages as a reduced fraction.
 - 11%
 - 65%
 - 16%
 - 12.5%
 - 87.5%
- Find the following.
 - 30% of 90
 - 16.7% of 199
 - 11.5% of 10.01
 - $y\%$ of $3x$
- A TV is advertised on sale. It is 35% off and now costs \$195. What was the pre-sale price?
- An employee at a store is currently paid \$9.50 per hour. If she works a full year she gets a 12% pay raise. What will her new hourly rate be after the raise?
- Store *A* and Store *B* both sell bikes, and both buy bikes from the same supplier at the same prices. Store *A* has a 40% mark-up for their prices, while store *B* has a 250% mark-up. Store *B* has a permanent sale and will always sell at 60% off the marked-up prices. Which store offers the better deal?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9613> <http://www.ck12.org/flexr/chapter/9613>.

Chapter 2

Graphs of Equations and Functions

2.1 The Coordinate Plane

Learning Objectives

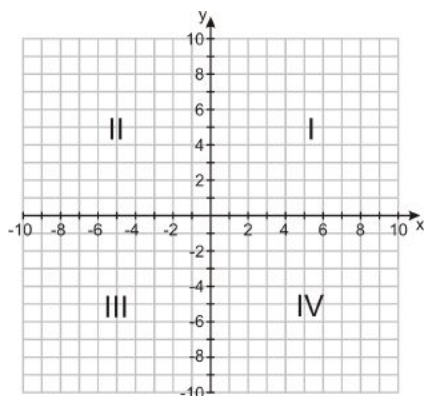
- Identify coordinates of points.
- Plot points in a coordinate plane.
- Graph a function given a table.
- Graph a function given a rule.

Introduction

Lydia lives 2 blocks north and one block east of school; Travis lives three blocks south and two blocks west of school. What's the shortest line connecting their houses?

The Coordinate Plane

We've seen how to represent numbers using number lines; now we'll see how to represent sets of numbers using a **coordinate plane**. The coordinate plane can be thought of as two number lines that meet at right angles. The horizontal line is called the **x-axis** and the vertical line is the **y-axis**. Together the lines are called the **axes**, and the point at which they cross is called the **origin**. The axes split the coordinate plane into four **quadrants**, which are numbered sequentially (I, II, III, IV) moving counter-clockwise from the upper right.



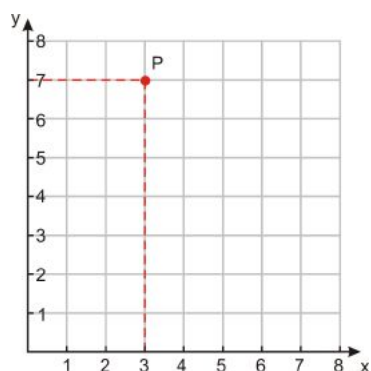
Identify Coordinates of Points

When given a point on a coordinate plane, it's easy to determine its **coordinates**. The coordinates of a point are two numbers - written together they are called an **ordered pair**. The numbers describe how far along the x -axis and y -axis the point is. The ordered pair is written in parentheses, with the x -**coordinate** (also called the **abscissa**) first and the y -**coordinate** (or the **ordinate**) second.

| | |
|-------------------|--|
| $(1, 7)$ | An ordered pair with an x - value of one and a y - value of seven |
| $(0, 5)$ | An ordered pair with an x - value of zero and a y - value of five |
| $(-2.5, 4)$ | An ordered pair with an x - value of -2.5 and a y - value of four |
| $(-107.2, -.005)$ | An ordered pair with an x - value of -107.2 and a y - value of $-.005$ |

Identifying coordinates is just like reading points on a number line, except that now the points do not actually lie **on** the number line! Look at the following example.

Example 1



Find the coordinates of the point labeled P in the diagram above

Solution

Imagine you are standing at the origin (the point where the x -axis meets the y -axis). In order to move to a position where P was directly above you, you would move 3 units to the **right** (we say this is in the **positive** x -direction).

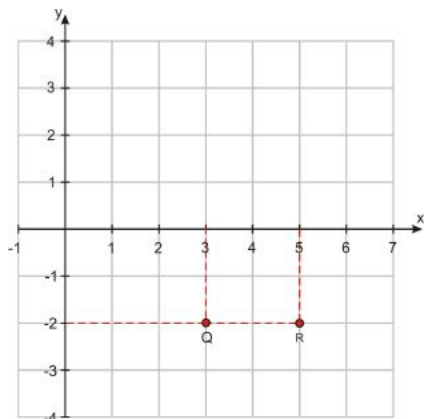
The x -coordinate of P is $+3$.

Now if you were standing at the 3 marker on the x -axis, point P would be 7 units **above** you (above the axis means it is in the **positive** y direction).

The y -coordinate of P is $+7$.

The coordinates of point P are $(3, 7)$.

Example 2



Find the coordinates of the points labeled Q and R in the diagram to the right.

Solution

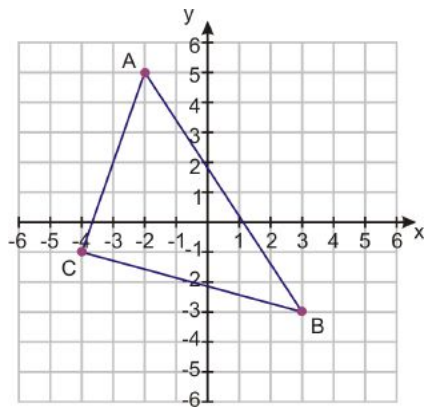
In order to get to Q we move three units to the right, in the positive x -direction, then two units **down**. This time we are moving in the **negative** y -direction. The x -coordinate of Q is $+3$, the y -coordinate of Q is -2 .

The coordinates of R are found in a similar way. The x -coordinate is $+5$ (five units in the positive x -direction) and the y -coordinate is again -2 .

The coordinates of Q are $(3, -2)$. The coordinates of R are $(5, -2)$.

Example 3

Triangle ABC is shown in the diagram to the right. Find the coordinates of the vertices A , B and C .



Point A :

x - coordinate = -2

y - coordinate = $+5$

Point B :

x - coordinate = $+3$

y - coordinate = -3

Point C:

x - coordinate = -4

y - coordinate = -1

Solution

$A(-2, 5)$

$B(3, -3)$

$C(-4, -1)$

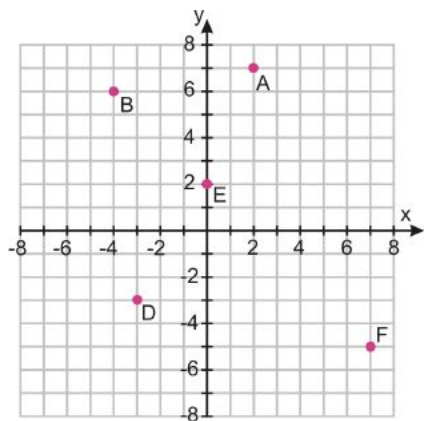
Plot Points in a Coordinate Plane

Plotting points is simple, once you understand how to read coordinates and read the scale on a graph. As a note on scale, in the next two examples pay close attention to the labels on the axes.

Example 4

Plot the following points on the coordinate plane.

$A(2, 7)$ $B(-4, 6)$ $D(-3, -3)$ $E(0, 2)$ $F(7, -5)$



Point $A(2, 7)$ is 2 units right, 7 units up. It is in Quadrant I.

Point $B(-4, 6)$ is 4 units left, 6 units up. It is in Quadrant II.

Point $D(-3, -3)$ is 3 units left, 3 units down. It is in Quadrant III.

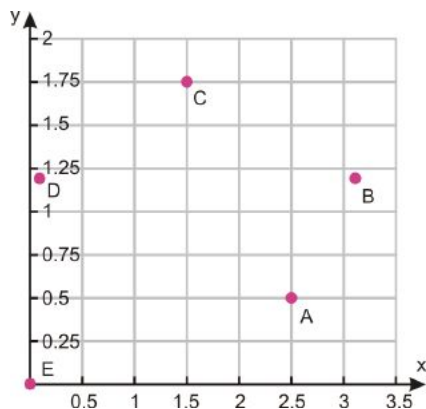
Point $E(0, 2)$ is 2 units up from the origin. It is right on the y -axis, between Quadrants I and II.

Point $F(7, -5)$ is 7 units right, 5 units down. It is in Quadrant IV.

Example 5

Plot the following points on the coordinate plane.

$A(2.5, 0.5)$ $B(\pi, 1.2)$ $C(2, 1.75)$ $D(0.1, 1.2)$ $E(0, 0)$



Here we see the importance of choosing the right scale and range for the graph. In Example 4, our points were scattered throughout the four quadrants. In this case, all the coordinates are positive, so we don't need to show the negative values of x or y . Also, there are no x -values bigger than about 3.14, and 1.75 is the largest value of y . We can therefore show just the part of the coordinate plane where $0 \leq x \leq 3.5$ and $0 \leq y \leq 2$.

Here are some other important things to notice about this graph:

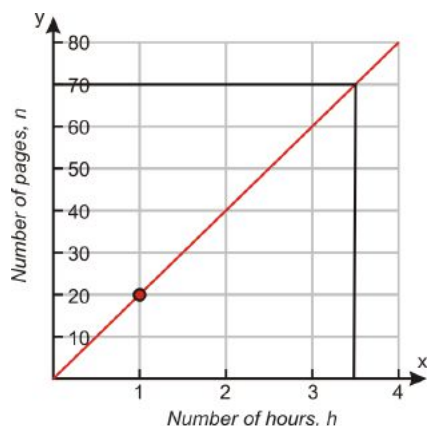
- The tick marks on the axes don't correspond to unit increments (i.e. the numbers do not go up by one each time). This is so that we can plot the points more precisely.
- The scale on the x -axis is different than the scale on the y -axis, so distances that look the same on both axes are actually greater in the x -direction. Stretching or shrinking the scale in one direction can be useful when the points we want to plot are farther apart in one direction than the other.

For more practice locating and naming points on the coordinate plane, try playing the Coordinate Plane Game at <http://www.math-play.com/Coordinate%20Plane%20Game/Coordinate%20Plane%20Game.html>.

Graph a Function Given a Table

Once we know how to plot points on a coordinate plane, we can think about how we'd go about plotting a relationship between x - and y -values. So far we've just been plotting sets of ordered pairs. A set like that is a **relation**, and there isn't necessarily a relationship between the x -values and y -values. If there is a relationship between the x - and y -values, and each x -value corresponds to exactly one y -value, then the relation is called a **function**. Remember that a function is a particular way to relate one quantity to another.

If you're reading a book and can read twenty pages an hour, there is a relationship between how many hours you read and how many pages you read. You may even know that you could write the formula as either $n = 20h$ or $h = \frac{n}{20}$, where h is the number of hours you spend reading and n is the number of pages you read. To find out, for example, how many pages you could read in $3\frac{1}{2}$ hours, or how many hours it would take you to read 46 pages, you could use one of those formulas. Or, you could make a graph of the function:



Once you know how to graph a function like this, you can simply read the relationship between the x - and y -values off the graph. You can see in this case that you could read 70 pages in $3\frac{1}{2}$ hours, and it would take you about $2\frac{1}{3}$ hours to read 46 pages.

Generally, the graph of a **function** appears as a line or curve that goes through all points that have the relationship that the function describes. If the domain of the function (the set of x -values we can plug into the function) is all real numbers, then we call it a **continuous function**. If the domain of the function is a particular set of values (such as whole numbers only), then it is called a **discrete function**. The graph will be a series of dots, but they will still often fall along a line or curve.

In graphing equations, we assume the domain is all real numbers, unless otherwise stated. Often, though, when we look at data in a table, the domain will be whole numbers (number of presents, number of days, etc.) and the function will be discrete. But sometimes we'll still draw the graph as a continuous line to make it easier to interpret. Be aware of the difference between discrete and continuous functions as you work through the examples.

Example 6

Sarah is thinking of the number of presents she receives as a function of the number of friends who come to her birthday party. She knows she will get a present from her parents, one from her grandparents and one each from her uncle and aunt. She wants to invite up to ten of her friends, who will each bring one present. She makes a table of how many presents she will get if one, two, three, four or five friends come to the party. Plot the points on a coordinate plane and graph the function that links the number of presents with the number of friends. Use your graph to determine how many presents she would get if eight friends show up.

Table 2.1:

| Number of Friends | Number of Presents |
|-------------------|--------------------|
| 0 | 4 |
| 1 | 5 |
| 2 | 6 |
| 3 | 7 |
| 4 | 8 |
| 5 | 9 |

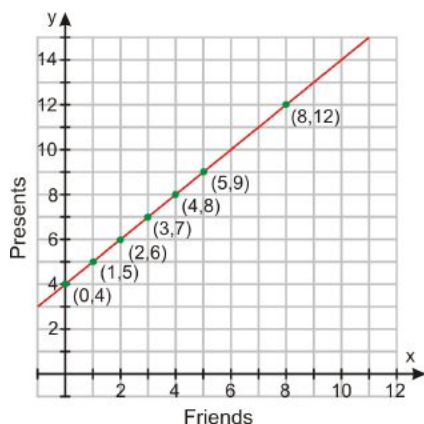
The first thing we need to do is decide how our graph should appear. We need to decide what the independent variable is, and what the dependant variable is. Clearly in this case, the number of friends can vary independently, but the number of presents must depend on the number of friends who show up.

So we'll plot friends on the x -axis and presents on the y -axis. Let's add another column to our table containing the coordinates that each (friends, presents) ordered pair gives us.

Table 2.2:

| Friends (x) | Presents (y) | Coordinates (x, y) |
|-----------------|------------------|------------------------|
| 0 | 4 | (0, 4) |
| 1 | 5 | (1, 5) |
| 2 | 6 | (2, 6) |
| 3 | 7 | (3, 7) |
| 4 | 8 | (4, 8) |
| 5 | 9 | (5, 9) |

Next we need to set up our axes. It is clear that the number of friends and number of presents both must be positive, so we only need to show points in Quadrant I. Now we need to choose a suitable scale for the x - and y -axes. We only need to consider eight friends (look again at the question to confirm this), but it always pays to allow a little extra room on your graph. We also need the y -scale to accommodate the presents for eight people. We can see that this is still going to be under 20!



The scale of this graph has room for up to 12 friends and 15 presents. This will be fine, but there are many other scales that would be equally good!

Now we proceed to plot the points. The first five points are the coordinates from our table. You can see they all lie on a straight line, so the function that describes the relationship between x and y will be linear. To graph the function, we simply draw a line that goes through all five points. This line represents the function.

This is a discrete problem since Sarah can only invite a positive whole number of friends. For instance, it would be impossible for 2.4 or -3 friends to show up. So although the line helps us see where the other values of the function are, the only points on the line that actually *are* values of the function are the ones with positive whole-number coordinates.

The graph easily lets us find other values for the function. For example, the question asks how many presents Sarah would get if eight friends come to her party. Don't forget that x represents the number of friends and y represents the number of presents. If we look at the graph where $x = 8$, we can see that the function has a y -value of 12.

Solution

If 8 friends show up, Sarah will receive a total of 12 presents.

Graph a Function Given a Rule

If we are given a rule instead of a table, we can proceed to graph the function in either of two ways. We will use the following example to show each way.

Example 7

Ali is trying to work out a trick that his friend showed him. His friend started by asking him to think of a number, then double it, then add five to the result. Ali has written down a rule to describe the first part of the trick. He is using the letter x to stand for the number he thought of and the letter y to represent the final result of applying the rule. He wrote his rule in the form of an equation: $y = 2x + 5$.

Help him visualize what is going on by graphing the function that this rule describes.

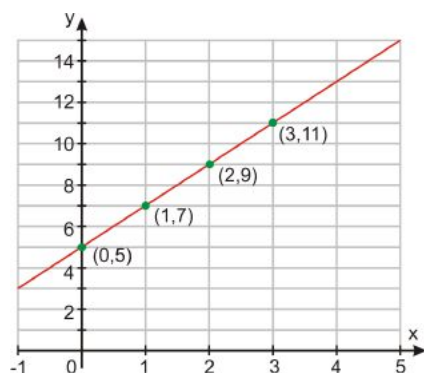
Method One - Construct a Table of Values

If we wish to plot a few points to see what is going on with this function, then the best way is to construct a table and populate it with a few (x, y) pairs. We'll use 0, 1, 2 and 3 for x -values.

Table 2.3:

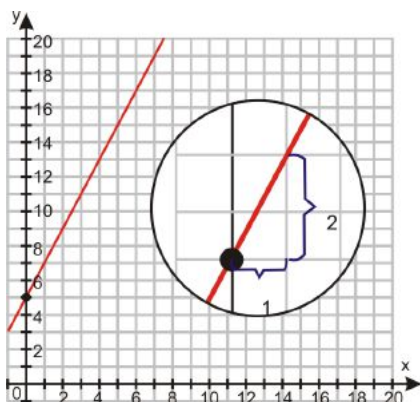
| x | y |
|-----|-----|
| 0 | 5 |
| 1 | 7 |
| 2 | 9 |
| 3 | 11 |

Next, we plot the points and join them with a line.



This method is nice and simple—especially with linear relationships, where we don't need to plot more than two or three points to see the shape of the graph. In this case, the function is continuous because the domain is all real numbers—that is, Ali could think of any real number, even though he may only be thinking of positive whole numbers.

Method Two - Intercept and Slope



Another way to graph this function (one that we'll learn in more detail in a later lesson) is the slope-intercept method. To use this method, follow these steps:

1. Find the y value when $y = 0$.

$y(0) = 2 \cdot 0 + 5 = 5$, so our y -intercept is $(0, 5)$.

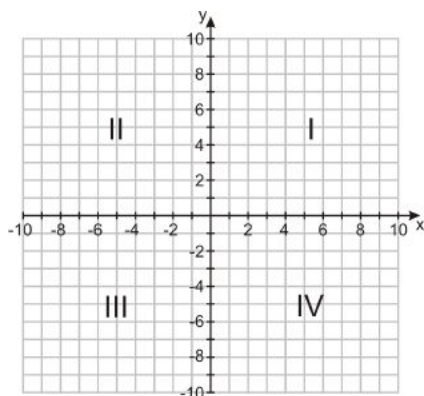
2. Look at the coefficient multiplying the x .

Every time we increase x by one, y increases by two, so our slope is $+2$.

3. Plot the line with the given slope that goes through the intercept. We start at the point $(0, 5)$ and move over one in the x -direction, then up two in the y -direction. This gives the slope for our line, which we extend in both directions.

We will properly examine this last method later in this chapter!

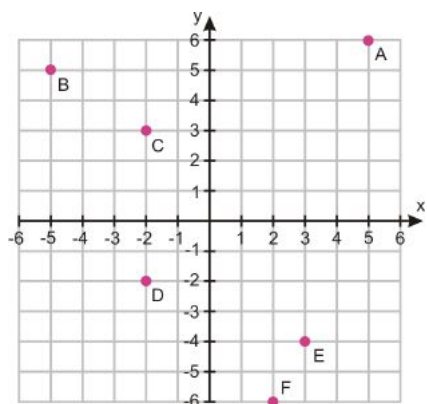
Lesson Summary



- The **coordinate plane** is a two-dimensional space defined by a horizontal number line (the x -axis) and a vertical number line (the y -axis). The **origin** is the point where these two lines meet. Four areas, or **quadrants**, are formed as shown in the diagram above.
- Each point on the coordinate plane has a set of **coordinates**, two numbers written as an **ordered pair** which describe how far along the x -axis and y -axis the point is. The x -coordinate is always written first, then the y -coordinate, in the form (x, y) .
- **Functions** are a way that we can relate one quantity to another. Functions can be plotted on the coordinate plane.

Review Questions

1. Identify the coordinates of each point, $A - F$, on the graph below.



2. Draw a line on the above graph connecting point B with the origin. Where does that line intersect the line connecting points C and D ?
3. Plot the following points on a graph and identify which quadrant each point lies in:
- (a) $(4, 2)$
 - (b) $(-3, 5.5)$
 - (c) $(4, -4)$
 - (d) $(-2, -3)$
4. Without graphing the following points, identify which quadrant each lies in:
- (a) $(5, 3)$
 - (b) $(-3, -5)$
 - (c) $(-4, 2)$
 - (d) $(2, -4)$
5. Consider the graph of the equation $y = 3$. Which quadrants does it pass through?
6. Consider the graph of the equation $y = x$. Which quadrants does it pass through?
7. Consider the graph of the equation $y = x + 3$. Which quadrants does it pass through?
8. The point $(4, 0)$ is on the boundary between which two quadrants?
9. The point $(0, -5)$ is on the boundary between which two quadrants?
10. If you moved the point $(3, 2)$ five units to the left, what quadrant would it be in?
11. The following three points are three vertices of square $ABCD$. Plot them on a graph, then determine what the coordinates of the fourth point, D , would be. Plot that point and label it.

$$A(-4, -4) \quad B(3, -4) \quad C(3, 3)$$

12. In what quadrant is the center of the square from problem 10? (You can find the center by drawing the square's diagonals.)
13. What point is halfway between $(1, 3)$ and $(1, 5)$?
14. What point is halfway between $(2, 8)$ and $(6, 8)$?
15. What point is halfway between the origin and $(10, 4)$?
16. What point is halfway between $(3, -2)$ and $(-3, 2)$?
17. Becky has a large bag of M&Ms that she knows she should share with Jaeyun. Jaeyun has a packet of Starburst. Becky tells Jaeyun that for every Starburst he gives her, she will give him three M&Ms in return. If x is the number of Starburst that Jaeyun gives Becky, and y is the number of M&Ms he gets in return, then complete each of the following.
- (a) Write an algebraic rule for y in terms of x .
 - (b) Make a table of values for y with x -values of 0, 1, 2, 3, 4, 5.

- (c) Plot the function linking x and y on the following scale: $0 \leq x \leq 10$, $0 \leq y \leq 10$.

2.2 Graphs of Linear Equations

Learning Objectives

- Graph a linear function using an equation.
- Write equations and graph horizontal and vertical lines.
- Analyze graphs of linear functions and read conversion graphs.

Introduction

You're stranded downtown late at night with only \$8 in your pocket, and your home is 6 miles away. Two cab companies serve this area; one charges \$1.20 per mile with an additional \$1 fee, and the other charges \$0.90 per mile with an additional \$2.50 fee. Which cab will be able to get you home?

Graph a Linear Equation

At the end of Lesson 4.1 we looked at ways to graph a function from a rule. A rule is a way of writing the relationship between the two quantities we are graphing. In mathematics, we tend to use the words **formula** and **equation** to describe the rules we get when we express relationships algebraically. Interpreting and graphing these equations is an important skill that you'll use frequently in math.

Example 1

A taxi costs more the further you travel. Taxis usually charge a fee on top of the per-mile charge to cover hire of the vehicle. In this case, the taxi charges \$3 as a set fee and \$0.80 per mile traveled. Here is the equation linking the cost in dollars (y) to hire a taxi and the distance traveled in miles (x).

$$y = 0.8x + 3$$

Graph the equation and use your graph to estimate the cost of a seven-mile taxi ride.

Solution

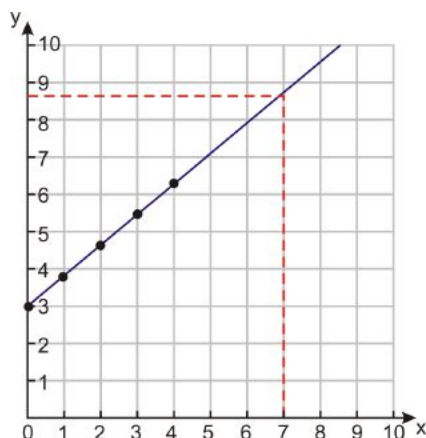
We'll start by making a table of values. We will take a few values for x (0, 1, 2, 3, and 4), find the corresponding y -values, and then plot them. Since the question asks us to find the cost for a seven-mile journey, we need to choose a scale that can accommodate this.

First, here's our table of values:

Table 2.4:

| x | y |
|-----|-----|
| 0 | 3 |
| 1 | 3.8 |
| 2 | 4.6 |
| 3 | 5.4 |
| 4 | 6.2 |

And here's our graph:



To find the cost of a seven-mile journey, first we find $x = 7$ on the horizontal axis and draw a line up to our graph. Next, we draw a horizontal line across to the y -axis and read where it hits. It appears to hit around half way between $y = 8$ and $y = 9$. Let's call it 8.5.

A seven mile taxi ride would cost approximately \$8.50 (\$8.60 exactly).

Here are some things you should notice about this graph and the formula that generated it:

- The graph is a straight line (this means that the equation is **linear**), although the function is **discrete** and really just consists of a series of points.
- The graph crosses the y -axis at $y = 3$ (notice that there's $a + 3$ in the equation—that's not a coincidence!). This is the base cost of the taxi.
- Every time we move **over** by one square we move **up** by 0.8 squares (notice that that's also the coefficient of x in the equation). This is the rate of charge of the taxi (cost per mile).
- If we move over by three squares, we move up by 3×0.8 squares.

Example 2

A small business has a debt of \$500,000 incurred from start-up costs. It predicts that it can pay off the debt at a rate of \$85,000 per year according to the following equation governing years in business (x) and debt measured in thousands of dollars (y).

$$y = -85x + 500$$

Graph the above equation and use your graph to predict when the debt will be fully paid.

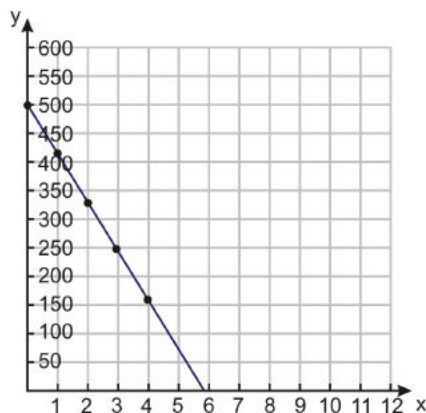
Solution

First, we start with our table of values:

Table 2.5:

| x | y |
|-----|-----|
| 0 | 500 |
| 1 | 415 |
| 2 | 330 |
| 3 | 245 |
| 4 | 160 |

Then we plot our points and draw the line that goes through them:



Notice the scale we've chosen here. There's no need to include any points above $y = 500$, but it's still wise to allow a little extra.

Next we need to determine how many years it takes the debt to reach zero, or in other words, what x -value will make the y -value equal 0. We know it's greater than four (since at $x = 4$ the y -value is still positive), so we need an x -scale that goes well past $x = 4$. Here we've chosen to show the x -values from 0 to 12, though there are many other places we could have chosen to stop.

To read the time that the debt is paid off, we simply read the point where the line hits $y = 0$ (the x -axis). It looks as if the line hits pretty close to $x = 6$. So **the debt will definitely be paid off in six years.**

To see more simple examples of graphing linear equations by hand, see the Khan Academy video on graphing lines at <http://www.youtube.com/watch?v=2UrcUfBizyw>. The narrator shows how to graph several linear equations, using a table of values to plot points and then connecting the points with a line.

Graphs and Equations of Horizontal and Vertical Lines

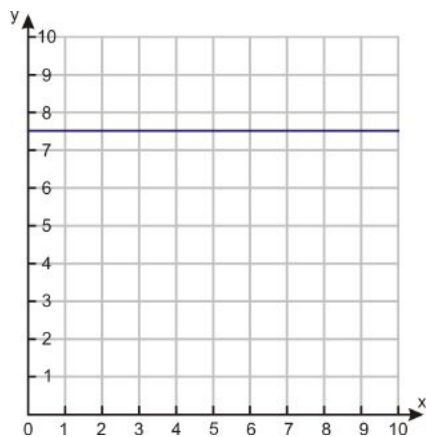
Example 3

"Mad-cabs" have an unusual offer going on. They are charging \$7.50 for a taxi ride of any length within the city limits. Graph the function that relates the cost of hiring the taxi (y) to the length of the journey in miles (x).

To proceed, the first thing we need is an **equation**. You can see from the problem that the cost of a journey doesn't depend on the length of the journey. It should come as no surprise that the equation then, does not have x in it. Since any value of x results in the same value of $y(7.5)$, the value you choose for x doesn't matter, so it isn't included in the equation. Here is the equation:

$$y = 7.5$$

The graph of this function is shown below. You can see that it's simply a horizontal line.



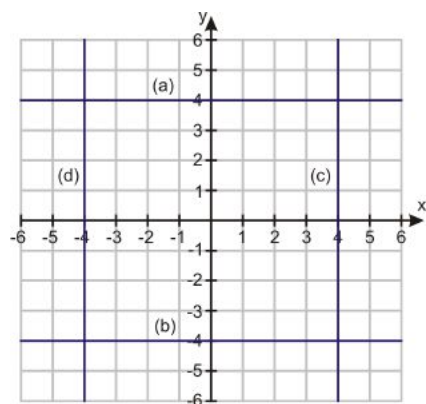
Any time you see an equation of the form “ $y = \text{constant}$,” the graph is a horizontal line that intercepts the y -axis at the value of the constant.

Similarly, when you see an equation of the form $x = \text{constant}$, then the graph is a vertical line that intercepts the x -axis at the value of the constant. (Notice that that kind of equation is a relation, and not a function, because each x -value (there’s only one in this case) corresponds to many (actually an infinite number) y -values.)

Example 4

Plot the following graphs.

- (a) $y = 4$
- (b) $y = -4$
- (c) $x = 4$
- (d) $x = -4$



- (a) $y = 4$ is a horizontal line that crosses the y -axis at 4.
- (b) $y = -4$ is a horizontal line that crosses the y -axis at -4.
- (c) $x = 4$ is a vertical line that crosses the x -axis at 4.
- (d) $x = -4$ is a vertical line that crosses the x -axis at -4.

Example 5

Find an equation for the x -axis and the y -axis.

Look at the axes on any of the graphs from previous examples. We have already said that they intersect at the origin (the point where $x = 0$ and $y = 0$). The following definition could easily work for each axis.

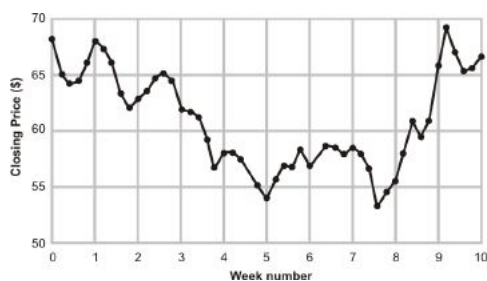
x-axis: A horizontal line crossing the *y*-axis at zero.

y-axis: A vertical line crossing the *x*-axis at zero.

So using example 3 as our guide, we could define the *x*-axis as the line $y = 0$ and the *y*-axis as the line $x = 0$.

Analyze Graphs of Linear Functions

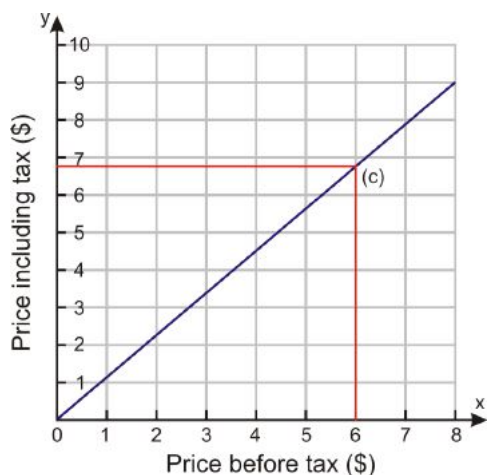
We often use graphs to represent relationships between two linked quantities. It's useful to be able to interpret the information that graphs convey. For example, the chart below shows a fluctuating stock price over ten weeks. You can read that the index closed the first week at about \$68, and at the end of the third week it was at about \$62. You may also see that in the first five weeks it lost about 20% of its value, and that it made about 20% gain between weeks seven and ten. Notice that this relationship is discrete, although the dots are connected to make the graph easier to interpret.



Analyzing graphs is a part of life - whether you are trying to decide to buy stock, figure out if your blog readership is increasing, or predict the temperature from a weather report. Many graphs are very complicated, so for now we'll start off with some simple linear conversion graphs. Algebra starts with basic relationships and builds to more complicated tasks, like reading the graph above.

Example 6

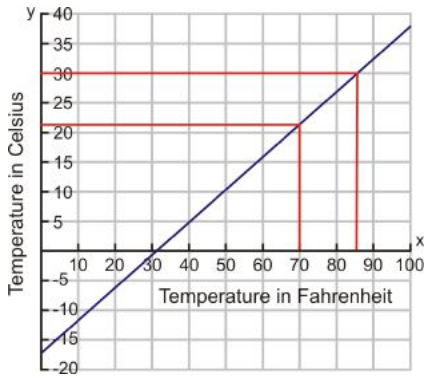
Below is a graph for converting marked prices in a downtown store into prices that include sales tax. Use the graph to determine the cost including sales tax for a \$6.00 pen in the store.



To find the relevant price with tax, first find the correct pre-tax price on the *x*-axis. This is the point $x = 6$.

Draw the line $x = 6$ up until it meets the function, then draw a horizontal line to the *y*-axis. This line hits at $y \approx 6.75$ (about three fourths of the way from $y = 6$ to $y = 7$).

The approximate cost including tax is \$6.75.



Example 7

The chart for converting temperature from Fahrenheit to Celsius is shown to the right. Use the graph to convert the following:

- a) 70° Fahrenheit to Celsius
- b) 0° Fahrenheit to Celsius
- c) 30° Celsius to Fahrenheit
- d) 0° Celsius to Fahrenheit

Solution

a) To find 70° Fahrenheit, we look along the Fahrenheit-axis (in other words the x -axis) and draw the line $x = 70$ up to the function. Then we draw a horizontal line to the Celsius-axis (y -axis). The horizontal line hits the axis at a little over 20 (21 or 22).

70° Fahrenheit is approximately equivalent to 21° Celsius.

b) To find 0° Fahrenheit, we just look at the y -axis. (Don't forget that this axis is simply the line $x = 0$.) The line hits the y -axis just below the half way point between 15 and 20.

0° Fahrenheit is approximately equivalent to -18° Celsius.

c) To find 30° Celsius, we look up the Celsius-axis and draw the line $y = 30$ along to the function. When this horizontal line hits the function, we draw a line straight down to the Fahrenheit-axis. The line hits the axis at approximately 85.

30° Celsius is approximately equivalent to 85° Fahrenheit.

d) To find 0° Celsius, we look at the Fahrenheit-axis (the line $y = 0$). The function hits the x -axis just right of 30.

0° Celsius is equivalent to 32° Fahrenheit.

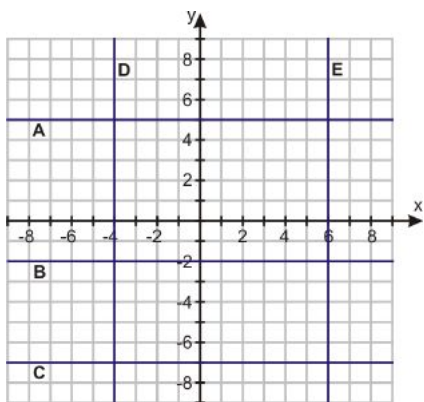
Lesson Summary

- Equations with the variables y and x can be graphed by making a chart of values that fit the equation and then plotting the values on a coordinate plane. This graph is simply another representation of the equation and can be analyzed to solve problems.
- Horizontal lines are defined by the equation $y = \text{constant}$ and vertical lines are defined by the equation $x = \text{constant}$.
- Be aware that although we graph the function as a line to make it easier to interpret, the function

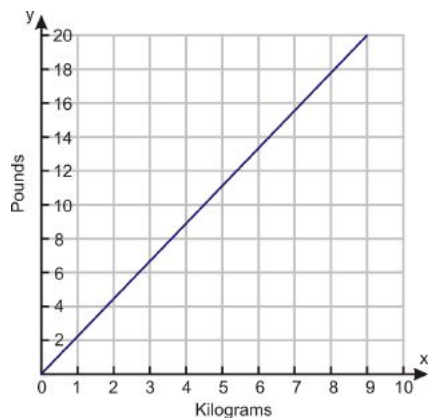
may actually be discrete.

Review Questions

- Make a table of values for the following equations and then graph them.
 - $y = 2x + 7$
 - $y = 0.7x - 4$
 - $y = 6 - 1.25x$
- “Think of a number. Multiply it by 20, divide the answer by 9, and then subtract seven from the result.”*
 - Make a table of values and plot the function that represents this sentence.
 - If you picked 0 as your starting number, what number would you end up with?
 - To end up with 12, what number would you have to start out with?
- Write the equations for the five lines (*A* through *E*) plotted in the graph below.



- In the graph above, at what points do the following lines intersect?
 - A* and *E*
 - A* and *D*
 - C* and *D*
 - B* and the *y*-axis
 - E* and the *x*-axis
 - C* and the line $y = x$
 - E* and the line $y = \frac{1}{2}x$
 - A* and the line $y = x + 3$
- At the airport, you can change your money from dollars into euros. The service costs \$5, and for every additional dollar you get 0.7 euros.
 - Make a table for this and plot the function on a graph.
 - Use your graph to determine how many euros you would get if you give the office \$50.
 - To get 35 euros, how many dollars would you have to pay?
 - The exchange rate drops so that you can only get 0.5 euros per additional dollar. Now how many dollars do you have to pay for 35 euros?
- The graph below shows a conversion chart for converting between weight in kilograms and weight in pounds. Use it to convert the following measurements.



- (a) 4 kilograms into weight in pounds
- (b) 9 kilograms into weight in pounds
- (c) 12 pounds into weight in kilograms
- (d) 17 pounds into weight in kilograms

7. Use the graph from problem 6 to answer the following questions.

- (a) An employee at a sporting goods store is packing 3-pound weights into a box that can hold 8 kilograms. How many weights can she place in the box?
- (b) After packing those weights, there is some extra space in the box that she wants to fill with one-pound weights. How many of those can she add?
- (c) After packing those, she realizes she misread the label and the box can actually hold 9 kilograms. How many more one-pound weights can she add?

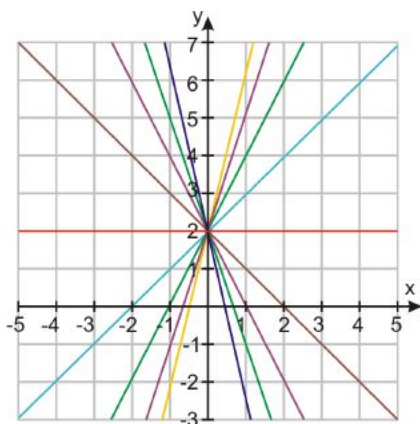
2.3 Graphing Using Intercepts

Learning Objectives

- Find intercepts of the graph of an equation.
- Use intercepts to graph an equation.
- Solve real-world problems using intercepts of a graph

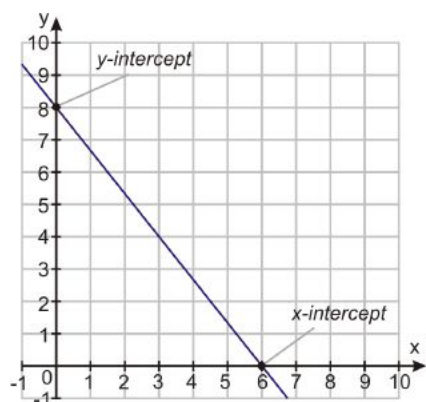
Introduction

Sanjit's office is 25 miles from home, and in traffic he expects the trip home to take him an hour if he starts at 5 PM. Today he hopes to stop at the post office along the way. If the post office is 6 miles from his office, when will Sanjit get there?



If you know just one of the points on a line, you'll find that isn't enough information to plot the line on a graph. As you can see in the graph above, there are many lines—in fact, infinitely many lines—that pass through a single point. But what if you know two points that are both on the line? Then there's only one way to graph that line; all you need to do is plot the two points and use a ruler to draw the line that passes through both of them.

There are a lot of options for choosing which two points on the line you use to plot it. In this lesson, we'll focus on two points that are rather convenient for graphing: the points where our line crosses the x - and y -axes, or **intercepts**. We'll see how to find intercepts algebraically and use them to quickly plot graphs.



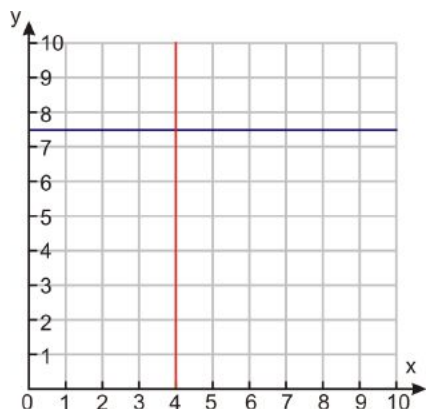
Look at the graph above. The **y-intercept** occurs at the point where the graph crosses the y -axis. The y -value at this point is 8, and the x -value is 0.

Similarly, the **x-intercept** occurs at the point where the graph crosses the x -axis. The x -value at this point is 6, and the y -value is 0.

So we know the coordinates of two points on the graph: (0, 8) and (6, 0). If we'd just been given those two coordinates out of the blue, we could quickly plot those points and join them with a line to recreate the above graph.

Note: Not all lines will have both an x - and a y -intercept, but most do. However, horizontal lines never cross the x -axis and vertical lines never cross the y -axis.

For examples of these special cases, see the graph below.



Finding Intercepts by Substitution

Example 1

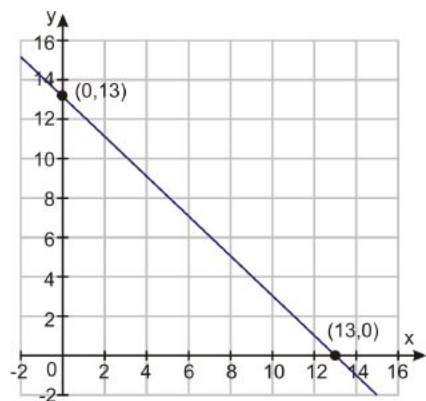
Find the intercepts of the line $y = 13 - x$ and use them to graph the function.

Solution

The first intercept is easy to find. The y -intercept occurs when $x = 0$. Substituting gives us $y = 13 - 0 = 13$, so the y -intercept is $(0, 13)$.

Similarly, the x -intercept occurs when $y = 0$. Plugging in 0 for y gives us $0 = 13 - x$, and adding x to both sides gives us $x = 13$. So $(13, 0)$ is the x -intercept.

To draw the graph, simply plot these points and join them with a line.



Example 2

Graph the following functions by finding intercepts.

- a) $y = 2x + 3$
- b) $y = 7 - 2x$
- c) $4x - 2y = 8$
- d) $2x + 3y = -6$

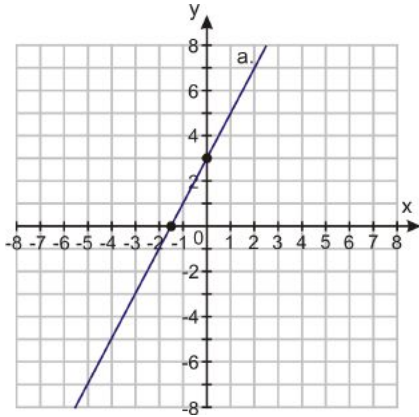
Solution

a) Find the y -intercept by plugging in $x = 0$:

$$y = 2 \cdot 0 + 3 = 3 \quad \text{— the } y\text{-intercept is } (0, 3)$$

Find the x -intercept by plugging in $y = 0$:

$$\begin{aligned} 0 &= 2x + 3 && - \text{subtract 3 from both sides :} \\ -3 &= 2x && - \text{divide by 2 :} \\ -\frac{3}{2} &= x && - \text{the } x - \text{intercept is } (-1.5, 0) \end{aligned}$$

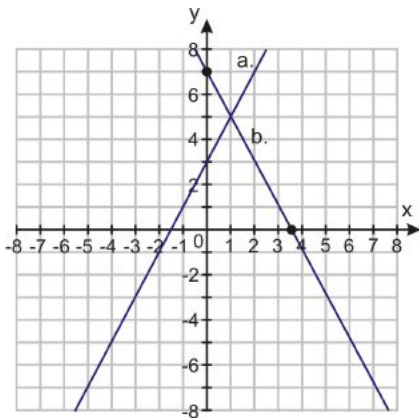


b) Find the y -intercept by plugging in $x = 0$:

$$y = 7 - 2 \cdot 0 = 7 \quad - \text{the } y - \text{intercept is } (0, 7)$$

Find the x -intercept by plugging in $y = 0$:

$$\begin{aligned} 0 &= 7 - 2x && - \text{subtract 7 from both sides :} \\ -7 &= -2x && - \text{divide by } -2 : \\ \frac{7}{2} &= x && - \text{the } x - \text{intercept is } (3.5, 0) \end{aligned}$$



c) Find the y -intercept by plugging in $x = 0$:

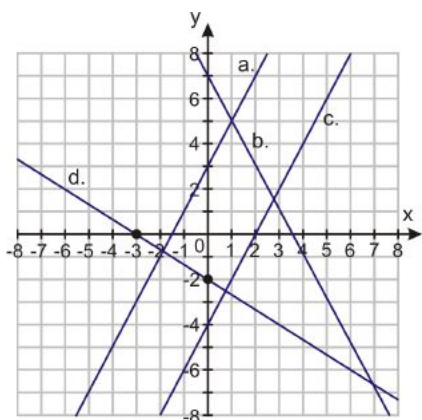
$$\begin{aligned} 4 \cdot 0 - 2y &= 8 \\ -2y &= 8 && - \text{divide by } -2 \\ y &= -4 && - \text{the } y - \text{intercept is } (0, -4) \end{aligned}$$

Find the x -intercept by plugging in $y = 0$:

$$4x - 2 \cdot 0 = 8$$

$$4x = 8 \quad - \text{divide by } 4 :$$

$$x = 2 \quad - \text{the } x - \text{intercept is } (2, 0)$$



d) Find the y-intercept by plugging in $x = 0$:

$$2 \cdot 0 + 3y = -6$$

$$3y = -6 \quad - \text{divide by } 3 :$$

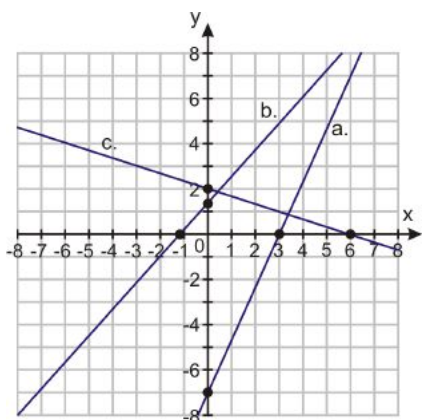
$$y = -2 \quad - \text{the } y - \text{intercept is } (0, -2)$$

Find the x-intercept by plugging in $y = 0$:

$$2x + 3 \cdot 0 = -6$$

$$2x = -6 \quad - \text{divide by } 2 :$$

$$x = -3 \quad - \text{the } x - \text{intercept is } (-3, 0)$$



Finding Intercepts for Standard Form Equations Using the Cover-Up Method

Look at the last two equations in example 2. These equations are written in **standard form**. Standard form equations are always written “**coefficient** times x plus (or minus) **coefficient** times y equals **value**”. In other words, they look like this:

$$ax + by = c$$

where a has to be positive, but b and c do not.

There is a neat method for finding intercepts in standard form, often referred to as the cover-up method.

Example 3

Find the intercepts of the following equations:

a) $7x - 3y = 21$

b) $12x - 10y = -15$

c) $x + 3y = 6$

Solution

To solve for each intercept, we realize that at the intercepts the value of **either** x **or** y is zero, and so any terms that contain that variable effectively drop out of the equation. To make a term disappear, simply cover it (a finger is an excellent way to cover up terms) and solve the resulting equation.

a) To solve for the y -intercept we set $x = 0$ and cover up the x -term:

$$\text{[finger covering } 7x\text{]} - 3y = 21$$

$$-3y = 21$$

$$y = -7 \quad (0, -7) \text{ is the } y\text{-intercept.}$$

Now we solve for the x -intercept:

$$7x - \text{[finger covering } -3y\text{]} = 21$$

$$7x = 21$$

$$x = 3 \quad (3, 0) \text{ is the } x\text{-intercept.}$$

b) To solve for the y -intercept ($x = 0$), cover up the x -term:

$$\text{[finger covering } 12x\text{]} - 10y = -15$$

$$-10y = -15$$

$$y = 1.5 \quad (0, 1.5) \text{ is the } y\text{-intercept.}$$

Now solve for the x -intercept ($y = 0$):

$$12x - \text{[finger covering } -10y\text{]} = -15$$

$$12x = -15$$

$$x = -\frac{5}{4} \quad (-1.25, 0) \text{ is the } x\text{-intercept.}$$

c) To solve for the y -intercept ($x = 0$), cover up the x -term:

$$\text{[finger covering } x\text{]} + 3y = 6$$

$$3y = 6$$

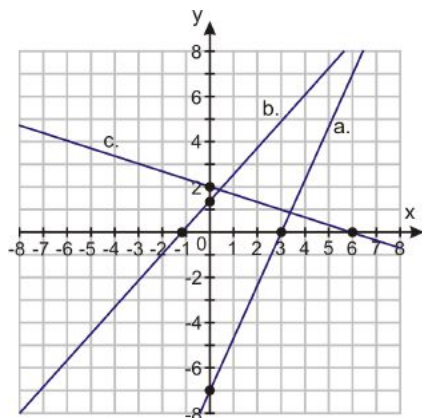
$$y = 2 \quad (0, 2) \text{ is the } y\text{-intercept.}$$

Solve for the y -intercept:

$$x \text{ (face)} = 6$$

$$x = 6 \quad (6, 0) \text{ is the } x\text{-intercept.}$$

The graph of these functions and the intercepts is below:



To learn more about equations in standard form, try the Java applet at <http://www.analyzemath.com/line/line.htm> (scroll down and click the “click here to start” button.) You can use the sliders to change the values of a , b , and c and see how that affects the graph.

Solving Real-World Problems Using Intercepts of a Graph

Example 4

Jesus has \$30 to spend on food for a class barbecue. Hot dogs cost \$0.75 each (including the bun) and burgers cost \$1.25 (including the bun). Plot a graph that shows all the combinations of hot dogs and burgers he could buy for the barbecue, without spending more than \$30.

This time we will find an equation first, and then we can think logically about finding the intercepts.

If the number of burgers that Jesus buys is x , then the money he spends on burgers is $1.25x$

If the number of hot dogs he buys is y , then the money he spends on hot dogs is $0.75y$

So the total cost of the food is $1.25x + 0.75y$.

The total amount of money he has to spend is \$30, so if he is to spend it ALL, we can use the following equation:

$$1.25x + 0.75y = 30$$

We can solve for the intercepts using the cover-up method. First the y -intercept ($x = 0$):

$$\text{face} + 0.75y = 30$$

$$0.75y = 30$$

$$y = 40 \quad y\text{-intercept: } (0, 40)$$

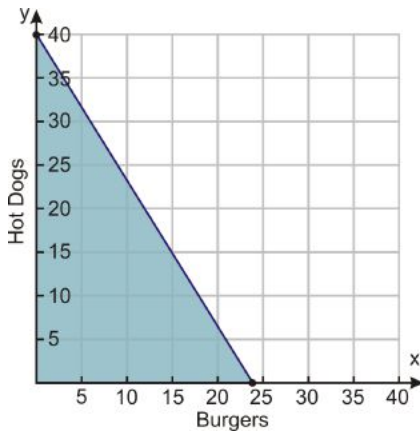
Then the x -intercept ($y = 0$):

$$1.25x + \text{🐕} = 30$$

$$1.25x = 30$$

$$x = 24 \quad x\text{-intercept: } (24, 0)$$

Now we plot those two points and join them to create our graph, shown here:



We could also have created this graph without needing to come up with an equation. We know that if John were to spend ALL the money on hot dogs, he could buy $\frac{30}{.75} = 40$ hot dogs. And if he were to buy only burgers he could buy $\frac{30}{1.25} = 24$ burgers. From those numbers, we can get 2 intercepts: (0 burgers, 40 hot dogs) and (24 burgers, 0 hot dogs). We could plot these just as we did above and obtain our graph that way.

As a final note, we should realize that Jesus' problem is really an example of an inequality. He can, in fact, spend any amount up to \$30. The only thing he cannot do is spend more than \$30. The graph above reflects this: the line is the set of solutions that involve spending exactly \$30, and the shaded region shows solutions that involve spending less than \$30.

Lesson Summary

- A **y-intercept** occurs at the point where a graph crosses the y -axis (where $x = 0$) and an **x-intercept** occurs at the point where a graph crosses the x -axis (where $y = 0$).
- The y -intercept can be found by **substituting** $x = 0$ into the equation and solving for y . Likewise, the x -intercept can be found by **substituting** $y = 0$ into the equation and solving for x .
- A linear equation is in **standard form** if it is written as "positive coefficient times x plus coefficient times y equals value". Equations in standard form can be solved for the intercepts by covering up the x (or y) term and solving the equation that remains.

Review Questions

1. Find the intercepts for the following equations using substitution.

- (a) $y = 3x - 6$
- (b) $y = -2x + 4$
- (c) $y = 14x - 21$
- (d) $y = 7 - 3x$

- (e) $y = 2.5x - 4$
 (f) $y = 1.1x + 2.2$
 (g) $y = \frac{3}{8}x + 7$
 (h) $y = \frac{5}{9} - \frac{2}{7}x$
2. Find the intercepts of the following equations using the cover-up method.
- (a) $5x - 6y = 15$
 (b) $3x - 4y = -5$
 (c) $2x + 7y = -11$
 (d) $5x + 10y = 25$
 (e) $5x - 1.3y = 12$
 (f) $1.4x - 3.5y = 7$
 (g) $\frac{3}{5}x + 2y = \frac{2}{5}$
 (h) $\frac{3}{4}x - \frac{2}{3}y = \frac{1}{5}$
3. Use any method to find the intercepts and then graph the following equations.
- (a) $y = 2x + 3$
 (b) $6(x - 1) = 2(y + 3)$
 (c) $x - y = 5$
 (d) $x + y = 8$
4. At the local grocery store strawberries cost \$3.00 per pound and bananas cost \$1.00 per pound.
- (a) If I have \$10 to spend on strawberries and bananas, draw a graph to show what combinations of each I can buy and spend exactly \$10.
 (b) Plot the point representing 3 pounds of strawberries and 2 pounds of bananas. Will that cost more or less than \$10?
 (c) Do the same for the point representing 1 pound of strawberries and 5 pounds of bananas.
5. A movie theater charges \$7.50 for adult tickets and \$4.50 for children. If the theater takes in \$900 in ticket sales for a particular screening, draw a graph which depicts the possibilities for the number of adult tickets and the number of child tickets sold.
6. Why can't we use the intercept method to graph the following equation? $3(x + 2) = 2(y + 3)$
7. Name two more equations that we can't use the intercept method to graph.

2.4 Slope and Rate of Change

Learning Objectives

- Find positive and negative slopes.
- Recognize and find slopes for horizontal and vertical lines.
- Understand rates of change.
- Interpret graphs and compare rates of change.

Introduction

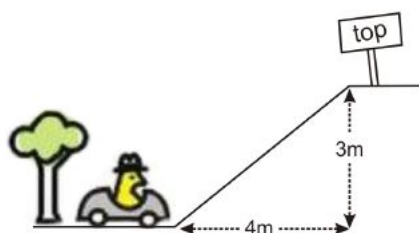
Wheelchair ramps at building entrances must have a slope between $\frac{1}{16}$ and $\frac{1}{20}$. If the entrance to a new office building is 28 inches off the ground, how long does the wheelchair ramp need to be?

We come across many examples of slope in everyday life. For example, a slope is in the pitch of a roof, the grade or incline of a road, or the slant of a ladder leaning on a wall. In math, we use the word **slope** to define steepness in a particular way.

$$\text{Slope} = \frac{\text{distance moved vertically}}{\text{distance moved horizontally}}$$

To make it easier to remember, we often word it like this:

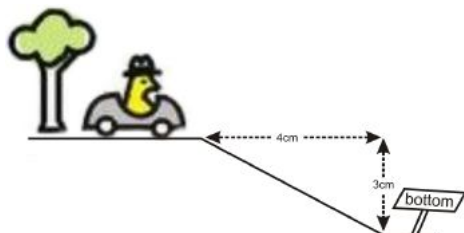
$$\text{Slope} = \frac{\text{rise}}{\text{run}}$$



In the picture above, the slope would be the ratio of the **height** of the hill to the horizontal **length** of the hill. In other words, it would be $\frac{3}{4}$, or 0.75.

If the car were driving to the **right** it would **climb** the hill - we say this is a positive slope. Any time you see the graph of a line that goes up as you move to the right, the slope is **positive**.

If the car kept driving after it reached the top of the hill, it might go down the other side. If the car is driving to the **right** and **descending**, then we would say that the slope is **negative**.



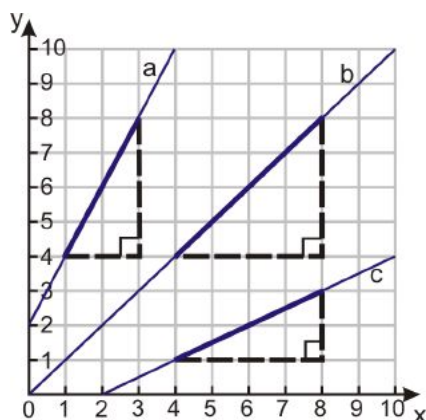
Here's where it gets tricky: If the car turned around instead and drove back down the left side of the hill, the slope of that side would still be positive. This is because the rise would be -3, but the run would be -4 (think of the x -axis - if you move from right to left you are moving in the **negative** x -direction). That means our slope ratio would be $\frac{-3}{-4}$, and the negatives cancel out to leave 0.75, the same slope as before. In other words, the slope of a line is the same no matter which direction you travel along it.

Find the Slope of a Line

A simple way to find a value for the slope of a line is to draw a right triangle whose hypotenuse runs along the line. Then we just need to measure the distances on the triangle that correspond to the rise (the vertical dimension) and the run (the horizontal dimension).

Example 1

Find the slopes for the three graphs shown.



Solution

There are already right triangles drawn for each of the lines - in future problems you'll do this part yourself. Note that it is easiest to make triangles whose vertices are **lattice points** (i.e. points whose coordinates are all integers).

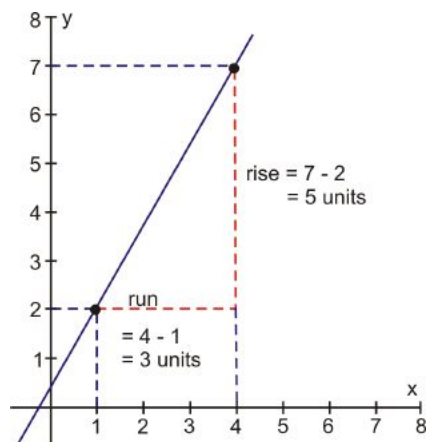
- a) The rise shown in this triangle is 4 units; the run is 2 units. The slope is $\frac{4}{2} = 2$.
- b) The rise shown in this triangle is 4 units, and the run is also 4 units. The slope is $\frac{4}{4} = 1$.
- c) The rise shown in this triangle is 2 units, and the run is 4 units. The slope is $\frac{2}{4} = \frac{1}{2}$.

Example 2

Find the slope of the line that passes through the points $(1, 2)$ and $(4, 7)$.

Solution

We already know how to graph a line if we're given two points: we simply plot the points and connect them with a line. Here's the graph:



Since we already have coordinates for the vertices of our right triangle, we can quickly work out that the rise is $7 - 2 = 5$ and the run is $4 - 1 = 3$ (see diagram). So the slope is $\frac{7-2}{4-1} = \frac{5}{3}$.

If you look again at the calculations for the slope, you'll notice that the 7 and 2 are the y -coordinates of the two points and the 4 and 1 are the x -coordinates. This suggests a pattern we can follow to get a general formula for the slope between two points (x_1, y_1) and (x_2, y_2) :

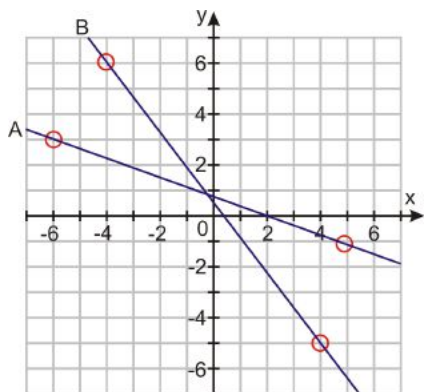
$$\text{Slope between } (x_1, y_1) \text{ and } (x_2, y_2) = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{or } m = \frac{\Delta y}{\Delta x}$$

In the second equation the letter m denotes the slope (this is a mathematical convention you'll see often) and the Greek letter delta (Δ) means **change**. So another way to express slope is *change in y* divided by *change in x*. In the next section, you'll see that it doesn't matter which point you choose as point 1 and which you choose as point 2.

Example 3

Find the slopes of the lines on the graph below.



Solution

Look at the lines - they both slant down (or decrease) as we move from left to right. Both these lines have **negative slope**.

The lines don't pass through very many convenient lattice points, but by looking carefully you can see a few points that look to have integer coordinates. These points have been circled on the graph, and we'll use them to determine the slope. We'll also do our calculations twice, to show that we get the same slope whichever way we choose point 1 and point 2.

For Line A:

$$\begin{aligned} (x_1, y_1) &= (-6, 3) & (x_2, y_2) &= (5, -1) \\ m &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-1) - (3)}{(5) - (-6)} = \frac{-4}{11} \approx -0.364 \end{aligned} \qquad \begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{(3) - (-1)}{(-6) - (5)} = \frac{4}{-11} \approx -0.364 \end{aligned}$$

For Line B

$$\begin{aligned} (x_1, y_1) &= (-4, 6) & (x_2, y_2) &= (4, -5) \\ m &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-5) - (6)}{(4) - (-4)} = \frac{-11}{8} = -1.375 \end{aligned} \qquad \begin{aligned} (x_1, y_1) &= (4, -5) & (x_2, y_2) &= (-4, 6) \\ m &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{(6) - (-5)}{(-4) - (4)} = \frac{11}{-8} = -1.375 \end{aligned}$$

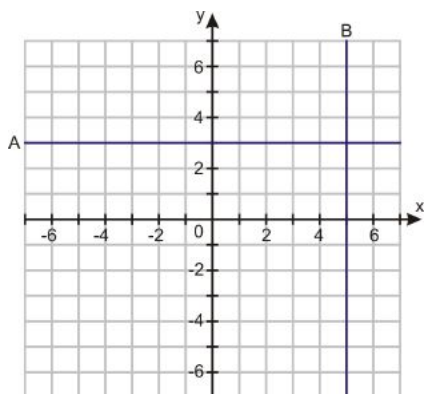
You can see that whichever way round you pick the points, the answers are the same. Either way, **Line A has slope -0.364, and Line B has slope -1.375.**

Khan Academy has a series of videos on finding the slope of a line, starting at <http://www.youtube.com/watch?v=hXP1>

Find the Slopes of Horizontal and Vertical lines

Example 4

Determine the slopes of the two lines on the graph below.



Solution

There are 2 lines on the graph: $A(y = 3)$ and $B(x = 5)$.

Let's pick 2 points on line A —say, $(x_1, y_1) = (-4, 3)$ and $(x_2, y_2) = (5, 3)$ —and use our equation for slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(3) - (3)}{(5) - (-4)} = \frac{0}{9} = 0.$$

If you think about it, this makes sense - if y doesn't change as x increases then there is no slope, or rather, the slope is zero. You can see that this must be true for all horizontal lines.

Horizontal lines ($y = \text{constant}$) all have a slope of 0.

Now let's consider line B . If we pick the points $(x_1, y_1) = (5, -3)$ and $(x_2, y_2) = (5, 4)$, our slope equation is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(4) - (-3)}{(5) - (5)} = \frac{7}{0}$. But dividing by zero isn't allowed!

In math we often say that a term which involves division by zero is **undefined**. (Technically, the answer can also be said to be infinitely large—or infinitely small, depending on the problem.)

Vertical lines ($x = \text{constant}$) all have an infinite (or undefined) slope.

Find a Rate of Change

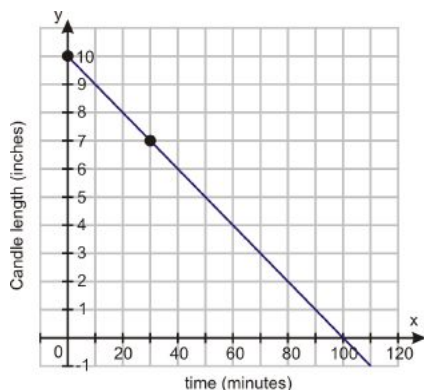
The slope of a function that describes real, measurable quantities is often called a **rate of change**. In that case the slope refers to a change in one quantity (y) **per** unit change in another quantity (x). (This is where the equation $m = \frac{\Delta y}{\Delta x}$ comes in—remember that Δy and Δx represent the change in y and x respectively.)

Example 5

A candle has a starting length of 10 inches. 30 minutes after lighting it, the length is 7 inches. Determine the rate of change in length of the candle as it burns. Determine how long the candle takes to completely burn to nothing.

Solution

First we'll graph the function to visualize what is happening. We have 2 points to start with: we know that at the moment the candle is lit ($\text{time} = 0$) the length of the candle is 10 inches, and after 30 minutes ($\text{time} = 30$) the length is 7 inches. Since the candle length depends on the time, we'll plot time on the horizontal axis, and candle length on the vertical axis.



The rate of change of the candle's length is simply the slope of the line. Since we have our 2 points $(x_1, y_1) = (0, 10)$ and $(x_2, y_2) = (30, 7)$, we can use the familiar version of the slope formula:

$$\text{Rate of change} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(7 \text{ inches}) - (10 \text{ inches})}{(30 \text{ minutes}) - (0 \text{ minutes})} = \frac{-3 \text{ inches}}{30 \text{ minutes}} = -0.1 \text{ inches per minute}$$

Note that the slope is negative. A negative rate of change means that the quantity is decreasing with time—just as we would expect the length of a burning candle to do.

To find the point when the candle reaches zero length, we can simply read the x -intercept off the graph (100 minutes). We can use the rate equation to verify this algebraically:

$$\begin{aligned} \text{Length burned} &= \text{rate} \times \text{time} \\ 10 &= 0.1 \times 100 \end{aligned}$$

Since the candle length was originally 10 inches, our equation confirms that 100 minutes is the time taken.

Example 6

The population of fish in a certain lake increased from 370 to 420 over the months of March and April. At what rate is the population increasing?

Solution

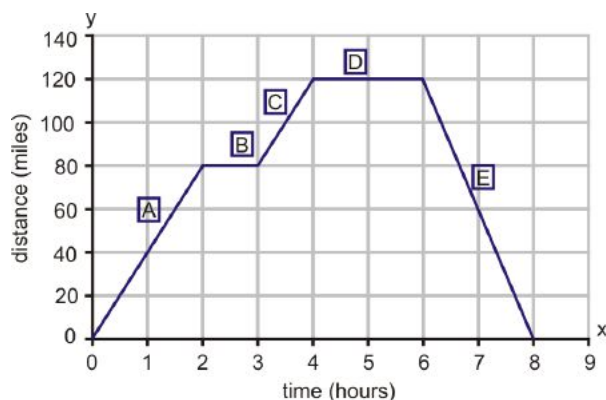
Here we don't have two points from which we can get x - and y -coordinates for the slope formula. Instead, we'll need to use the alternate formula, $m = \frac{\Delta y}{\Delta x}$.

The change in y -values, or Δy , is the change in the number of fish, which is $420 - 370 = 50$. The change in x -values, Δx , is the amount of time over which this change took place: two months. So $\frac{\Delta y}{\Delta x} = \frac{50 \text{ fish}}{2 \text{ months}}$, or **25 fish per month**.

Interpret a Graph to Compare Rates of Change

Example 7

The graph below represents a trip made by a large delivery truck on a particular day. During the day the truck made two deliveries, one taking an hour and the other taking two hours. Identify what is happening at each stage of the trip (stages A through E).



Solution

Here are the stages of the trip:

- a) The truck sets off and travels 80 miles in 2 hours.
- b) The truck covers no distance for 2 hours.
- c) The truck covers $(120 - 80) = 40$ miles in 1 hour.
- d) The truck covers no distance for 1 hour.
- e) The truck covers -120 miles in 2 hours.

Let's look at each section more closely.

A. Rate of change = $\frac{\Delta y}{\Delta x} = \frac{80 \text{ miles}}{2 \text{ hours}} = 40$ miles per hour

Notice that the rate of change is a **speed**—or rather, a **velocity**. (The difference between the two is that velocity has a direction, and speed does not. In other words, velocity can be either positive or negative, with negative velocity representing travel in the opposite direction. You'll see the difference more clearly in part E.)

Since velocity equals distance divided by time, the slope (or rate of change) of a distance-time graph is always a velocity.

So during the first part of the trip, the truck travels at a constant speed of 40 mph for 2 hours, covering a distance of 80 miles.

B. The slope here is 0, so the rate of change is 0 mph. The truck is stationary for one hour. This is the first delivery stop.

C. Rate of change = $\frac{\Delta y}{\Delta x} = \frac{(120-80) \text{ miles}}{(4-3) \text{ hours}} = 40$ miles per hour. The truck is traveling at 40 mph.

D. Once again the slope is 0, so the rate of change is 0 mph. The truck is stationary for two hours. This is the second delivery stop. At this point the truck is 120 miles from the start position.

E. Rate of change = $\frac{\Delta y}{\Delta x} = \frac{(0-120) \text{ miles}}{(8-6) \text{ hours}} = \frac{-120 \text{ miles}}{2 \text{ hours}} = -60$ miles per hour. The truck is traveling at *negative* 60 mph.

Wait – a negative speed? Does that mean that the truck is reversing? Well, probably not. It's actually the *velocity* and not the speed that is negative, and a negative velocity simply means that the distance *from the starting position* is decreasing with time. The truck is driving in the opposite direction – back to where it started from. Since it no longer has 2 heavy loads, it travels faster (60 mph instead of 40 mph), covering the 120 mile return trip in 2 hours. Its *speed* is 60 mph, and its *velocity* is -60 mph, because it is traveling in the opposite direction from when it started out.

Lesson Summary

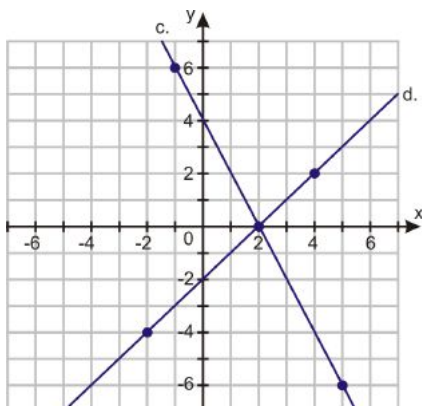
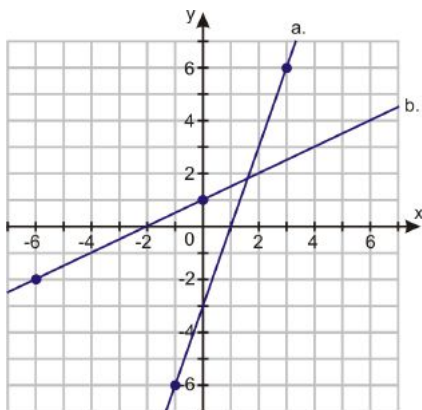
- **Slope** is a measure of change in the vertical direction for each step in the horizontal direction. Slope is often represented as “ m ”.
- Slope can be expressed as $\frac{\text{rise}}{\text{run}}$, or $\frac{\Delta y}{\Delta x}$.
- The slope between two points (x_1, y_1) and (x_2, y_2) is equal to $\frac{y_2 - y_1}{x_2 - x_1}$.
- **Horizontal lines** (where $y = a$ constant) all have a slope of 0.
- **Vertical lines** (where $x = a$ constant) all have an infinite (or undefined) slope.
- The slope (or **rate of change**) of a distance-time graph is a **velocity**.

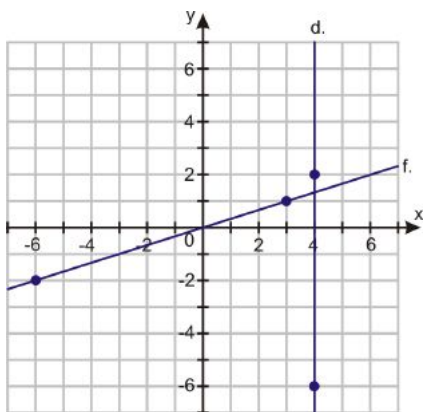
Review Questions

1. Use the slope formula to find the slope of the line that passes through each pair of points.

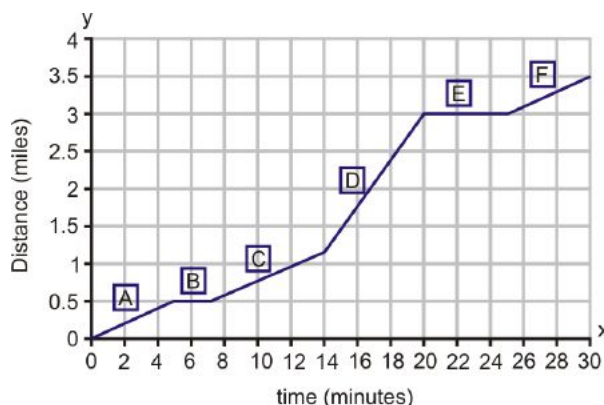
- (a) $(-5, 7)$ and $(0, 0)$
- (b) $(-3, -5)$ and $(3, 11)$
- (c) $(3, -5)$ and $(-2, 9)$
- (d) $(-5, 7)$ and $(-5, 11)$
- (e) $(9, 9)$ and $(-9, -9)$
- (f) $(3, 5)$ and $(-2, 7)$
- (g) $(2.5, 3)$ and $(8, 3.5)$

2. For each line in the graphs below, use the points indicated to determine the slope.





- For each line in the graphs above, imagine another line with the same slope that passes through the point $(1, 1)$, and name one more point on that line.
- The graph below is a distance-time graph for Mark's three and a half mile cycle ride to school. During this ride, he rode on cycle paths but the terrain was hilly. He rode slower up hills and faster down them. He stopped once at a traffic light and at one point he stopped to mend a punctured tire. The graph shows his distance from home at any given time. Identify each section of the graph accordingly.



2.5 Graphs Using Slope-Intercept Form

Learning Objectives

- Identify the slope and y -intercept of equations and graphs.
- Graph an equation in slope-intercept form.
- Understand what happens when you change the slope or intercept of a line.
- Identify parallel lines from their equations.

Introduction

The total profit of a business is described by the equation $y = 15000x - 80000$, where x is the number of months the business has been running. How much profit is the business making per month, and what were its start-up costs? How much profit will it have made in a year?

Identify Slope and y-intercept

So far, we've been writing a lot of our equations in **slope-intercept form**—that is, we've been writing them in the form $y = mx + b$, where m and b are both constants. It just so happens that m is the slope and the point $(0, b)$ is the y-intercept of the graph of the equation, which gives us enough information to draw the graph quickly.

Example 1

Identify the slope and y-intercept of the following equations.

a) $y = 3x + 2$

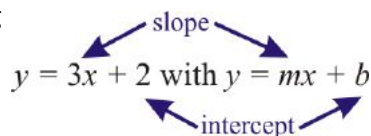
b) $y = 0.5x - 3$

c) $y = -7x$

d) $y = -4$

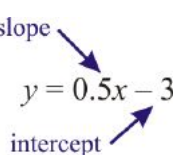
Solution

a) Comparing $y = 3x + 2$ with $y = mx + b$, we can see that $m = 3$ and $b = 2$. So $y = 3x + 2$ has a **slope**



of **3** and a y-intercept of **(0, 2)**.

b) $y = 0.5x - 3$ has a **slope of 0.5** and a y-intercept of **(0, -3)**.



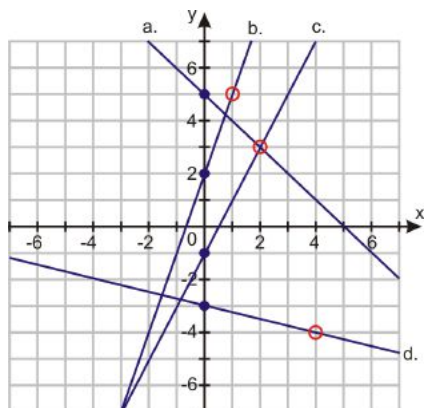
Notice that the intercept is **negative**. The b -term includes the sign of the operator (plus or minus) in front of the number—for example, $y = 0.5x - 3$ is identical to $y = 0.5x + (-3)$, and that means that b is -3 , not just 3.

c) At first glance, this equation doesn't look like it's in slope-intercept form. But we can rewrite it as $y = -7x + 0$, and that means it has a **slope of -7** and a y-intercept of **(0, 0)**. Notice that the slope is negative and the line passes through the origin.

d) We can rewrite this one as $y = 0x - 4$, giving us a **slope of 0** and a y-intercept of **(0, -4)**. This is a horizontal line.

Example 2

Identify the slope and y-intercept of the lines on the graph shown below.



The intercepts have been marked, as well as some convenient lattice points that the lines pass through.

Solution

- a) **The y-intercept is (0, 5).** The line also passes through (2, 3), so the slope is $\frac{\Delta y}{\Delta x} = \frac{-2}{2} = -1$.
- b) **The y-intercept is (0, 2).** The line also passes through (1, 5), so the slope is $\frac{\Delta y}{\Delta x} = \frac{3}{1} = 3$.
- c) **The y-intercept is (0, -1).** The line also passes through (2, 3), so the slope is $\frac{\Delta y}{\Delta x} = \frac{4}{2} = 2$.
- d) **The y-intercept is (0, -3).** The line also passes through (4, -4), so the slope is $\frac{\Delta y}{\Delta x} = \frac{-1}{4} = -\frac{1}{4}$ or -0.25.

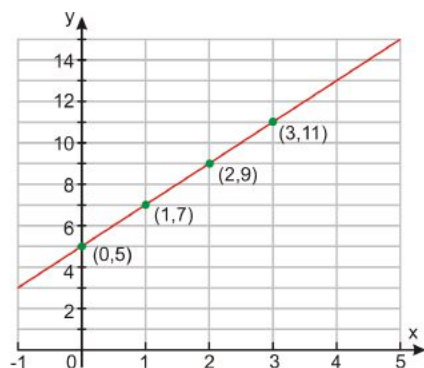
Graph an Equation in Slope-Intercept Form

Once we know the slope and intercept of a line, it's easy to graph it. Just remember what slope means. Let's look back at this example from Lesson 4.1.

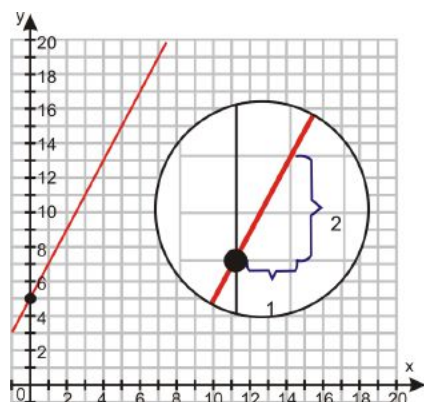
Ali is trying to work out a trick that his friend showed him. His friend started by asking him to think of a number, then double it, then add five to the result. Ali has written down a rule to describe the first part of the trick. He is using the letter x to stand for the number he thought of and the letter y to represent the final result of applying the rule. He wrote his rule in the form of an equation: $y = 2x + 5$.

Help him visualize what is going on by graphing the function that this rule describes.

In that example, we constructed a table of values, and used that table to plot some points to create our graph.



We also saw another way to graph this equation. Just by looking at the equation, we could see that the y-intercept was (0, 5), so we could start by plotting that point. Then we could also see that the slope was 2, so we could find another point on the graph by going over 1 unit and up 2 units. The graph would then be the line between those two points.



Here's another problem where we can use the same method.

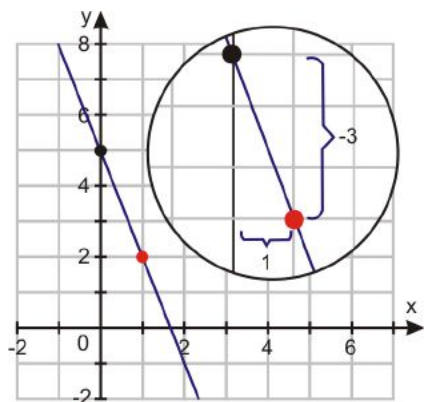
Example 3

Graph the following function: $y = -3x + 5$

Solution

To graph the function without making a table, follow these steps:

1. Identify the y-intercept: $b = 5$
2. Plot the intercept: $(0, 5)$
3. Identify the slope: $m = -3$. (This is equal to $\frac{-3}{1}$, so the **rise** is -3 and the **run** is 1.)
4. Move **over** 1 unit and **down** 3 units to find another point on the line: $(1, 2)$
5. Draw the line through the points $(0, 5)$ and $(1, 2)$.



Notice that to graph this equation based on its slope, we had to find the rise and run—and it was easiest to do that when the slope was expressed as a fraction. That's true in general: to graph a line with a particular slope, it's easiest to first express the slope as a fraction in simplest form, and then read off the numerator and the denominator of the fraction to get the rise and run of the graph.

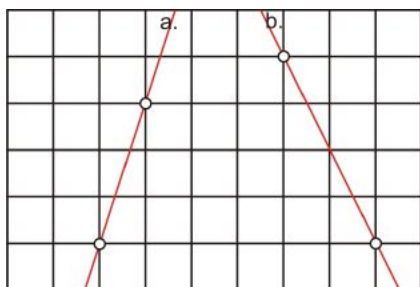
Example 4

Find integer values for the **rise** and **run** of the following slopes, then graph lines with corresponding slopes.

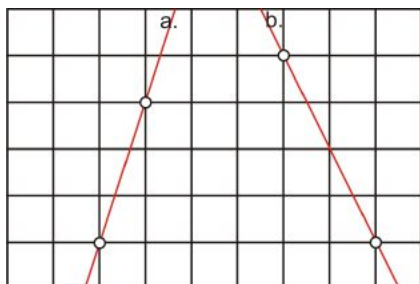
- a) $m = 3$
- b) $m = -2$
- c) $m = 0.75$
- d) $m = -0.375$

Solution

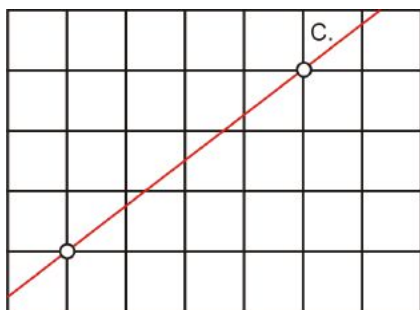
- a)
- $3 = \frac{3}{1}$ As we move **across** 1 unit we move **up** by 3



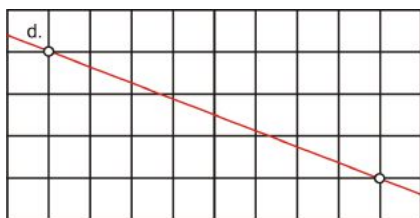
b) $-2 = \frac{-2}{1}$ As we move **across** 1 unit we move **down** by 2



c) $0.75 = \frac{3}{4}$ As we move **across** 4 units we move **up** by 3

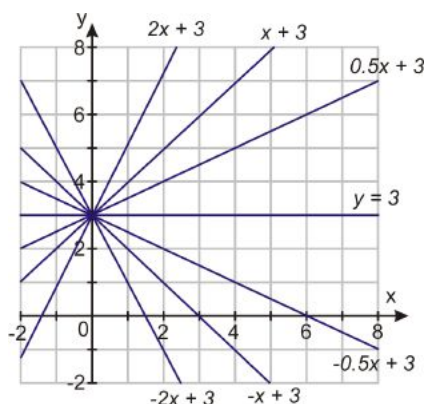


d) $-0.375 = \frac{-3}{8}$ As we move **across** 8 units we move **down** by 3



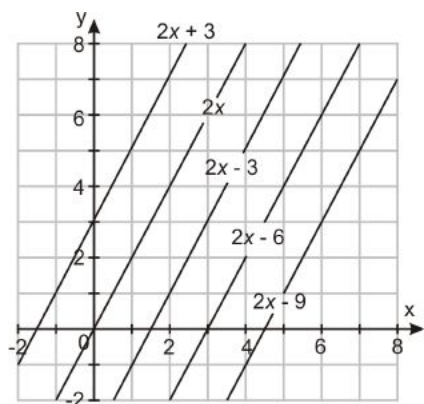
Changing the Slope or Intercept of a Line

The following graph shows a number of lines with different slopes, but all with the same y-intercept: (0, 3).



You can see that all the functions with positive slopes increase as we move from left to right, while all functions with negative slopes decrease as we move from left to right. Another thing to notice is that the greater the slope, the steeper the graph.

This graph shows a number of lines with the same slope, but different y-intercepts.



Notice that changing the intercept simply translates (shifts) the graph up or down. Take a point on the graph of $y = 2x$, such as $(1, 2)$. The corresponding point on $y = 2x + 3$ would be $(1, 5)$. Adding 3 to the y-intercept means we also add 3 to every other y-value on the graph. Similarly, the corresponding point on the $y = 2x - 3$ line would be $(1, -1)$; we would subtract 3 from the y-value and from every other y-value.

Notice also that these lines all appear to be parallel. Are they truly parallel?

To answer that question, we'll use a technique that you'll learn more about in a later chapter. We'll take 2 of the equations—say, $y = 2x$ and $y = 2x + 3$ —and solve for values of x and y that satisfy both equations. That will tell us at what point those two lines intersect, if any. (Remember that **parallel lines**, by definition, are lines that don't intersect.)

So what values would satisfy both $y = 2x$ and $y = 2x + 3$? Well, if both of those equations were true, then y would be equal to both $2x$ and $2x + 3$, which means those two expressions would also be equal to each other. So we can get our answer by solving the equation $2x = 2x + 3$.

But what happens when we try to solve that equation? If we subtract $2x$ from both sides, we end up with $0 = 3$. That can't be true no matter what x equals. And that means that there just isn't any value for x that will make both of the equations we started out with true. In other words, there isn't any point where those two lines intersect. They are parallel, just as we thought.

And we'd find out the same thing no matter which two lines we'd chosen. In general, since changing the intercept of a line just results in shifting the graph up or down, the new line will always be parallel to the old line as long as the slope stays the same.

Any two lines with identical slopes are parallel.

Further Practice

To get a better understanding of what happens when you change the slope or the y-intercept of a linear equation, try playing with the Java applet at <http://standards.nctm.org/document/eexamples/chap7/7.5/index.htm>.

Lesson Summary

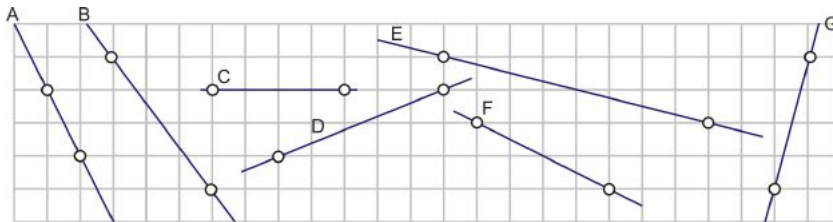
- A common form of a line (linear equation) is **slope-intercept form**: $y = mx + b$, where m is the slope and the point $(0, b)$ is the y-intercept
- Graphing a line in slope-intercept form is a matter of first plotting the y-intercept $(0, b)$, then finding a second point based on the slope, and using those two points to graph the line.
- Any two lines with identical slopes are **parallel**.

Review Questions

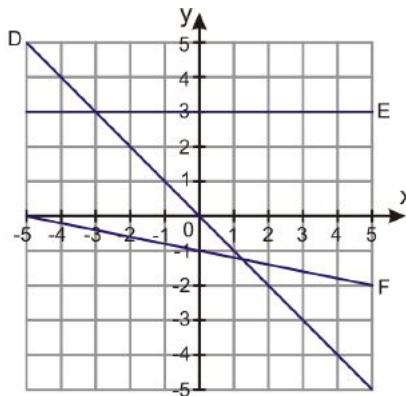
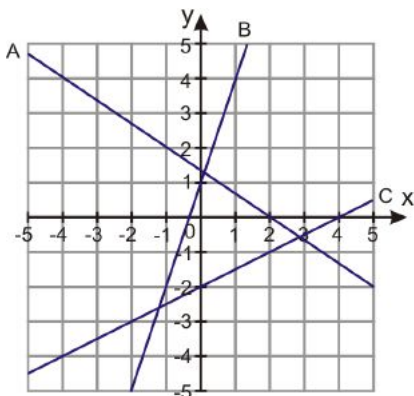
1. Identify the slope and y-intercept for the following equations.

- (a) $y = 2x + 5$
- (b) $y = -0.2x + 7$
- (c) $y = x$
- (d) $y = 3.75$

2. Identify the slope of the following lines.



3. Identify the slope and y-intercept for the following functions.



4. Plot the following functions on a graph.

- (a) $y = 2x + 5$
- (b) $y = -0.2x + 7$
- (c) $y = x$
- (d) $y = 3.75$

5. Which two of the following lines are parallel?
- (a) $y = 2x + 5$
 - (b) $y = -0.2x + 7$
 - (c) $y = x$
 - (d) $y = 3.75$
 - (e) $y = -\frac{1}{5}x - 11$
 - (f) $y = -5x + 5$
 - (g) $y = -3x + 11$
 - (h) $y = 3x + 3.5$
6. What is the y -intercept of the line passing through $(1, -4)$ and $(3, 2)$?
7. What is the y -intercept of the line with slope -2 that passes through $(3, 1)$?
8. Line A passes through the points $(2, 6)$ and $(-4, 3)$. Line B passes through the point $(3, 2.5)$, and is parallel to line A
- (a) Write an equation for line A in slope-intercept form.
 - (b) Write an equation for line B in slope-intercept form.
9. Line C passes through the points $(2, 5)$ and $(1, 3.5)$. Line D is parallel to line C , and passes through the point $(2, 6)$. Name another point on line D . (Hint: you can do this without graphing or finding an equation for either line.)

2.6 Direct Variation Models

Learning Objectives

- Identify direct variation.
- Graph direct variation equations.
- Solve real-world problems using direct variation models.

Introduction

Suppose you see someone buy five pounds of strawberries at the grocery store. The clerk weighs the strawberries and charges \$12.50 for them. Now suppose you wanted two pounds of strawberries for yourself. How much would you expect to pay for them?

Identify Direct Variation

The preceding problem is an example of a **direct variation**. We would expect that the strawberries are priced on a “per pound” basis, and that if you buy two-fifths the amount of strawberries, you would pay two-fifths of \$12.50 for your strawberries, or \$5.00.

Similarly, if you bought 10 pounds of strawberries (twice the amount) you would pay twice \$12.50, and if you did not buy any strawberries you would pay nothing.

If variable y varies directly with variable x , then we write the relationship as $y = k \cdot x$. k is called the **constant of proportionality**.

If we were to graph this function, you can see that it would pass through the origin, because $y = 0$ when $x = 0$, whatever the value of k . So we know that a direct variation, when graphed, has a single intercept at $(0, 0)$.

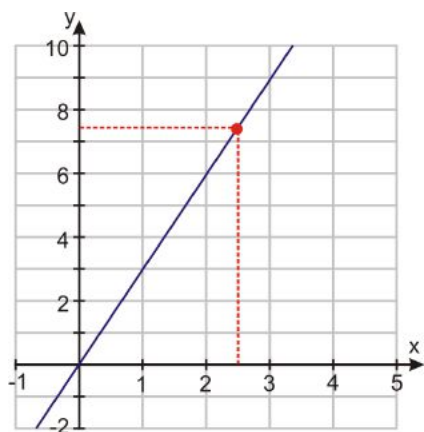
Example 1

If y varies directly with x according to the relationship $y = k \cdot x$, and $y = 7.5$ when $x = 2.5$, determine the constant of proportionality, k .

Solution

We can solve for the constant of proportionality using substitution. Substitute $x = 2.5$ and $y = 7.5$ into the equation $y = k \cdot x$ to get $7.5 = k(2.5)$. Then divide both sides by 2.5 to get $k = \frac{7.5}{2.5} = 3$. **The constant of proportionality, k , is 3.**

We can graph the relationship quickly, using the intercept $(0, 0)$ and the point $(2.5, 7.5)$. The graph is shown below. It is a straight line with slope 3.



The graph of a direct variation always passes through the origin, and always has a slope that is equal to the constant of proportionality, k .

Example 2

The volume of water in a fish-tank, V , varies directly with depth, d . If there are 15 gallons in the tank when the depth is 8 inches, calculate how much water is in the tank when the depth is 20 inches.

Solution

This is a good example of a direct variation, but for this problem we'll have to determine the equation of the variation ourselves. Since the volume, V , depends on depth, d , we'll use an equation of the form $y = k \cdot x$, but in place of y we'll use V and in place of x we'll use d :

$$V = k \cdot d$$

We know that when the depth is 8 inches the volume is 15 gallons, so to solve for k , we plug in 15 for V and 8 for d to get $15 = k(8)$. Dividing by 8 gives us $k = \frac{15}{8} = 1.875$.

Now to find the volume of water at the final depth, we use $V = k \cdot d$ again, but this time we can plug in our new d and the value we found for k :

$$V = 1.875 \times 20$$

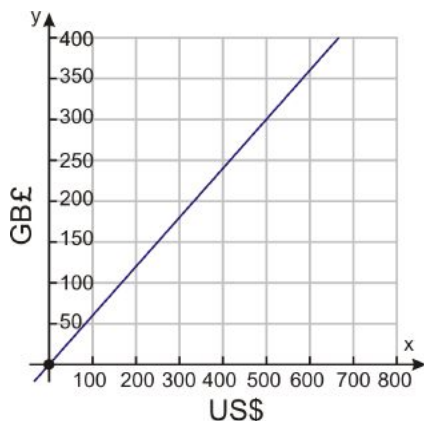
$$V = 37.5$$

At a depth of 20 inches, the volume of water in the tank is 37.5 gallons.

Example 3

The graph shown below is a conversion chart used to convert U.S. dollars (US\$) to British pounds (GB£) in a bank on a particular day. Use the chart to determine:

- a) the number of pounds you could buy for \$600
- b) the number of dollars it would cost to buy £200
- c) the exchange rate in pounds per dollar



Solution

We can read the answers to a) and b) right off the graph. It looks as if at $x = 600$ the graph is about one fifth of the way between £350 and £400. So \$600 would buy £360.

Similarly, the line $y = 200$ appears to intersect the graph about a third of the way between \$300 and \$400. We can round this to \$330, so it would cost approximately \$330 to buy £200.

To solve for the exchange rate, we should note that as this is a direct variation - the graph is a straight line passing through the origin. The slope of the line gives the constant of proportionality (in this case the **exchange rate**) and it is equal to the ratio of the y -value to x -value at any point. Looking closely at the graph, we can see that the line passes through one convenient lattice point: (500, 300). This will give us the most accurate value for the slope and so the exchange rate.

$$y = k \cdot x \Rightarrow \frac{y}{x} \quad \text{And so rate} = \frac{300 \text{ pounds}}{500 \text{ dollars}} = 0.60 \text{ pounds per dollar}$$

Graph Direct Variation Equations

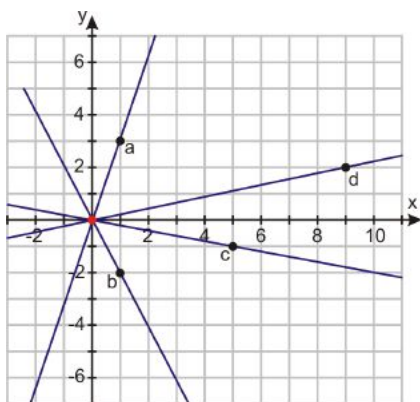
We know that all direct variation graphs pass through the origin, and also that the slope of the line is equal to the constant of proportionality, k . Graphing is a simple matter of using the point-slope or point-point methods discussed earlier in this chapter.

Example 4

Plot the following direct variations on the same graph.

- a) $y = 3x$
- b) $y = -2x$
- c) $y = -0.2x$
- d) $y = \frac{2}{9}x$

Solution



- a) The line passes through $(0, 0)$, as will all these functions. This function has a slope of 3. When we move across by one unit, the function increases by three units.
- b) The line has a slope of -2. When we move across the graph by one unit, the function **falls** by two units.
- c) The line has a slope of -0.2. As a fraction this is equal to $-\frac{1}{5}$. When we move across by five units, the function **falls** by one unit.
- d) The line passes through $(0, 0)$ and has a slope of $\frac{2}{9}$. When we move across the graph by 9 units, the function increases by two units.

For more examples of how to plot and identify direct variation functions, see the video at <http://neaportal.k12.ar.us/indirect-variation-and-direct-variation/>.

Solve Real-World Problems Using Direct Variation Models

Direct variations are seen everywhere in everyday life. Any time one quantity increases at the same rate another quantity increases (for example, doubling when it doubles and tripling when it triples), we say that they follow a direct variation.

Newton's Second Law

In 1687 Sir Isaac Newton published the famous *Principia Mathematica*. It contained, among other things, his second law of motion. This law is often written as $F = m \cdot a$, where a force of F Newtons applied to a mass of m kilograms results in acceleration of a meters per second². Notice that if the mass stays constant, then this formula is basically the same as the direct variation equation, just with different variables—and m is the constant of proportionality.

Example 5

If a 175 Newton force causes a shopping cart to accelerate down the aisle with an acceleration of 2.5 m/s^2 , calculate:

- a) The mass of the shopping cart.
- b) The force needed to accelerate the same cart at 6 m/s^2 .

Solution

a) We can solve for m (the mass) by plugging in our given values for force and acceleration. $F = m \cdot a$ becomes $175 = m(2.5)$, and then we divide both sides by 2.5 to get $70 = m$.

So the mass of the shopping cart is 70 kg.

b) Once we have solved for the mass, we simply substitute that value, plus our required acceleration, back into the formula $F = m \cdot a$ and solve for F . We get $F = 70 \times 6 = 420$.

So the force needed to accelerate the cart at 6 m/s^2 is 420 Newtons.

Ohm's Law

The electrical current, I (amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship $V = I \cdot R$, where R is the resistance (measured in Ohms). The resistance is considered to be a constant for all values of V and I , so once again, this formula is a version of the direct variation formula, with R as the constant of proportionality.

Example 6

A certain electronics component was found to pass a current of 1.3 amps at a voltage of 2.6 volts. When the voltage was increased to 12.0 volts the current was found to be 6.0 amps.

- a) *Does the component obey Ohm's law?*
- b) *What would the current be at 6 volts?*

Solution

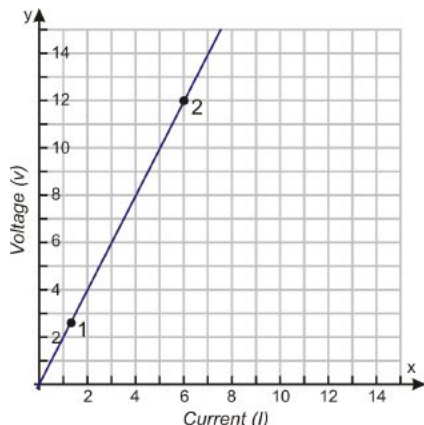
Ohm's law is a simple direct proportionality law, with the resistance R as our constant of proportionality. To know if this component obeys Ohm's law, we need to know if it follows a direct proportionality rule. In other words, **is V directly proportional to I ?**

We can determine this in two different ways.

Graph It: If we plot our two points on a graph and join them with a line, does the line pass through (0, 0)?

Voltage is the independent variable and current is the dependent variable, so normally we would graph V on the horizontal axis and I on the vertical axis. However, if we swap the variables around just this once, we'll get a graph whose slope conveniently happens to be equal to the resistance, R . So we'll treat I as the independent variable, and our two points will be (1.3, 2.6) and (6, 12).

Plotting those points and joining them gives the following graph:



The graph does appear to pass through the origin, so **yes, the component obeys Ohm's law.**

Solve for R : If this component does obey Ohm's law, the constant of proportionality (R) should be the same when we plug in the second set of values as when we plug in the first set. Let's see if it is. (We can quickly find the value of R in each case; since $V = I \cdot R$, that means $R = \frac{V}{I}$.)

$$\text{Case 1: } R = \frac{V}{I} = \frac{2.6}{1.3} = 2 \text{ Ohms}$$

$$\text{Case 2: } R = \frac{V}{I} = \frac{12}{6} = 2 \text{ Ohms}$$

The values for R agree! This means that we are indeed looking at a direct variation. **The component obeys Ohm's law.**

b) Now to find the current at 6 volts, simply substitute the values for V and R into $V = I \cdot R$. We found that $R = 2$, so we plug in 2 for R and 6 for V to get $6 = I(2)$, and divide both sides by 2 to get $3 = I$.

So the current through the component at a voltage of 6 volts is 3 amps.

Lesson Summary

- If a variable y varies *directly* with variable x , then we write the relationship as $y = k \cdot x$, where k is a constant called the **constant of proportionality**.
- **Direct variation** is very common in many areas of science.

Review Questions

- Plot the following direct variations on the same graph.
 - $y = \frac{4}{3}x$
 - $y = -\frac{2}{3}x$
 - $y = -\frac{1}{6}x$
 - $y = 1.75x$
- Dasan's mom takes him to the video arcade for his birthday.
 - In the first 10 minutes, he spends \$3.50 playing games. If his allowance for the day is \$20, how long can he keep playing games before his money is gone?
 - He spends the next 15 minutes playing Alien Invaders. In the first two minutes, he shoots 130 aliens. If he keeps going at this rate, how many aliens will he shoot in fifteen minutes?
 - The high score on this machine is 120000 points. If each alien is worth 100 points, will Dasan beat the high score? What if he keeps playing for five more minutes?
- The current standard for low-flow showerheads is 2.5 gallons per minute.
 - How long would it take to fill a 30-gallon bathtub using such a showerhead to supply the water?
 - If the bathtub drain were not plugged all the way, so that every minute 0.5 gallons ran out as 2.5 gallons ran in, how long would it take to fill the tub?
 - After the tub was full and the showerhead was turned off, how long would it take the tub to empty through the partly unplugged drain?
 - If the drain were immediately unplugged all the way when the showerhead was turned off, so that it drained at a rate of 1.5 gallons per minute, how long would it take to empty?
- Amin is using a hose to fill his new swimming pool for the first time. He starts the hose at 10 PM and leaves it running all night.
 - At 6 AM he measures the depth and calculates that the pool is four sevenths full. At what time will his new pool be full?
 - At 10 AM he measures again and realizes his earlier calculations were wrong. The pool is still only three quarters full. When will it actually be full?

- (c) After filling the pool, he needs to chlorinate it to a level of 2.0 ppm (parts per million). He adds two gallons of chlorine solution and finds that the chlorine level is now 0.7 ppm. How many more gallons does he need to add?
 - (d) If the chlorine level in the pool decreases by 0.05 ppm per day, how much solution will he need to add each week?
5. Land in Wisconsin is for sale to property investors. A 232-acre lot is listed for sale for \$200,500.
 - (a) Assuming the same price per acre, how much would a 60-acre lot sell for?
 - (b) Again assuming the same price, what size lot could you purchase for \$100,000?
6. The force (F) needed to stretch a spring by a distance x is given by the equation $F = k \cdot x$, where k is the spring constant (measured in Newtons per centimeter, or N/cm). If a 12 Newton force stretches a certain spring by 10 cm, calculate:
 - (a) The spring constant, k
 - (b) The force needed to stretch the spring by 7 cm.
 - (c) The distance the spring would stretch with a 23 Newton force.
7. Angela's cell phone is completely out of power when she puts it on the charger at 3 PM. An hour later, it is 30% charged. When will it be completely charged?
8. It costs \$100 to rent a recreation hall for three hours and \$150 to rent it for five hours.
 - (a) Is this a direct variation?
 - (b) Based on the cost to rent the hall for three hours, what would it cost to rent it for six hours, assuming it is a direct variation?
 - (c) Based on the cost to rent the hall for five hours, what would it cost to rent it for six hours, assuming it is a direct variation?
 - (d) Plot the costs given for three and five hours and graph the line through those points. Based on that graph, what would you expect the cost to be for a six-hour rental?

2.7 Linear Function Graphs

Learning Objectives

- Recognize and use function notation.
- Graph a linear function.
- Analyze arithmetic progressions.

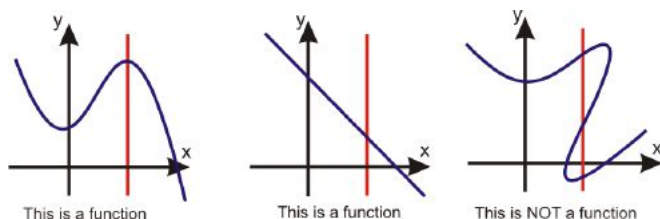
Introduction

The highly exclusive Fellowship of the Green Mantle allows in only a limited number of new members a year. In its third year of membership it has 28 members, in its fourth year it has 33, and in its fifth year it has 38. How many members are admitted a year, and how many founding members were there?

Functions

So far we've used the term **function** to describe many of the equations we've been graphing, but in mathematics it's important to remember that not all equations are functions. In order to be a function, a relationship between two variables, x and y , must map each x -value to **exactly one** y -value.

Visually this means the graph of y versus x must pass the **vertical line test**, meaning that a vertical line drawn through the graph of the function must never intersect the graph in more than one place:



Use Function Notation

When we write functions we often use the notation “ $f(x)$ =” in place of “ y =”. $f(x)$ is pronounced “ f of x ”.

Example 1

Rewrite the following equations so that y is a function of x and is written $f(x)$:

- a) $y = 2x + 5$
- b) $y = -0.2x + 7$
- c) $x = 4y - 5$
- d) $9x + 3y = 6$

Solution

- a) Simply replace y with $f(x)$: $f(x) = 2x + 5$
- b) Again, replace y with $f(x)$: $f(x) = -0.2x + 7$
- c) First we need to solve for y . Starting with $x = 4y - 5$, we add 5 to both sides to get $x + 5 = 4y$, divide by 4 to get $\frac{x+5}{4} = y$, and then replace y with $f(x)$: $f(x) = \frac{x+5}{4}$.
- d) Solve for y : take $9x + 3y = 6$, subtract $9x$ from both sides to get $3y = 6 - 9x$, divide by 3 to get $y = \frac{6-9x}{3} = 2 - 3x$, and express as a function: $f(x) = 2 - 3x$.

Using the functional notation in an equation gives us more information. For instance, the expression $f(x) = mx + b$ shows clearly that x is the independent variable because you **plug in** values of x into the function and perform a series of operations on the value of x in order to calculate the values of the dependent variable, y .

We can also plug in expressions rather than just numbers. For example, if our function is $f(x) = x + 2$, we can plug in the expression $(x + 5)$. We would express this as $f(x + 5) = (x + 5) + 2 = x + 7$.

Example 2

A function is defined as $f(x) = 6x - 36$. Evaluate the following:

- a) $f(2)$
- b) $f(0)$
- c) $f(z)$
- d) $f(x + 3)$
- e) $f(2r - 1)$

Solution

- a) Substitute $x = 2$ into the function $f(x)$: $f(2) = 6 \cdot 2 - 36 = 12 - 36 = -24$
- b) Substitute $x = 0$ into the function $f(x)$: $f(0) = 6 \cdot 0 - 36 = 0 - 36 = -36$

c) Substitute $x = z$ into the function $f(x) : f(z) = 6z + 36$

d) Substitute $x = (x + 3)$ into the function $f(x) : f(x + 3) = 6(x + 3) + 36 = 6x + 18 + 36 = 6x + 54$

e) Substitute $x = (2r + 1)$ into the function $f(x) : f(2r + 1) = 6(2r + 1) + 36 = 12r + 6 + 36 = 12r + 42$

Graph a Linear Function

Since the notations “ $f(x) =$ ” and “ $y =$ ” are interchangeable, we can use all the concepts we have learned so far to graph functions.

Example 3

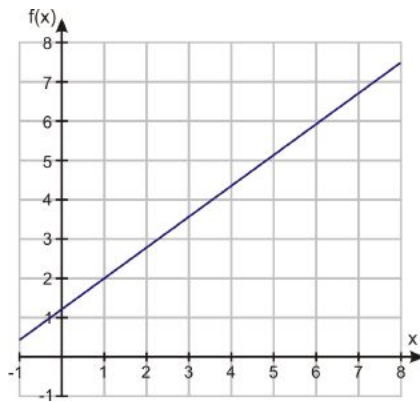
Graph the function $f(x) = \frac{3x+5}{4}$.

Solution

We can write this function in **slope-intercept** form:

$$f(x) = \frac{3}{4}x + \frac{5}{4} = 0.75x + 1.25$$

So our graph will have a y -intercept of $(0, 1.25)$ and a slope of 0.75.



Example 4

Graph the function $f(x) = \frac{7(5-x)}{5}$.

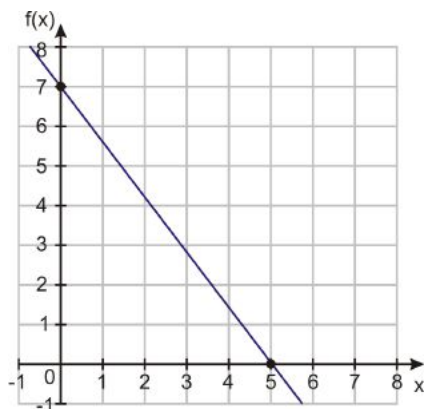
Solution

This time we'll solve for the x - and y -intercepts.

To solve for the y -intercept, plug in $x = 0$: $f(0) = \frac{7(5-0)}{5} = \frac{35}{5} = 7$, so the x -intercept is **(0, 7)**.

To solve for the x -intercept, set $f(x) = 0$: $0 = \frac{7(5-x)}{5}$, so $0 = 35 - 7x$, therefore $7x = 35$ and $x = 5$. The y -intercept is **(5, 0)**.

We can graph the function from those two points:



Arithmetic Progressions

You may have noticed that with linear functions, when you increase the x -value by 1 unit, the y -value increases by a fixed amount, equal to the slope. For example, if we were to make a table of values for the function $f(x) = 2x + 3$, we might start at $x = 0$ and then add 1 to x for each row:

Table 2.6:

| x | $f(x)$ |
|-----|--------|
| 0 | 3 |
| 1 | 5 |
| 2 | 7 |
| 3 | 9 |
| 4 | 11 |

Notice that the values for $f(x)$ go up by 2 (the slope) each time. When we repeatedly add a fixed value to a starting number, we get a sequence like $\{3, 5, 7, 9, 11, \dots\}$. We call this an **arithmetic progression**, and it is characterized by the fact that each number is bigger (or smaller) than the preceding number by a fixed amount. This amount is called the **common difference**. We can find the common difference for a given sequence by taking 2 consecutive terms in the sequence and subtracting the first from the second.

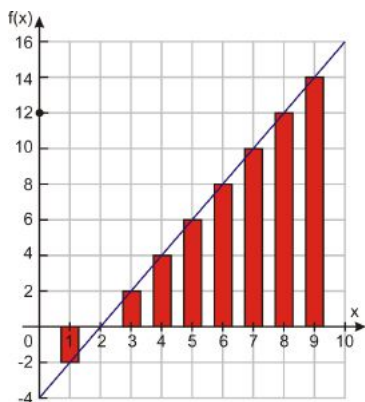
Example 5

Find the common difference for the following arithmetic progressions:

- a) $\{7, 11, 15, 19, \dots\}$
- b) $\{12, 1, -10, -21, \dots\}$
- c) $\{7, _, 12, _, 17, \dots\}$

Solution

- a) $11 - 7 = 4$; $15 - 11 = 4$; $19 - 15 = 4$. The common difference is **4**.
- b) $1 - 12 = -11$. The common difference is **-11**.
- c) There are not 2 consecutive terms here, but we know that to get the term after 7 we would add the common difference, and then to get to 12 we would add the common difference again. So *twice* the common difference is $12 - 7 = 5$, and so the common difference is **2.5**.



Arithmetic sequences and linear functions are very closely related. To get to the next term in an arithmetic sequence, you add the *common difference* to the last term; similarly, when the x -value of a linear function increases by one, the y -value increases by the amount of the *slope*. So arithmetic sequences are very much like linear functions, with the common difference playing the same role as the slope.

The graph below shows the arithmetic progression $\{-2, 0, 2, 4, 6\ldots\}$ along with the function $y = 2x - 4$. The only major difference between the two graphs is that an arithmetic sequence is **discrete** while a linear function is **continuous**.

We can write a formula for an arithmetic progression: if we define the first term as a_1 and d as the common difference, then the other terms are as follows:

$$\begin{array}{ccccccccccc}
 1 & a_2 & a_3 & a_4 & a_5 & & a_n \\
 & 1 & a_1 + d & a_1 + 2d & a_1 + 3d & a_1 + 4d & \dots & a_1 + (n-1) \cdot d
 \end{array}$$

The online calculator at <http://planetcalc.com/177/> will tell you the n th term in an arithmetic progression if you tell it the first term, the common difference, and what value to use for n (in other words, which term in the sequence you want to know). It will also tell you the sum of all the terms up to that point. Finding sums of sequences is something you will learn to do in future math classes.

Lesson Summary

- In order for an equation to be a **function**, the relationship between the two variables, x and y , must map each x -value to *exactly* one y -value.
- The graph of a function of y versus x must pass the **vertical line test**: any vertical line will only cross the graph of the function in one place.
- Functions can be expressed in function notation using $f(x)$ = in place of y =.
- The sequence of $f(x)$ values for a linear function form an arithmetic progression. Each number is greater than (or less than) the preceding number by a fixed amount, or **common difference**.

Review Questions

1. When an object falls under gravity, it gains speed at a constant rate of 9.8 m/s every second. An item dropped from the top of the Eiffel Tower, which is 300 meters tall, takes 7.8 seconds to hit the ground. How fast is it moving on impact?
2. A prepaid phone card comes with \$20 worth of calls on it. Calls cost a flat rate of \$0.16 per minute.
 - (a) Write the value left on the card as a function of minutes used so far.
 - (b) Use the function to determine how many minutes of calls you can make with the card.

3. For each of the following functions evaluate:

(a) $f(x) = -2x + 3$

(b) $f(x) = 0.7x + 3.2$

(c) $f(x) = \frac{5(2-x)}{11}$

i. $f(-3)$

ii. $f(0)$

iii. $f(z)$

iv. $f(x + 3)$

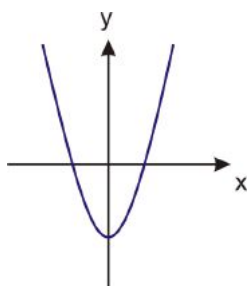
v. $f(2n)$

vi. $f(3y + 8)$

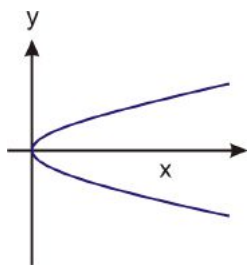
vii. $f\left(\frac{q}{2}\right)$

4. Determine whether the following could be graphs of **functions**.

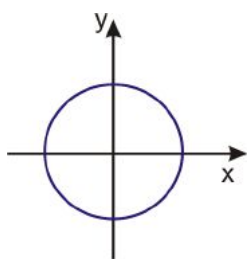
(a)



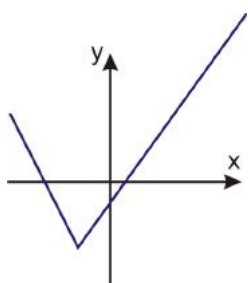
(b)



(c)



(d)



5. The roasting guide for a turkey suggests cooking for 100 minutes plus an additional 8 minutes per pound.

(a) Write a function for the roasting time the given the turkey weight in pounds (x).

(b) Determine the time needed to roast a 10 lb turkey.

- (c) Determine the time needed to roast a 27 lb turkey.
 - (d) Determine the maximum size turkey you could roast in 4.5 hours.
6. Determine the missing terms in the following arithmetic progressions.
- (a) $\{-11, 17, _, 73\}$
 - (b) $\{2, _, -4\}$
 - (c) $\{13, _, _, _, 0\}$

2.8 Problem-Solving Strategies - Graphs

Learning Objectives

- Read and understand given problem situations.
- Use the strategy “Read a Graph.”
- Use the strategy “Make a Graph.”
- Solve real-world problems using selected strategies as part of a plan.

Introduction

In this chapter, we’ve been solving problems where quantities are linearly related to each other. In this section, we’ll look at a few examples of linear relationships that occur in real-world problems, and see how we can solve them using graphs. Remember back to our Problem Solving Plan:

1. **Understand the Problem**
2. **Devise a Plan—Translate**
3. **Carry Out the Plan—Solve**
4. **Look—Check and Interpret**

Example 1

A cell phone company is offering its costumers the following deal: You can buy a new cell phone for \$60 and pay a monthly flat rate of \$40 per month for unlimited calls. How much money will this deal cost you after 9 months?

Solution

Let’s follow the problem solving plan.

Step 1: The phone costs \$60; the calling plan costs \$40 per month.

Let x = number of months.

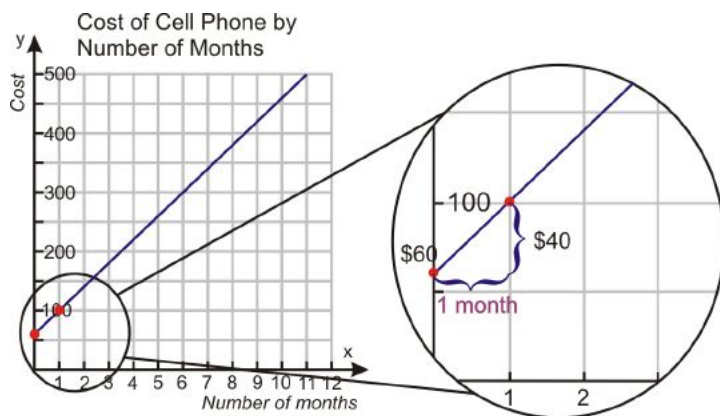
Let y = total cost.

Step 2: We can solve this problem by making a graph that shows the number of months on the horizontal axis and the cost on the vertical axis.

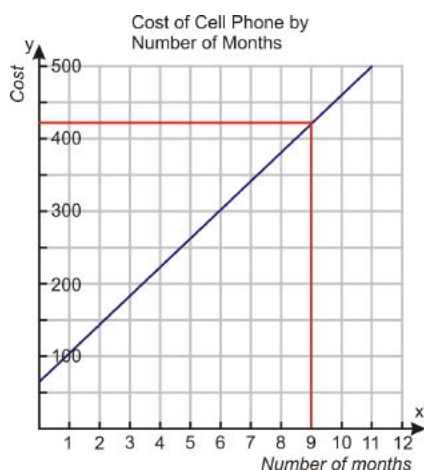
Since you pay \$60 when you get the phone, the y -intercept is $(0, 60)$.

You pay \$40 for each month, so the cost rises by \$40 for 1 month, so the slope is 40.

We can graph this line using the slope-intercept method.



Step 3: The question was “How much will this deal cost after 9 months?” We can now read the answer from the graph. We draw a vertical line from 9 months until it meets the graph, and then draw a horizontal line until it meets the vertical axis.



We see that after 9 months **you pay approximately \$420.**

Step 4: To check if this is correct, let’s think of the deal again.

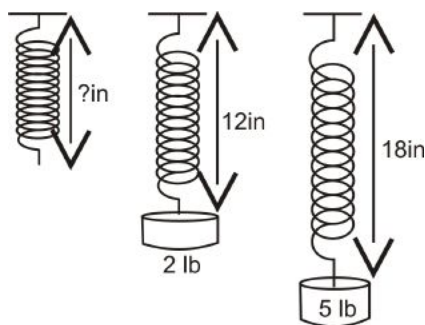
Originally, you pay \$60 and then \$40 a month for 9 months.

$$\begin{aligned}\text{Phone} &= \$60 \\ \text{Calling plan} &= \$40 \times 9 = \$360 \\ \text{Total cost} &= \$420.\end{aligned}$$

The answer checks out.

Example 2

A stretched spring has a length of 12 inches when a weight of 2 lbs is attached to the spring. The same spring has a length of 18 inches when a weight of 5 lbs is attached to the spring. What is the length of the spring when no weights are attached?



Solution

Step 1: We know: the length of the spring = 12 inches when weight = 2 lbs

the length of the spring = 18 inches when weight = 5 lbs

We want: the length of the spring when weight = 0 lbs

Let x = the weight attached to the spring.

Let y = the length of the spring.

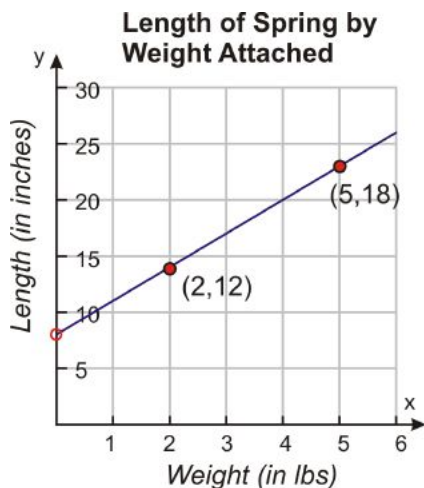
Step 2: We can solve this problem by making a graph that shows the weight on the horizontal axis and the length of the spring on the vertical axis.

We have two points we can graph:

When the weight is 2 lbs, the length of the spring is 12 inches. This gives point (2, 12).

When the weight is 5 lbs, the length of the spring is 18 inches. This gives point (5, 18).

Graphing those two points and connecting them gives us our line.



Step 3: The question was: “What is the length of the spring when no weights are attached?”

We can answer this question by reading the graph we just made. When there is no weight on the spring, the x -value equals zero, so we are just looking for the y -intercept of the graph. On the graph, the y -intercept appears to be approximately 8 inches.

Step 4: To check if this correct, let's think of the problem again.

You can see that the length of the spring goes up by 6 inches when the weight is increased by 3 lbs, so the slope of the line is $\frac{6 \text{ inches}}{3 \text{ lbs}} = 2 \text{ inches/lb}$.

To find the length of the spring when there is no weight attached, we can look at the spring when there are 2 lbs attached. For each pound we take off, the spring will shorten by 2 inches. If we take off 2 lbs, the spring will be shorter by 4 inches. So, the length of the spring with no weights is 12 inches - 4 inches = 8 inches.

The answer checks out.

Example 3

Christine took 1 hour to read 22 pages of Harry Potter. She has 100 pages left to read in order to finish the book. How much time should she expect to spend reading in order to finish the book?

Solution

Step 1: We know - Christine takes 1 hour to read 22 pages.

We want - How much time it takes to read 100 pages.

Let x = the time expressed in hours.

Let y = the number of pages.

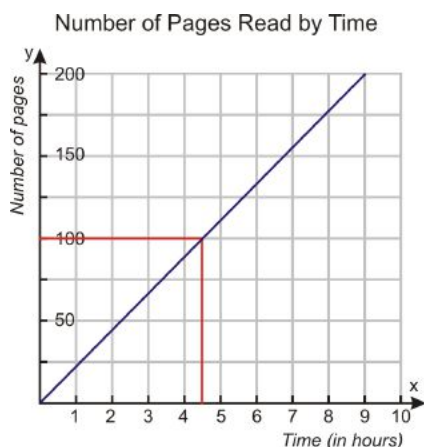
Step 2: We can solve this problem by making a graph that shows the number of hours spent reading on the horizontal axis and the number of pages on the vertical axis.

We have two points we can graph:

Christine takes 1 hour to read 22 pages. This gives point (1, 22).

A second point is not given, but we know that Christine would take 0 hours to read 0 pages. This gives point (0, 0).

Graphing those two points and connecting them gives us our line.



Step 3: The question was: “How much time should Christine expect to spend reading 100 pages?” We can find the answer from reading the graph - we draw a horizontal line from 100 pages until it meets the graph and then we draw the vertical until it meets the horizontal axis. We see that it takes **approximately 4.5 hours** to read the remaining 100 pages.

Step 4: To check if this correct, let's think of the problem again.

We know that Christine reads 22 pages per hour - this is the slope of the line or the rate at which she is reading. To find how many hours it takes her to read 100 pages, we divide the number of pages by the rate. In this case, $\frac{100 \text{ pages}}{22 \text{ pages/hour}} = 4.54 \text{ hours}$. This is very close to the answer we got from reading the graph.

The answer checks out.

Example 4

Aatif wants to buy a surfboard that costs \$249. He was given a birthday present of \$50 and he has a summer job that pays him \$6.50 per hour. To be able to buy the surfboard, how many hours does he need to work?

Solution

Step 1: We know - The surfboard costs \$249.

Aatif has \$50.

His job pays \$6.50 per hour.

We want - How many hours Aatif needs to work to buy the surfboard.

Let x = the time expressed in hours

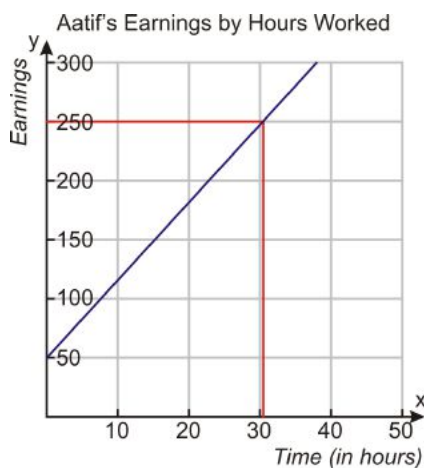
Let y = Aatif's earnings

Step 2: We can solve this problem by making a graph that shows the number of hours spent working on the horizontal axis and Aatif's earnings on the vertical axis.

Aatif has \$50 at the beginning. This is the y -intercept: $(0, 50)$.

He earns \$6.50 per hour. This is the slope of the line.

We can graph this line using the slope-intercept method. We graph the y -intercept of $(0, 50)$, and we know that for each unit in the horizontal direction, the line rises by 6.5 units in the vertical direction. Here is the line that describes this situation.



Step 3: The question was: “How many hours does Aatif need to work to buy the surfboard?”

We find the answer from reading the graph - since the surfboard costs \$249, we draw a horizontal line from \$249 on the vertical axis until it meets the graph and then we draw a vertical line downwards until it meets the horizontal axis. We see that it takes **approximately 31 hours** to earn the money.

Step 4: To check if this correct, let's think of the problem again.

We know that Aatif has \$50 and needs \$249 to buy the surfboard. So, he needs to earn $\$249 - \$50 = \$199$ from his job.

His job pays \$6.50 per hour. To find how many hours he need to work we divide: $\frac{\$199}{\$6.50/\text{hour}} = 30.6 \text{ hours}$. This is very close to the answer we got from reading the graph.

The answer checks out.

Lesson Summary

The four steps of the **problem solving plan** when using graphs are:

1. **Understand the Problem**
2. **Devise a Plan—Translate:** Make a graph.
3. **Carry Out the Plan—Solve:** Use the graph to answer the question asked.
4. **Look—Check and Interpret**

Review Questions

Solve the following problems by making a graph and reading it.

1. A gym is offering a deal to new members. Customers can sign up by paying a registration fee of \$200 and a monthly fee of \$39.
 - (a) How much will this membership cost a member by the end of the year?
 - (b) The old membership rate was \$49 a month with a registration fee of \$100. How much more would a year's membership cost at that rate?
 - (c) **Bonus:** For what number of months would the two membership rates be the same?
2. A candle is burning at a linear rate. The candle measures five inches two minutes after it was lit. It measures three inches eight minutes after it was lit.
 - (a) What was the original length of the candle?
 - (b) How long will it take to burn down to a half-inch stub?
 - (c) Six half-inch stubs of candle can be melted together to make a new candle measuring $2\frac{5}{6}$ inches (a little wax gets lost in the process). How many stubs would it take to make three candles the size of the original candle?
3. A dipped candle is made by taking a wick and dipping it repeatedly in melted wax. The candle gets a little bit thicker with each added layer of wax. After it has been dipped three times, the candle is 6.5 mm thick. After it has been dipped six times, it is 11 mm thick.
 - (a) How thick is the wick before the wax is added?
 - (b) How many times does the wick need to be dipped to create a candle 2 cm thick?
4. Tali is trying to find the thickness of a page of his telephone book. In order to do this, he takes a measurement and finds out that 55 pages measures $\frac{1}{8}$ inch. What is the thickness of one page of the phone book?
5. Bobby and Petra are running a lemonade stand and they charge 45 cents for each glass of lemonade. In order to break even they must make \$25.
 - (a) How many glasses of lemonade must they sell to break even?
 - (b) When they've sold \$18 worth of lemonade, they realize that they only have enough lemons left to make 10 more glasses. To break even now, they'll need to sell those last 10 glasses at a higher price. What does the new price need to be?
6. Dale is making cookies using a recipe that calls for 2.5 cups of flour for two dozen cookies. How many cups of flour does he need to make five dozen cookies?
7. To buy a car, Jason makes a down payment of \$1500 and pays \$350 per month in installments.
 - (a) How much money has Jason paid at the end of one year?
 - (b) If the total cost of the car is \$8500, how long will it take Jason to finish paying it off?
 - (c) The resale value of the car decreases by \$100 each month from the original purchase price. If Jason sells the car as soon as he finishes paying it off, how much will he get for it?

8. Anne transplants a rose seedling in her garden. She wants to track the growth of the rose so she measures its height every week. On the third week, she finds that the rose is 10 inches tall and on the eleventh week she finds that the rose is 14 inches tall. Assuming the rose grows linearly with time, what was the height of the rose when Anne planted it?
9. Ravi hangs from a giant spring whose length is 5 m. When his child Nimi hangs from the spring its length is 2 m. Ravi weighs 160 lbs and Nimi weighs 40 lbs. Write the equation for this problem in slope-intercept form. What should we expect the length of the spring to be when his wife Amardeep, who weighs 140 lbs, hangs from it?
10. Nadia is placing different weights on a spring and measuring the length of the stretched spring. She finds that for a 100 gram weight the length of the stretched spring is 20 cm and for a 300 gram weight the length of the stretched spring is 25 cm.
 - (a) What is the unstretched length of the spring?
 - (b) If the spring can only stretch to twice its unstretched length before it breaks, how much weight can it hold?
11. Andrew is a submarine commander. He decides to surface his submarine to periscope depth. It takes him 20 minutes to get from a depth of 400 feet to a depth of 50 feet.
 - (a) What was the submarine's depth five minutes after it started surfacing?
 - (b) How much longer would it take at that rate to get all the way to the surface?
12. Kiersta's phone has completely run out of battery power when she puts it on the charger. Ten minutes later, when the phone is 40% recharged, Kiersta's friend Danielle calls and Kiersta takes the phone off the charger to talk to her. When she hangs up 45 minutes later, her phone has 10% of its charge left. Then she gets another call from her friend Kwan.
 - (a) How long can she spend talking to Kwan before the battery runs out again?
 - (b) If she puts the phone back on the charger afterward, how long will it take to recharge completely?
13. Marji is painting a 75-foot fence. She starts applying the first coat of paint at 2 PM, and by 2:10 she has painted 30 feet of the fence. At 2:15, her husband, who paints about $\frac{2}{3}$ as fast as she does, comes to join her.
 - (a) How much of the fence has Marji painted when her husband joins in?
 - (b) When will they have painted the whole fence?
 - (c) How long will it take them to apply the second coat of paint if they work together the whole time?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9614>.

Chapter 3

Writing Linear Equations

3.1 Forms of Linear Equations

Learning Objectives

- Write equations in slope-intercept form.
- Write equations in point-slope form.
- Write equations in standard form.
- Solve real-world problems using linear models in all three forms.

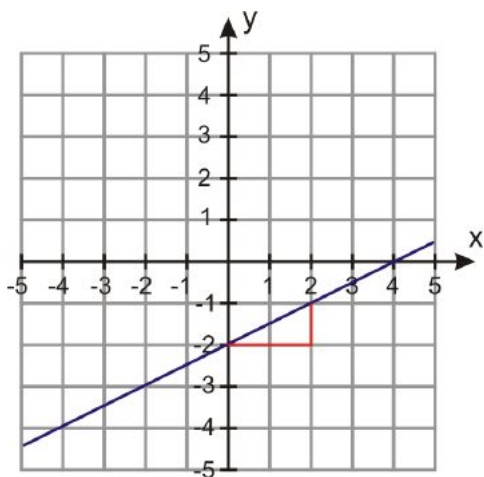
Introduction

We saw in the last chapter that many real-world situations can be described with linear graphs and equations. In this chapter, we'll see how to find those equations in a variety of situations.

Write an Equation Given Slope and y-Intercept

You've already learned how to write an equation in slope-intercept form: simply start with the general equation for the slope-intercept form of a line, $y = mx + b$, and then plug the given values of m and b into the equation. For example, a line with a slope of 4 and a y-intercept of -3 would have the equation $y = 4x - 3$.

If you are given just the graph of a line, you can read off the slope and y-intercept from the graph and write the equation from there. For example, on the graph below you can see that the line rises by 1 unit as it moves 2 units to the right, so its slope is $\frac{1}{2}$. Also, you can see that the y-intercept is -2, so the equation of the line is $y = \frac{1}{2}x - 2$.



Write an Equation Given the Slope and a Point

Often, we don't know the value of the y -intercept, but we know the value of y for a non-zero value of x . In this case, it's often easier to write an equation of the line in **point-slope form**. An equation in point-slope form is written as $y - y_0 = m(x - x_0)$, where m is the slope and (x_0, y_0) is a point on the line.

Example 1

A line has a slope of $\frac{3}{5}$, and the point $(2, 6)$ is on the line. Write the equation of the line in point-slope form.

Solution

Start with the formula $y - y_0 = m(x - x_0)$.

Plug in $\frac{3}{5}$ for m , 2 for x_0 and 6 for y_0 .

The equation in point-slope form is $y - 6 = \frac{3}{5}(x - 2)$.

Notice that the equation in point-slope form is not solved for y . If we did solve it for y , we'd have it in y -intercept form. To do that, we would just need to distribute the $\frac{3}{5}$ and add 6 to both sides. That means that the equation of this line in slope-intercept form is $y = \frac{3}{5}x - \frac{6}{5} + 6$, or simply $y = \frac{3}{5}x + \frac{24}{5}$.

Write an Equation Given Two Points

Point-slope form also comes in useful when we need to find an equation given just two points on a line.

For example, suppose we are told that the line passes through the points $(-2, 3)$ and $(5, 2)$. To find the equation of the line, we can start by finding the slope.

Starting with the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$, we plug in the x - and y -values of the two points to get $m = \frac{2 - 3}{5 - (-2)} = \frac{-1}{7}$. We can plug that value of m into the point-slope formula to get $y - y_0 = -\frac{1}{7}(x - x_0)$.

Now we just need to pick one of the two points to plug into the formula. Let's use $(5, 2)$; that gives us $y - 2 = -\frac{1}{7}(x - 5)$.

What if we'd picked the other point instead? Then we'd have ended up with the equation $y - 3 = -\frac{1}{7}(x + 2)$, which doesn't look the same. That's because there's more than one way to write an equation for a given line in point-slope form. But let's see what happens if we solve each of those equations for y .

Starting with $y - 2 = -\frac{1}{7}(x - 5)$, we distribute the $-\frac{1}{7}$ and add 2 to both sides. That gives us $y = -\frac{1}{7}x + \frac{5}{7} + 2$,

or $y = -\frac{1}{7}x + \frac{19}{7}$.

On the other hand, if we start with $y - 3 = -\frac{1}{7}(x + 2)$, we need to distribute the $-\frac{1}{7}$ and add 3 to both sides. That gives us $y = -\frac{1}{7}x - \frac{2}{7} + 3$, which also simplifies to $y = -\frac{1}{7}x + \frac{19}{7}$.

So whichever point we choose to get an equation in point-slope form, the equation is still mathematically the same, and we can see this when we convert it to y-intercept form.

Example 2

A line contains the points (3, 2) and (-2, 4). Write an equation for the line in point-slope form; then write an equation in y-intercept form.

Solution

Find the slope of the line: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 2}{-2 - 3} = -\frac{2}{5}$

Plug in the value of the slope: $y - y_0 = -\frac{2}{5}(x - x_0)$.

Plug point (3, 2) into the equation: $y - 2 = -\frac{2}{5}(x - 3)$.

The equation in point-slope form is $y - 2 = -\frac{2}{5}(x - 3)$.

To convert to y-intercept form, simply solve for y:

$$y - 2 = -\frac{2}{5}(x - 3) \rightarrow y - 2 = -\frac{2}{5}x - \frac{6}{5} \rightarrow y = -\frac{2}{5}x - \frac{6}{5} + 2 = -\frac{2}{5}x + \frac{4}{5}.$$

The equation in y-intercept form is $y = -\frac{2}{5}x + \frac{4}{5}$.

Graph an Equation in Point-Slope Form

Another useful thing about point-slope form is that you can use it to graph an equation without having to convert it to slope-intercept form. From the equation $y - y_0 = m(x - x_0)$, you can just read off the slope m and the point (x_0, y_0) . To draw the graph, all you have to do is plot the point, and then use the slope to figure out how many units up and over you should move to find another point on the line.

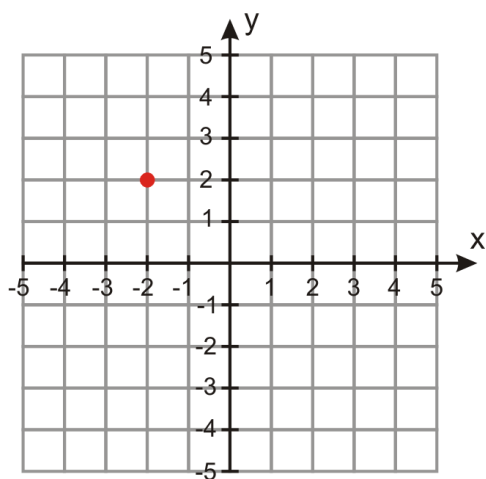
Example 5

Make a graph of the line given by the equation $y + 2 = \frac{2}{3}(x - 2)$.

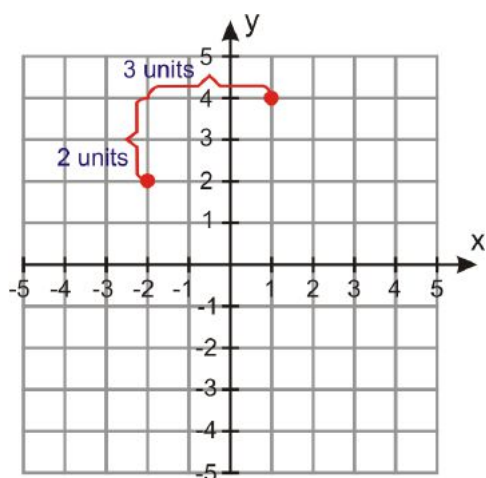
Solution

To read off the right values, we need to rewrite the equation slightly: $y - (-2) = \frac{2}{3}(x - 2)$. Now we see that point (2, -2) is on the line and that the slope is $\frac{2}{3}$.

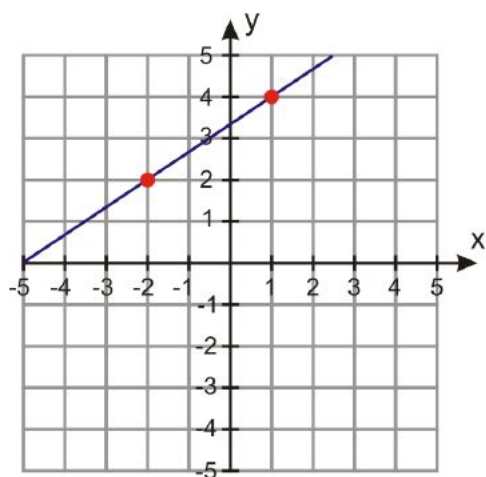
First plot point (2, -2) on the graph:



A slope of $\frac{2}{3}$ tells you that from that point you should move 2 units up and 3 units to the right and draw another point:



Now draw a line through the two points and extend it in both directions:



Linear Equations in Standard Form

You've already encountered another useful form for writing linear equations: **standard form**. An equation in standard form is written $ax + by = c$, where a, b , and c are all integers and a is positive. (Note that the b in the standard form is different than the b in the slope-intercept form.)

One useful thing about standard form is that it allows us to write equations for vertical lines, which we can't do in slope-intercept form.

For example, let's look at the line that passes through points $(2, 6)$ and $(2, 9)$. How would we find an equation for that line in slope-intercept form?

First we'd need to find the slope: $m = \frac{9-6}{0-0} = \frac{3}{0}$. But that slope is undefined because we can't divide by zero. And if we can't find the slope, we can't use point-slope form either.

If we just graph the line, we can see that x equals 2 no matter what y is. There's no way to express that in slope-intercept or point-slope form, but in standard form we can just say that $x + 0y = 2$, or simply $x = 2$.

Converting to Standard Form

To convert an equation from another form to standard form, all you need to do is rewrite the equation so that all the variables are on one side of the equation and the coefficient of x is not negative.

Example 1

Rewrite the following equations in standard form:

a) $y = 5x - 7$

b) $y - 2 = -3(x + 3)$

c) $y = \frac{2}{3}x + \frac{1}{2}$

Solution

We need to rewrite each equation so that all the variables are on one side and the coefficient of x is not negative.

a) $y = 5x - 7$

Subtract y from both sides to get $0 = 5x - y - 7$.

Add 7 to both sides to get $7 = 5x - y$.

Flip the equation around to put it in standard form: $5x - y = 7$.

b) $y - 2 = -3(x + 3)$

Distribute the -3 on the right-hand-side to get $y - 2 = -3x - 9$.

Add $3x$ to both sides to get $y + 3x - 2 = -9$.

Add 2 to both sides to get $y + 3x = -7$. Flip that around to get $3x + y = -7$.

c) $y = \frac{2}{3}x + \frac{1}{2}$

Find the common denominator for all terms in the equation – in this case that would be 6.

Multiply all terms in the equation by 6: $6\left(y = \frac{2}{3}x + \frac{1}{2}\right) \Rightarrow 6y = 4x + 3$

Subtract $6y$ from both sides: $0 = 4x - 6y + 3$

Subtract 3 from both sides: $-3 = 4x - 6y$

The equation in standard form is $4x - 6y = -3$.

Graphing Equations in Standard Form

When an equation is in slope-intercept form or point-slope form, you can tell right away what the slope is. How do you find the slope when an equation is in standard form?

Well, you could rewrite the equation in slope-intercept form and read off the slope. But there's an even easier way. Let's look at what happens when we rewrite an equation in standard form.

Starting with the equation $ax + by = c$, we would subtract ax from both sides to get $by = -ax + c$. Then we would divide all terms by b and end up with $y = -\frac{a}{b}x + \frac{c}{b}$.

That means that the slope is $-\frac{a}{b}$ and the y -intercept is $\frac{c}{b}$. So next time we look at an equation in standard form, we don't have to rewrite it to find the slope; we know the slope is just $-\frac{a}{b}$, where a and b are the coefficients of x and y in the equation.

Example 2

Find the slope and the y -intercept of the following equations written in standard form.

a) $3x + 5y = 6$

b) $2x - 3y = -8$

c) $x - 5y = 10$

Solution

a) $a = 3, b = 5$, and $c = 6$, so the slope is $-\frac{a}{b} = -\frac{3}{5}$, and the y -intercept is $\frac{c}{b} = \frac{6}{5}$.

b) $a = 2, b = -3$, and $c = -8$, so the slope is $-\frac{a}{b} = \frac{2}{3}$, and the y -intercept is $\frac{c}{b} = \frac{8}{3}$.

c) $a = 1, b = -5$, and $c = 10$, so the slope is $-\frac{a}{b} = \frac{1}{5}$, and the y -intercept is $\frac{c}{b} = \frac{10}{-5} = -2$.

Once we've found the slope and y -intercept of an equation in standard form, we can graph it easily. But if we start with a graph, how do we find an equation of that line in standard form?

First, remember that we can also use the cover-up method to graph an equation in standard form, by finding the intercepts of the line. For example, let's graph the line given by the equation $3x - 2y = 6$.

To find the x -intercept, cover up the y term (remember, the x -intercept is where $y = 0$):

$$3x - \text{[cover up]} = 6$$

$$3x = 6 \Rightarrow x = 2$$

The x -intercept is $(2, 0)$.

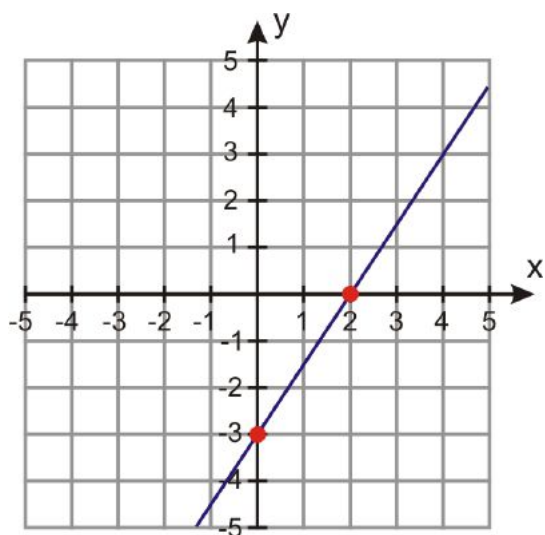
To find the y -intercept, cover up the x term (remember, the y -intercept is where $x = 0$):

$$\text{[cover up]} - 2y = 6$$

$$-2y = 6 \Rightarrow y = -3$$

The y -intercept is $(0, -3)$.

We plot the intercepts and draw a line through them that extends in both directions:

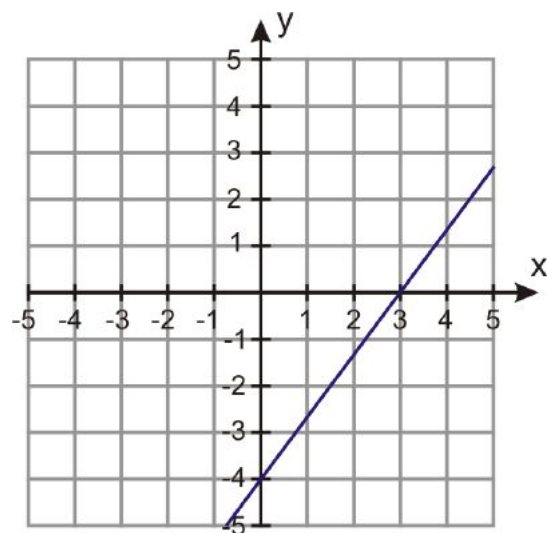


Now we want to apply this process in reverse—to start with the graph of the line and write the equation of the line in standard form.

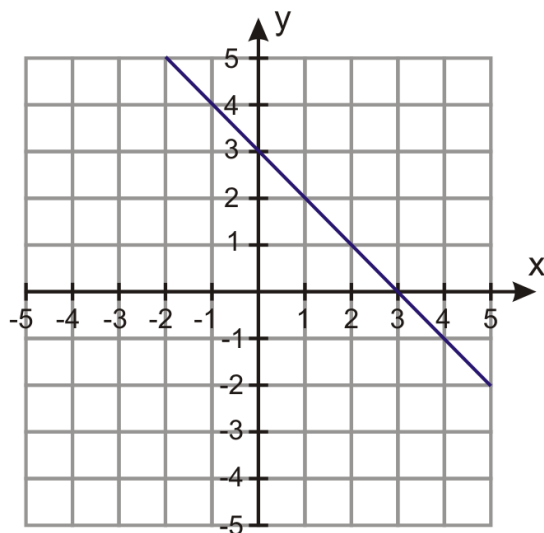
Example 3

Find the equation of each line and write it in standard form.

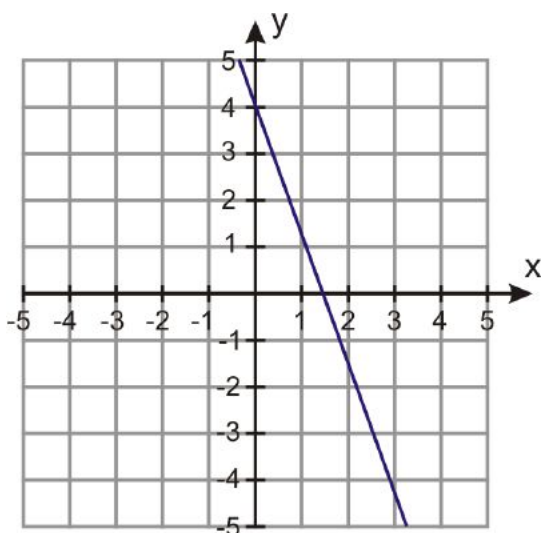
a)



b)



c)



Solution

a) We see that the x -intercept is $(3, 0) \Rightarrow x = 3$ and the y -intercept is $(0, -4) \Rightarrow y = -4$

We saw that in standard form $ax + by = c$: if we “cover up” the y term, we get $ax = c$, and if we “cover up” the x term, we get $by = c$.

So we need to find values for a and b so that we can plug in 3 for x and -4 for y and get the same value for c in both cases. This is like finding the least common multiple of the x - and y -intercepts.

In this case, we see that multiplying $x = 3$ by 4 and multiplying $y = -4$ by -3 gives the same result:

$$(x = 3) \times 4 \Rightarrow 4x = 12 \quad \text{and} \quad (y = -4) \times (-3) \Rightarrow -3y = 12$$

Therefore, $a = 4$, $b = -3$ and $c = 12$ and **the equation in standard form is $4x - 3y = 12$.**

b) We see that the x -intercept is $(3, 0) \Rightarrow x = 3$ and the y -intercept is $(0, 3) \Rightarrow y = 3$

The values of the intercept equations are already the same, so $a = 1$, $b = 1$ and $c = 3$. **The equation in standard form is $x + y = 3$.**

c) We see that the x -intercept is $(\frac{3}{2}, 0) \Rightarrow x = \frac{3}{2}$ and the y -intercept is $(0, 4) \Rightarrow y = 4$

Let's multiply the x -intercept equation by 2 $\Rightarrow 2x = 3$

Then we see we can multiply the x -intercept again by 4 and the y -intercept by 3, so we end up with $8x = 12$ and $3y = 12$.

The equation in standard form is $8x + 3y = 12$.

Solving Real-World Problems Using Linear Models in Point-Slope Form

Let's solve some word problems where we need to write the equation of a straight line in point-slope form.

Example 4

Marciel rented a moving truck for the day. Marciel only remembers that the rental truck company charges \$40 per day and some number of cents per mile. Marciel drives 46 miles and the final amount of the bill (before tax) is \$63. What is the amount per mile the truck rental company charges? Write an equation in point-slope form that describes this situation. How much would it cost to rent this truck if Marciel drove 220 miles?

Solution

Let's define our variables:

x = distance in miles

y = cost of the rental truck

Peter pays a flat fee of \$40 for the day; this is the y -intercept.

He pays \$63 for 46 miles; this is the coordinate point (46,63).

Start with the point-slope form of the line: $y - y_0 = m(x - x_0)$

Plug in the coordinate point: $63 - y_0 = m(46 - x_0)$

Plug in the point (0, 40): $63 - 40 = m(46 - 0)$

Solve for the slope: $23 = 46m \rightarrow m = \frac{23}{46} = 0.5$

The slope is 0.5 dollars per mile, so the truck company charges 50 cents per mile ($\$0.5 = 50$ cents). Plugging in the slope and the y -intercept, the equation of the line is $y = 0.5x + 40$.

To find out the cost of driving the truck 220 miles, we plug in $x = 220$ to get $y - 40 = 0.5(220) \Rightarrow y = \150 .

Driving 220 miles would cost \$150.

Example 5

Anne got a job selling window shades. She receives a monthly base salary and a \$6 commission for each window shade she sells. At the end of the month she adds up sales and she figures out that she sold 200 window shades and made \$2500. Write an equation in point-slope form that describes this situation. How much is Anne's monthly base salary?

Solution

Let's define our variables:

x = number of window shades sold

y = Anne's earnings

We see that we are given the slope and a point on the line:

Nadia gets \$6 for each shade, so the slope is 6.

She made \$2500 when she sold 200 shades, so the point is (200, 2500).

Start with the point-slope form of the line: $y - y_0 = m(x - x_0)$

Plug in the slope: $y - y_0 = 6(x - x_0)$

Plug in the point (200, 2500): $y - 2500 = 6(x - 200)$

To find Anne's base salary, we plug in $x = 0$ and get $y - 2500 = -1200 \Rightarrow y = \1300 .

Anne's monthly base salary is \$1300.

Solving Real-World Problems Using Linear Models in Standard Form

Here are two examples of real-world problems where the standard form of an equation is useful.

Example 6

Nadia buys fruit at her local farmer's market. This Saturday, oranges cost \$2 per pound and cherries cost \$3 per pound. She has \$12 to spend on fruit. Write an equation in standard form that describes this situation. If she buys 4 pounds of oranges, how many pounds of cherries can she buy?

Solution

Let's define our variables:

x = pounds of oranges

y = pounds of cherries

The equation that describes this situation is $2x + 3y = 12$.

If she buys 4 pounds of oranges, we can plug $x = 4$ into the equation and solve for y :

$$2(4) + 3y = 12 \Rightarrow 3y = 12 - 8 \Rightarrow 3y = 4 \Rightarrow y = \frac{4}{3}$$

Nadia can buy $1\frac{1}{3}$ pounds of cherries.

Example 7

Peter skateboards part of the way to school and walks the rest of the way. He can skateboard at 7 miles per hour and he can walk at 3 miles per hour. The distance to school is 6 miles. Write an equation in standard form that describes this situation. If he skateboards for $\frac{1}{2}$ an hour, how long does he need to walk to get to school?

Solution

Let's define our variables:

x = time Peter skateboards

y = time Peter walks

The equation that describes this situation is: $7x + 3y = 6$

If Peter skateboards $\frac{1}{2}$ an hour, we can plug $x = 0.5$ into the equation and solve for y :

$$7(0.5) + 3y = 6 \Rightarrow 3y = 6 - 3.5 \Rightarrow 3y = 2.5 \Rightarrow y = \frac{5}{6}$$

Peter must walk $\frac{5}{6}$ of an hour.

Further Practice

Now that you've worked with equations in all three basic forms, check out the Java applet at <http://www.ronblond.com/>. You can use it to manipulate graphs of equations in all three forms, and see how the graphs change when you vary the terms of the equations.

Another applet at <http://www.cut-the-knot.org/Curriculum/Calculus/StraightLine.shtml> lets you create multiple lines and see how they intersect. Each line is defined by two points; you can change the slope of a line by moving either of the points, or just drag the whole line around without changing its slope. To create another line, just click Duplicate and then drag one of the lines that are already there.

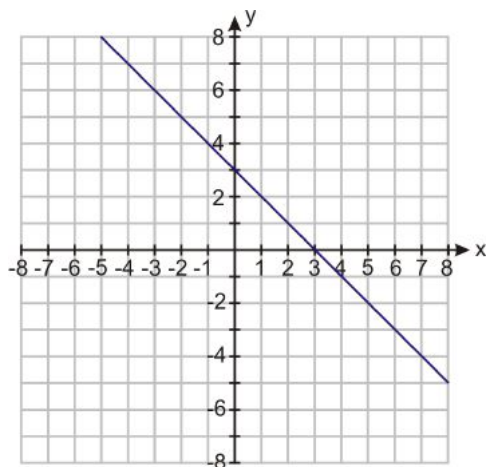
Review Questions

Find the equation of each line in slope-intercept form.

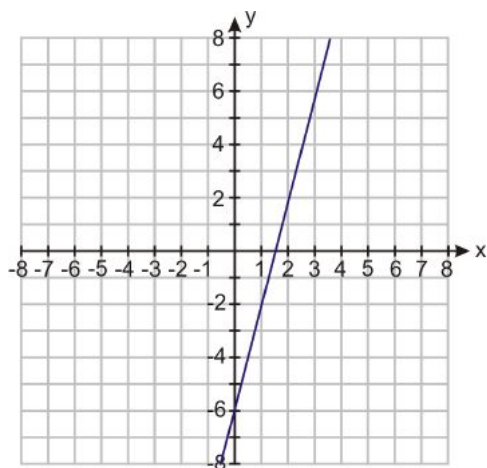
1. The line has a slope of 7 and a y-intercept of -2.
2. The line has a slope of -5 and a y-intercept of 6.
3. The line has a slope of $-\frac{1}{4}$ and contains the point (4, -1).
4. The line has a slope of $\frac{2}{3}$ and contains the point $(\frac{1}{2}, 1)$.
5. The line has a slope of -1 and contains the point $(\frac{4}{5}, 0)$.
6. The line contains points (2, 6) and (5, 0).
7. The line contains points (5, -2) and (8, 4).
8. The line contains points (3, 5) and (-3, 0).
9. The line contains points (10, 15) and (12, 20).

Write the equation of each line in slope-intercept form.

10.



11.



Find the equation of each linear function in slope-intercept form.

12. $m = 5, f(0) = -3$
13. $m = -7, f(2) = -1$
14. $m = \frac{1}{3}, f(-1) = \frac{2}{3}$
15. $m = 4.2, f(-3) = 7.1$
16. $f\left(\frac{1}{4}\right) = \frac{3}{4}, f(0) = \frac{5}{4}$
17. $f(1.5) = -3, f(-1) = 2$

Write the equation of each line in point-slope form.

18. The line has slope $-\frac{1}{10}$ and goes through the point (10, 2).
19. The line has slope -75 and goes through the point (0, 125).
20. The line has slope 10 and goes through the point (8, -2).
21. The line goes through the points (-2, 3) and (-1, -2).
22. The line contains the points (10, 12) and (5, 25).
23. The line goes through the points (2, 3) and (0, 3).
24. The line has a slope of $\frac{3}{5}$ and a y-intercept of -3.
25. The line has a slope of -6 and a y-intercept of 0.5.

Write the equation of each linear function in point-slope form.

26. $m = -\frac{1}{5}$ and $f(0) = 7$
27. $m = -12$ and $f(-2) = 5$
28. $f(-7) = 5$ and $f(3) = -4$
29. $f(6) = 0$ and $f(0) = 6$
30. $m = 3$ and $f(2) = -9$
31. $m = -\frac{9}{5}$ and $f(0) = 32$

Rewrite the following equations in standard form.

32. $y = 3x - 8$
33. $y - 7 = -5(x - 12)$
34. $2y = 6x + 9$

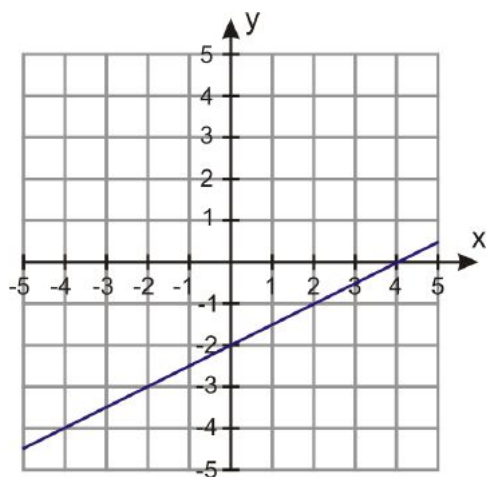
35. $y = \frac{9}{4}x + \frac{1}{4}$
 36. $y + \frac{8}{5} = \frac{2}{3}(x - 2)$
 37. $3y + 5 = 4(x - 9)$

Find the slope and y-intercept of the following lines.

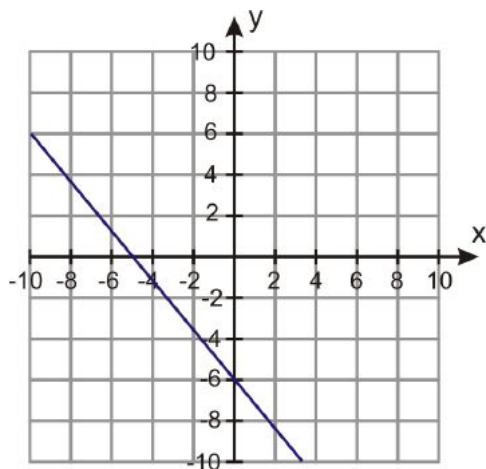
38. $5x - 2y = 15$
 39. $3x + 6y = 25$
 40. $x - 8y = 12$
 41. $3x - 7y = 20$
 42. $9x - 9y = 4$
 43. $6x + y = 3$

Find the equation of each line and write it in standard form.

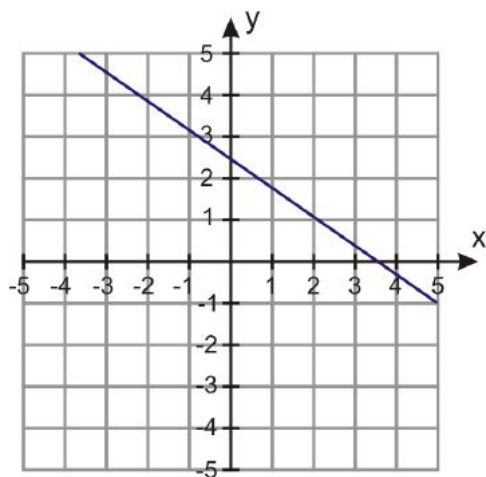
44.



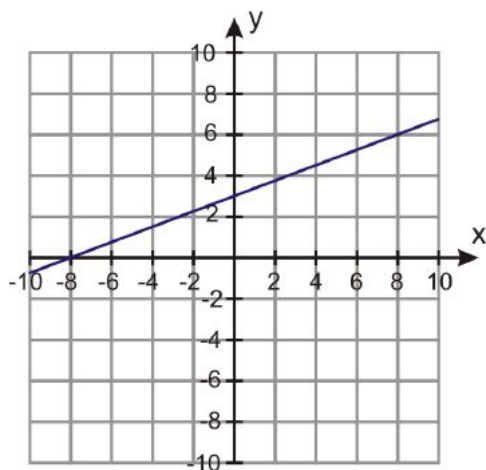
45.



46.



47.



48. Andrew has two part time jobs. One pays \$6 per hour and the other pays \$10 per hour. He wants to make \$366 per week. Write an equation in standard form that describes this situation. If he is only allowed to work 15 hours per week at the \$10 per hour job, how many hours does he need to work per week in his \$6 per hour job in order to achieve his goal?
49. Anne invests money in two accounts. One account returns 5% annual interest and the other returns 7% annual interest. In order not to incur a tax penalty, she can make no more than \$400 in interest per year. Write an equation in standard form that describes this problem. If she invests \$5000 in the 5% interest account, how much money does she need to invest in the other account?

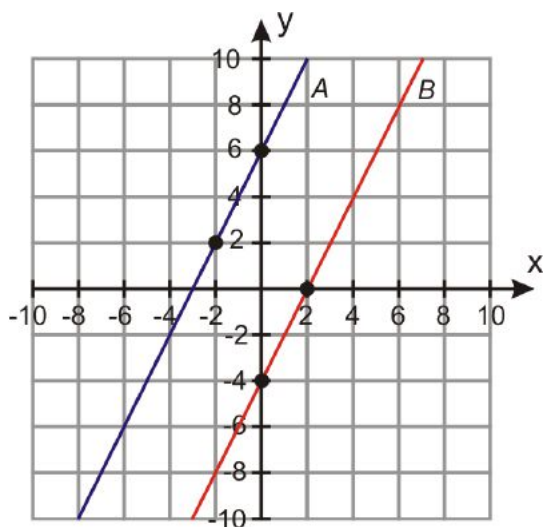
3.2 Equations of Parallel and Perpendicular Lines

Learning Objectives

- Determine whether lines are parallel or perpendicular
- Write equations of perpendicular lines
- Write equations of parallel lines
- Investigate families of lines

Introduction

In this section you will learn how **parallel lines** and **perpendicular lines** are related to each other on the coordinate plane. Let's start by looking at a graph of two parallel lines.



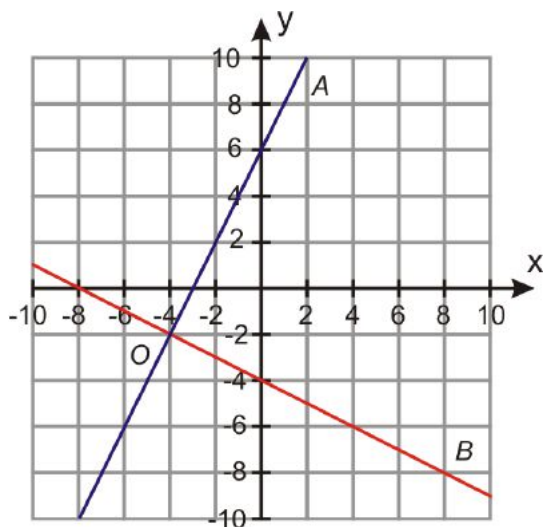
We can clearly see that the two lines have different y-intercepts: 6 and -4.

How about the slopes of the lines? The slope of line *A* is $\frac{6-2}{0-(-2)} = \frac{4}{2} = 2$, and the slope of line *B* is $\frac{0-(-4)}{2-0} = \frac{4}{2} = 2$. The slopes are the same.

Is that significant? Yes. By definition, parallel lines never meet. That means that when one of them slopes up by a certain amount, the other one has to slope up by the same amount so the lines will stay the same distance apart. If you look at the graph above, you can see that for any *x*-value you pick, the *y*-values of lines *A* and *B* are the same vertical distance apart—which means that both lines go up by the same vertical distance every time they go across by the same horizontal distance. In order to stay parallel, their slopes must stay the same.

All parallel lines have the same slopes and different y-intercepts.

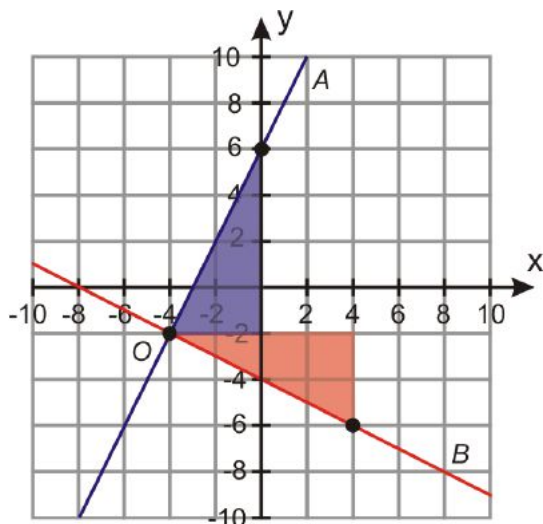
Now let's look at a graph of two perpendicular lines.



We can't really say anything about the y-intercepts. In this example, the y-intercepts are different, but if

we moved the lines four units to the right, they would both intercept the y -axis at $(0, -2)$. So perpendicular lines can have the same or different y -intercepts.

What about the relationship between the slopes of the two lines?



To find the slope of line A , we pick two points on the line and draw the blue (upper) right triangle. The legs of the triangle represent the rise and the run. We can see that the slope is $\frac{8}{4}$, or 2.

To find the slope of line B , we pick two points on the line and draw the red (lower) right triangle. Notice that the two triangles are identical, only rotated by 90° . Where line A goes 8 units up and 4 units right, line B goes 8 units right and 4 units down. Its slope is $-\frac{4}{8}$, or $-\frac{1}{2}$.

This is always true for perpendicular lines; where one line goes a units up and b units right, the other line will go a units right and b units down, so the slope of one line will be $\frac{a}{b}$ and the slope of the other line will be $-\frac{b}{a}$.

The slopes of **perpendicular lines** are always negative reciprocals of each other.

The Java applet at <http://members.shaw.ca/ron.blond/perp.APPLET/index.html> lets you drag around a pair of perpendicular lines to see how their slopes change. Click “Show Grid” to see the x - and y -axes, and click “Show Constructors” to see the triangles that are being used to calculate the slopes of the lines (you can then drag the circle to make it bigger or smaller, and click on a triangle to see the slope calculations in detail.)

Determine When Lines are Parallel or Perpendicular

You can find whether lines are parallel or perpendicular by comparing the slopes of the lines. If you are given points on the lines, you can find their slopes using the formula. If you are given the equations of the lines, re-write each equation in a form that makes it easy to read the slope, such as the slope-intercept form.

Example 1

Determine whether the lines are parallel or perpendicular or neither.

- One line passes through the points $(2, 11)$ and $(-1, 2)$; another line passes through the points $(0, -4)$ and $(-2, -10)$.
- One line passes through the points $(-2, -7)$ and $(1, 5)$; another line passes through the points $(4, 1)$ and $(-8, 4)$.

c) One line passes through the points (3, 1) and (-2, -2); another line passes through the points (5, 5) and (4, -6).

Solution

Find the slope of each line and compare them.

$$\text{a) } m_1 = \frac{2-11}{-1-2} = \frac{-9}{-3} = 3 \quad \text{and} \quad m_2 = \frac{-10-(-4)}{-2-0} = \frac{-6}{-2} = 3$$

The slopes are equal, so **the lines are parallel**.

$$\text{b) } m_1 = \frac{5-(-7)}{1-(-2)} = \frac{12}{3} = 4 \quad \text{and} \quad m_2 = \frac{4-1}{-8-4} = \frac{3}{-12} = -\frac{1}{4}$$

The slopes are negative reciprocals of each other, so **the lines are perpendicular**.

$$\text{c) } m_1 = \frac{-2-1}{-2-3} = \frac{-3}{-5} = \frac{3}{5} \quad \text{and} \quad m_2 = \frac{-6-5}{4-5} = \frac{-11}{-1} = 11$$

The slopes are not the same or negative reciprocals of each other, so **the lines are neither parallel nor perpendicular**.

Example 2

Determine whether the lines are parallel or perpendicular or neither:

$$\text{a) } 3x + 4y = 2 \text{ and } 8x - 6y = 5$$

$$\text{b) } 2x = y - 10 \text{ and } y = -2x + 5$$

$$\text{c) } 7y + 1 = 7x \text{ and } x + 5 = y$$

Solution

Write each equation in slope-intercept form:

$$\text{a) line 1: } 3x + 4y = 2 \Rightarrow 4y = -3x + 2 \Rightarrow y = -\frac{3}{4}x + \frac{1}{2} \Rightarrow \text{slope} = -\frac{3}{4}$$

$$\text{line 2: } 8x - 6y = 5 \Rightarrow 8x - 5 = 6y \Rightarrow y = \frac{8}{6}x - \frac{5}{6} \Rightarrow y = \frac{4}{3}x - \frac{5}{6} \Rightarrow \text{slope} = \frac{4}{3}$$

The slopes are negative reciprocals of each other, so **the lines are perpendicular**.

$$\text{b) line 1: } 2x = y - 10 \Rightarrow y = 2x + 10 \Rightarrow \text{slope} = 2$$

$$\text{line 2: } y = -2x + 5 \Rightarrow \text{slope} = -2$$

The slopes are not the same or negative reciprocals of each other, so **the lines are neither parallel nor perpendicular**.

$$\text{c) line 1: } 7y + 1 = 7x \Rightarrow 7y = 7x - 1 \Rightarrow y = x - \frac{1}{7} \Rightarrow \text{slope} = 1$$

$$\text{line 2: } x + 5 = y \Rightarrow y = x + 5 \Rightarrow \text{slope} = 1$$

The slopes are the same, so **the lines are parallel**.

Write Equations of Parallel and Perpendicular Lines

We can use the properties of parallel and perpendicular lines to write an equation of a line parallel or perpendicular to a given line. You might be given a line and a point, and asked to find the line that goes through the given point and is parallel or perpendicular to the given line. Here's how to do this:

1. Find the slope of the given line from its equation. (You might need to re-write the equation in a form such as the slope-intercept form.)
2. Find the slope of the parallel or perpendicular line—which is either the same as the slope you found in step 1 (if it's parallel), or the negative reciprocal of the slope you found in step 1 (if it's perpendicular).
3. Use the slope you found in step 2, along with the point you were given, to write an equation of the

new line in slope-intercept form or point-slope form.

Example 3

Find an equation of the line perpendicular to the line $y = -3x + 5$ that passes through the point $(2, 6)$.

Solution

The slope of the given line is -3 , so the perpendicular line will have a slope of $\frac{1}{3}$.

Now to find the equation of a line with slope $\frac{1}{3}$ that passes through $(2, 6)$:

Start with the slope-intercept form: $y = mx + b$.

Plug in the slope: $y = \frac{1}{3}x + b$.

Plug in the point $(2, 6)$ to find b : $6 = \frac{1}{3}(2) + b \Rightarrow b = 6 - \frac{2}{3} \Rightarrow b = \frac{20}{3}$.

The equation of the line is $y = \frac{1}{3}x + \frac{20}{3}$.

Example 4

Find the equation of the line perpendicular to $x - 5y = 15$ that passes through the point $(-2, 5)$.

Solution

Re-write the equation in slope-intercept form: $x - 5y = 15 \Rightarrow -5y = -x + 15 \Rightarrow y = \frac{1}{5}x - 3$.

The slope of the given line is $\frac{1}{5}$, so we're looking for a line with slope -5 .

Start with the slope-intercept form: $y = mx + b$.

Plug in the slope: $y = -5x + b$.

Plug in the point $(-2, 5)$: $5 = -5(-2) + b \Rightarrow b = 5 - 10 \Rightarrow b = -5$

The equation of the line is $y = -5x - 5$.

Example 5

Find the equation of the line parallel to $6x - 5y = 12$ that passes through the point $(-5, -3)$.

Solution

Rewrite the equation in slope-intercept form: $6x - 5y = 12 \Rightarrow 5y = 6x - 12 \Rightarrow y = \frac{6}{5}x - \frac{12}{5}$.

The slope of the given line is $\frac{6}{5}$, so we are looking for a line with slope $\frac{6}{5}$ that passes through the point $(-5, -3)$.

Start with the slope-intercept form: $y = mx + b$.

Plug in the slope: $y = \frac{6}{5}x + b$.

Plug in the point $(-5, -3)$: $-3 = \frac{6}{5}(-5) + b \Rightarrow -3 = -6 + b \Rightarrow b = 3$

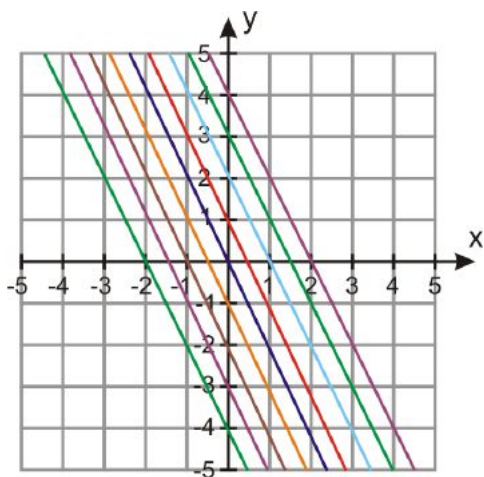
The equation of the line is $y = \frac{6}{5}x + 3$.

Investigate Families of Lines

A **family of lines** is a set of lines that have something in common with each other. Straight lines can belong to two types of families: one where the slope is the same and one where the y -intercept is the same.

Family 1: Keep the slope unchanged and vary the y -intercept.

The figure below shows the family of lines with equations of the form $y = -2x + b$:

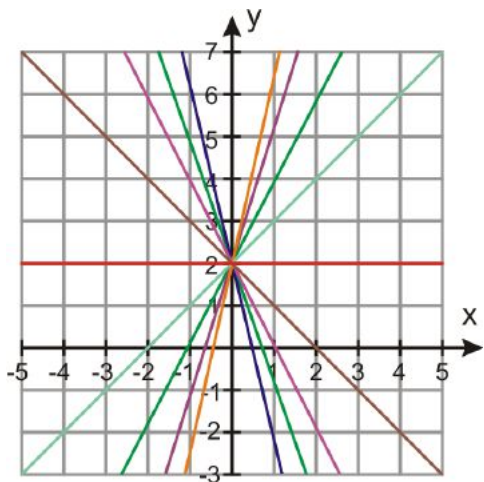


All the lines have a slope of -2 , but the value of b is different for each line.

Notice that in such a family all the lines are parallel. All the lines look the same, except that they are shifted up and down the y -axis. As b gets larger the line rises on the y -axis, and as b gets smaller the line goes lower on the y -axis. This behavior is often called a **vertical shift**.

Family 2: Keep the y -intercept unchanged and vary the slope.

The figure below shows the family of lines with equations of the form $y = mx + 2$:



All the lines have a y -intercept of two, but the slope is different for each line. The steeper lines have higher values of m .

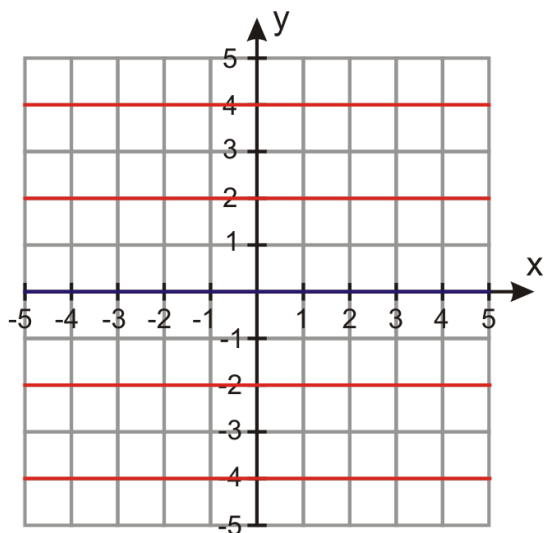
Example 6

Write the equation of the family of lines satisfying the given condition.

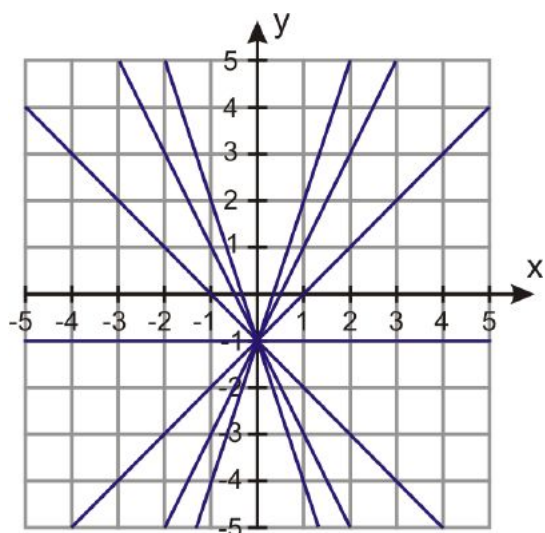
- parallel to the x -axis
- through the point $(0, -1)$
- perpendicular to $2x + 7y - 9 = 0$
- parallel to $x + 4y - 12 = 0$

Solution

a) All lines parallel to the x -axis have a slope of zero; the y -intercept can be anything. So the family of lines is $y = 0x + b$ or just $y = b$.

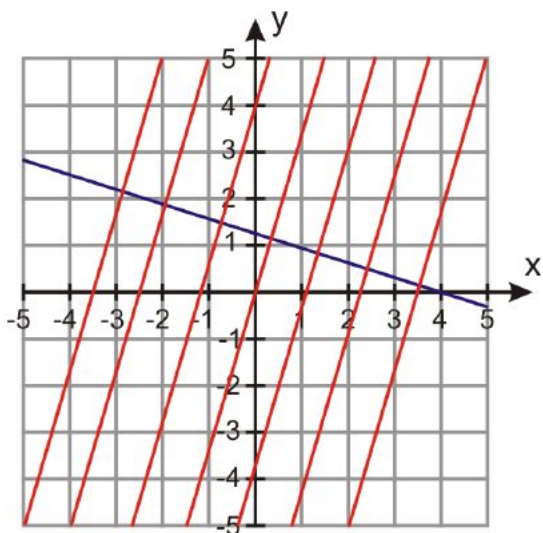


b) All lines passing through the point $(0, -1)$ have the same y-intercept, $b = -1$. The family of lines is: $y = mx - 1$.



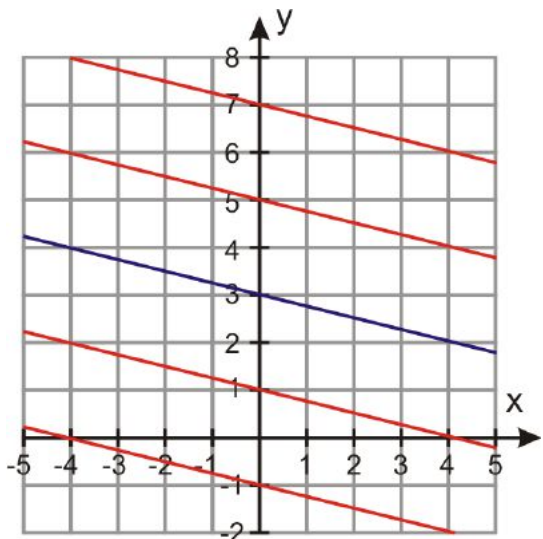
c) First we need to find the slope of the given line. Rewriting $2x + 7y - 9 = 0$ in slope-intercept form, we get $y = -\frac{2}{7}x + \frac{9}{7}$. The slope of the line is $-\frac{2}{7}$, so we're looking for the family of lines with slope $\frac{7}{2}$.

The family of lines is $y = \frac{7}{2}x + b$.



d) Rewrite $x + 4y - 12 = 0$ in slope-intercept form: $y = -\frac{1}{4}x + 3$. The slope is $-\frac{1}{4}$, so that's also the slope of the family of lines we are looking for.

The family of lines is $y = -\frac{1}{4}x + b$.



Review Questions

For questions 1-10, determine whether the lines are parallel, perpendicular or neither.

1. One line passes through the points $(-1, 4)$ and $(2, 6)$; another line passes through the points $(2, -3)$ and $(8, 1)$.
2. One line passes through the points $(4, -3)$ and $(-8, 0)$; another line passes through the points $(-1, -1)$ and $(-2, 6)$.
3. One line passes through the points $(-3, 14)$ and $(1, -2)$; another line passes through the points $(0, -3)$ and $(-2, 5)$.
4. One line passes through the points $(3, 3)$ and $(-6, -3)$; another line passes through the points $(2, -8)$ and $(-6, 4)$.
5. One line passes through the points $(2, 8)$ and $(6, 0)$; another line has the equation $x - 2y = 5$.

6. One line passes through the points $(-5, 3)$ and $(2, -1)$; another line has the equation $2x + 3y = 6$.
7. Both lines pass through the point $(2, 8)$; one line also passes through $(3, 5)$, and the other line has slope 3.
8. Line 1: $4y + x = 8$ Line 2: $12y + 3x = 1$
9. Line 1: $5y + 3x = 1$ Line 2: $6y + 10x = -3$
10. Line 1: $2y - 3x + 5 = 0$ Line 2: $y + 6x = -3$
11. Lines A, B, C, D , and E all pass through the point $(3, 6)$. Line A also passes through $(7, 12)$; line B passes through $(8, 4)$; line C passes through $(-1, -3)$; line D passes through $(1, 1)$; and line E passes through $(6, 12)$.
 - (a) Are any of these lines perpendicular? If so, which ones? If not, why not?
 - (b) Are any of these lines parallel? If so, which ones? If not, why not?
12. Find the equation of the line parallel to $5x - 2y = 2$ that passes through point $(3, -2)$.
13. Find the equation of the line perpendicular to $y = -\frac{2}{5}x - 3$ that passes through point $(2, 8)$.
14. Find the equation of the line parallel to $7y + 2x - 10 = 0$ that passes through the point $(2, 2)$.
15. Find the equation of the line perpendicular to $y + 5 = 3(x - 2)$ that passes through the point $(6, 2)$.
16. Line S passes through the points $(2, 3)$ and $(4, 7)$. Line T passes through the point $(2, 5)$. If Lines S and T are parallel, name one more point on line T . (**Hint:** you don't need to find the slope of either line.)
17. Lines P and Q both pass through $(-1, 5)$. Line P also passes through $(-3, -1)$. If P and Q are perpendicular, name one more point on line Q . (This time you will have to find the slopes of both lines.)
18. Write the equation of the family of lines satisfying the given condition.
 - (a) All lines that pass through point $(0, 4)$.
 - (b) All lines that are perpendicular to $4x + 3y - 1 = 0$.
 - (c) All lines that are parallel to $y - 3 = 4x + 2$.
 - (d) All lines that pass through the point $(0, -1)$.
19. Name two lines that pass through the point $(3, -1)$ and are perpendicular to each other.
20. Name two lines that are each perpendicular to $y = -4x - 2$. What is the relationship of those two lines to each other?
21. Name two perpendicular lines that both pass through the point $(3, -2)$. Then name a line parallel to one of them that passes through the point $(-2, 5)$.

3.3 Fitting a Line to Data

Learning Objectives

- Make a scatter plot.
- Fit a line to data and write an equation for that line.
- Perform linear regression with a graphing calculator.
- Solve real-world problems using linear models of scattered data.

Introduction

Katja has noticed that sales are falling off at her store lately. She plots her sales figures for each week on a graph and sees that the points are trending downward, but they don't quite make a straight line. How can she predict what her sales figures will be over the next few weeks?

In real-world problems, the relationship between our dependent and independent variables is linear, but not perfectly so. We may have a number of data points that don't quite fit on a straight line, but we may still want to find an equation representing those points. In this lesson, we'll learn how to find linear equations to fit real-world data.

Make a Scatter Plot

A **scatter plot** is a plot of all the ordered pairs in a table. Even when we expect the relationship we're analyzing to be linear, we usually can't expect that all the points will fit perfectly on a straight line. Instead, the points will be "scattered" about a straight line.

There are many reasons why the data might not fall perfectly on a line. Small errors in measurement are one reason; another reason is that the real world isn't always as simple as a mathematical abstraction, and sometimes math can only describe it approximately.

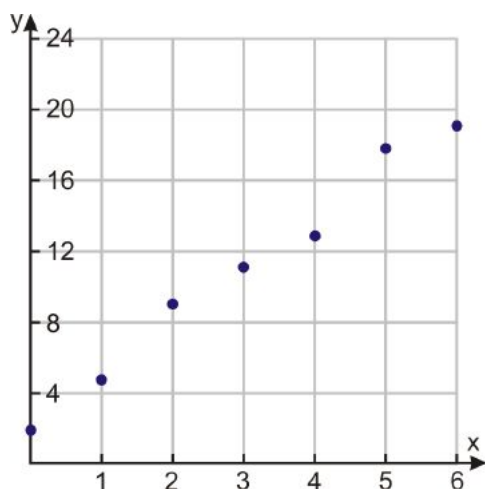
Example 1

Make a scatter plot of the following ordered pairs:

$(0, 2)$; $(1, 4.5)$; $(2, 9)$; $(3, 11)$; $(4, 13)$; $(5, 18)$; $(6, 19.5)$

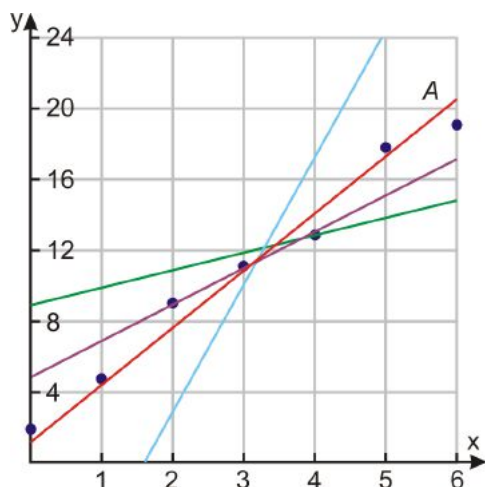
Solution

We make a scatter plot by graphing all the ordered pairs on the coordinate axis:

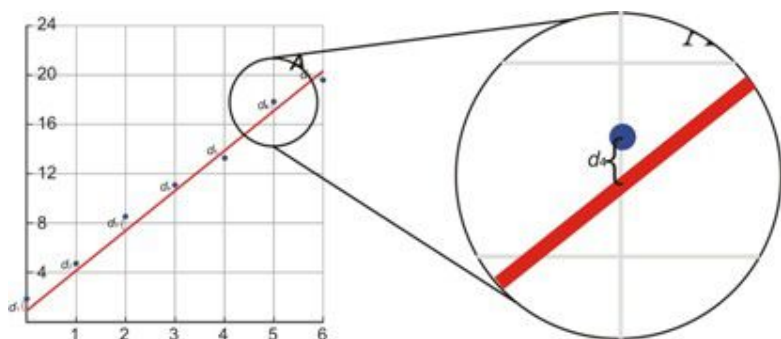


Fit a Line to Data

Notice that the points look like they might be part of a straight line, although they wouldn't fit perfectly on a straight line. If the points were perfectly lined up, we could just draw a line through any two of them, and that line would go right through all the other points as well. When the points aren't lined up perfectly, we just have to find a line that is as close to all the points as possible.



Here you can see that we could draw many lines through the points in our data set. However, the red line A is the line that best fits the points. To prove this mathematically, we would measure all the distances from each data point to line A: and then we would show that the sum of all those distances—or rather, the square root of the sum of the squares of the distances—is less than it would be for any other line.



Actually proving this is a lesson for a much more advanced course, so we won't do it here. And finding the best fit line in the first place is even more complex; instead of doing it by hand, we'll use a graphing calculator or just “eyeball” the line, as we did above—using our visual sense to guess what line fits best.

For more practice eyeballing lines of best fit, try the Java applet at <http://mste.illinois.edu/activity/regression/>. Click on the green field to place up to 50 points on it, then use the slider to adjust the slope of the red line to try and make it fit the points. (The thermometer shows how far away the line is from the points, so you want to try to make the thermometer reading as *low* as possible.) Then click “Show Best Fit” to show the actual best fit line in blue. Refresh the page or click “Reset” if you want to try again. For more of a challenge, try scattering the points in a less obvious pattern.

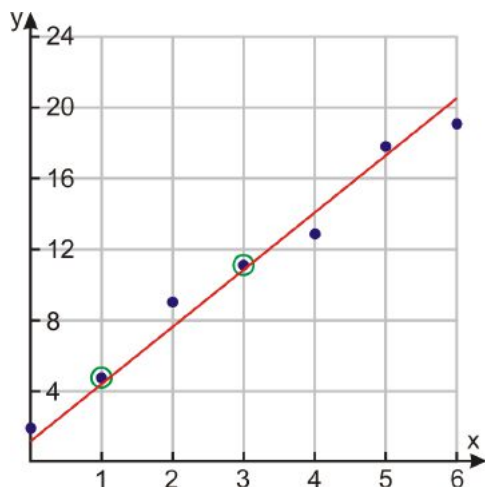
Write an Equation For a Line of Best Fit

Once you draw the line of best fit, you can find its equation by using two points on the line. Finding the equation of the line of best fit is also called **linear regression**.

Caution: Make sure you don't get caught making a common mistake. Sometimes the line of best fit won't pass straight through any of the points in the original data set. This means that you can't just use two points from the data set – **you need to use two points that are on the line**, which might not be in the data set at all.

In Example 1, it happens that two of the data points *are* very close to the line of best fit, so we can just

use these points to find the equation of the line: (1, 4.5) and (3, 11).



Start with the slope-intercept form of a line: $y = mx + b$

Find the slope: $m = \frac{11-4.5}{3-1} = \frac{6.5}{2} = 3.25$.

So $y = 3.25x + b$.

Plug (3, 11) into the equation: $11 = 3.25(3) + b \Rightarrow b = 1.25$

So the equation for the line that fits the data best is $y = 3.25x + 1.25$.

Perform Linear Regression With a Graphing Calculator

The problem with eyeballing a line of best fit, of course, is that you can't be sure how accurate your guess is. To get the most accurate equation for the line, we can use a graphing calculator instead. The calculator uses a mathematical algorithm to find the line that minimizes the sum of the squares.

Example 2

Use a graphing calculator to find the equation of the line of best fit for the following data:

(3, 12), (8, 20), (1, 7), (10, 23), (5, 18), (8, 24), (11, 30), (2, 10)

Solution

Step 1: Input the data in your calculator.

Press [STAT] and choose the [EDIT] option. Input the data into the table by entering the x -values in the first column and the y -values in the second column.

| L1 | L2 | L3 | 2 |
|------------|-------|----|---|
| 1 | 7 | | |
| 10 | 23 | | |
| 5 | 18 | | |
| 8 | 24 | | |
| 11 | 30 | | |
| 2 | 10 | | |
| ----- | ----- | | |
| L2(8) = 10 | | | |

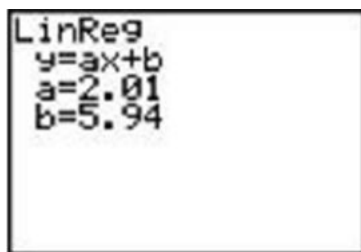
Step 2: Find the equation of the line of best fit.

Press [STAT] again use right arrow to select [CALC] at the top of the screen.

Chose option number 4, $\text{LinReg}(ax + b)$, and press [ENTER]

The calculator will display $LinReg(ax + b)$.

Press **[ENTER]** and you will be given the a - and b -values.



LinReg
 $y = ax + b$
 $a = 2.01$
 $b = 5.94$

Here a represents the slope and b represents the y -intercept of the equation. The linear regression line is $y = 2.01x + 5.94$.

Step 3. Draw the scatter plot.

To draw the scatter plot press **[STATPLOT]** **[2nd]** **[Y=]**.



Plot1 Plot2 Plot3
On Off
Type:   
Xlist: L1
Ylist: L2
Mark:   

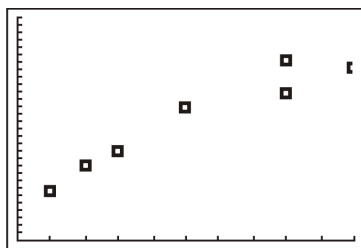
Choose Plot 1 and press **[ENTER]**.

Press the On option and set the Type as scatter plot (the one highlighted in black).

Make sure that the X list and Y list names match the names of the columns of the table in Step 1.

Choose the box or plus as the mark, since the simple dot may make it difficult to see the points.

Press **[GRAPH]** and adjust the window size so you can see all the points in the scatter plot.

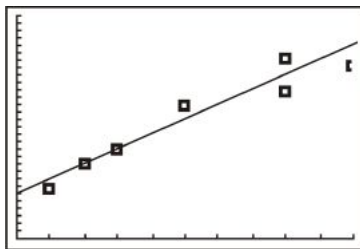


Step 4. Draw the line of best fit through the scatter plot.

Press **[Y=]**

Enter the equation of the line of best fit that you just found: $y = 2.01x + 5.94$.

Press **[GRAPH]**.



Solve Real-World Problems Using Linear Models of Scattered Data

Once we've found the line of best fit for a data set, we can use the equation of that line to predict other data points.

Example 3

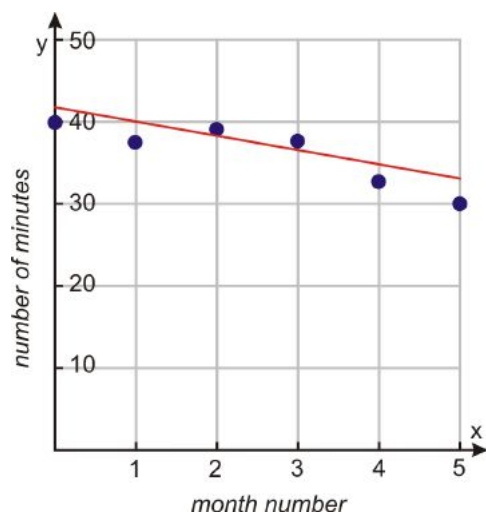
Nadia is training for a 5K race. The following table shows her times for each month of her training program. Find an equation of a line of fit. Predict her running time if her race is in August.

Table 3.1:

| Month | Month number | Average time (minutes) |
|----------|--------------|------------------------|
| January | 0 | 40 |
| February | 1 | 38 |
| March | 2 | 39 |
| April | 3 | 38 |
| May | 4 | 33 |
| June | 5 | 30 |

Solution

Let's make a scatter plot of Nadia's running times. The independent variable, x , is the month number and the dependent variable, y , is the running time. We plot all the points in the table on the coordinate plane, and then sketch a line of fit.



Two points on the line are (0, 42) and (4, 34). We'll use them to find the equation of the line:

$$m = \frac{34 - 42}{4 - 0} = -\frac{8}{4} = -2$$

$$y = -2x + b$$

$$42 = -2(0) + b \Rightarrow b = 42$$

$$y = -2x + 42$$

In a real-world problem, the slope and y-intercept have a physical significance. In this case, the slope tells us how Nadia's running time changes each month she trains. Specifically, it decreases by 2 minutes per month. Meanwhile, the y-intercept tells us that when Nadia started training, she ran a distance of 5K in 42 minutes.

The problem asks us to predict Nadia's running time in August. Since June is defined as month number 5, August will be month number 7. We plug $x = 7$ into the equation of the line of best fit:

$$y = -2(7) + 42 = -14 + 42 = 28$$

The equation predicts that **Nadia will run the 5K race in 28 minutes.**

In this solution, we eyeballed a line of fit. Using a graphing calculator, we can find this equation for a line of fit instead: $y = -2.2x + 43.7$

If we plug $x = 7$ into this equation, we get $y = -2.2(7) + 43.7 = 28.3$. This means that **Nadia will run her race in 28.3 minutes.** You see that the graphing calculator gives a different equation and a different answer to the question. The graphing calculator result is more accurate, but the line we drew by hand still gives a good approximation to the result. And of course, there's no guarantee that Nadia will actually finish the race in that exact time; both answers are estimates, it's just that the calculator's estimate is slightly more likely to be right.

Example 4

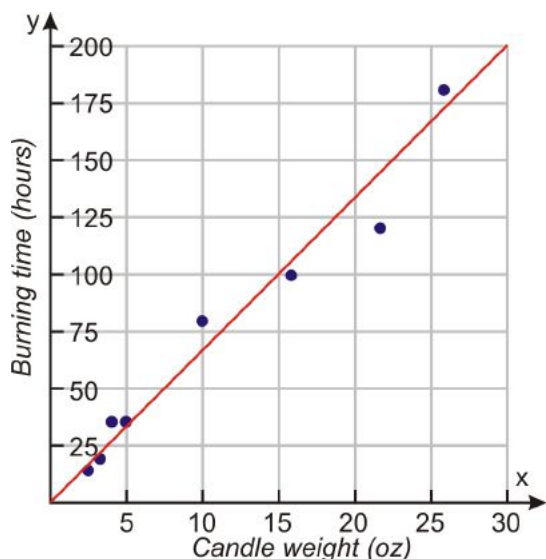
Peter is testing the burning time of "BriteGlo" candles. The following table shows how long it takes to burn candles of different weights. Assume it's a linear relation, so we can use a line to fit the data. If a candle burns for 95 hours, what must be its weight in ounces?

Table 3.2:

| Candle weight (oz) | Time (hours) |
|--------------------|--------------|
| 2 | 15 |
| 3 | 20 |
| 4 | 35 |
| 5 | 36 |
| 10 | 80 |
| 16 | 100 |
| 22 | 120 |
| 26 | 180 |

Solution

Let's make a scatter plot of the data. The independent variable, x , is the candle weight and the dependent variable, y , is the time it takes the candle to burn. We plot all the points in the table on the coordinate plane, and draw a line of fit.



Two convenient points on the line are (0,0) and (30, 200). Find the equation of the line:

$$\begin{aligned}
 m &= \frac{200}{30} = \frac{20}{3} \\
 y &= \frac{20}{3}x + b \\
 0 &= \frac{20}{3}(0) + b \Rightarrow b = 0 \\
 y &= \frac{20}{3}x
 \end{aligned}$$

A slope of $\frac{20}{3} = 6\frac{2}{3}$ tells us that for each extra ounce of candle weight, the burning time increases by $6\frac{2}{3}$ hours. A y -intercept of zero tells us that a candle of weight 0 oz will burn for 0 hours.

The problem asks for the weight of a candle that burns 95 hours; in other words, what's the x -value that gives a y -value of 95? Plugging in $y = 95$:

$$y = \frac{20}{3}x \Rightarrow 95 = \frac{20}{3}x \Rightarrow x = \frac{285}{20} = \frac{57}{4} = 14\frac{1}{4}$$

A candle that burns 95 hours weighs 14.25 oz.

A graphing calculator gives the linear regression equation as $y = 6.1x + 5.9$ and a result of **14.6 oz**.

Review Questions

For problems 1-4, draw the scatter plot and find an equation that fits the data set by hand.

- (57, 45); (65, 61); (34, 30); (87, 78); (42, 41); (35, 36); (59, 35); (61, 57); (25, 23); (35, 34)
- (32, 43); (54, 61); (89, 94); (25, 34); (43, 56); (58, 67); (38, 46); (47, 56); (39, 48)
- (12, 18); (5, 24); (15, 16); (11, 19); (9, 12); (7, 13); (6, 17); (12, 14)
- (3, 12); (8, 20); (1, 7); (10, 23); (5, 18); (8, 24); (2, 10)
- Use the graph from problem 1 to predict the y -values for two x -values of your choice that are not in the data set.
- Use the graph from problem 2 to predict the x -values for two y -values of your choice that are not in the data set.

7. Use the equation from problem 3 to predict the y -values for two x -values of your choice that are not in the data set.
8. Use the equation from problem 4 to predict the x -values for two y -values of your choice that are not in the data set.

For problems 9-11, use a graphing calculator to find the equation of the line of best fit for the data set.

9. (57, 45); (65, 61); (34, 30); (87, 78); (42, 41); (35, 36); (59, 35); (61, 57); (25, 23); (35, 34)
10. (32, 43); (54, 61); (89, 94); (25, 34); (43, 56); (58, 67); (38, 46); (47, 56); (95, 105); (39, 48)
11. (12, 18); (3, 26); (5, 24); (15, 16); (11, 19); (0, 27); (9, 12); (7, 13); (6, 17); (12, 14)
12. Graph the best fit line on top of the scatter plot for problem 10. Then pick a data point that's close to the line, and change its y -value to move it much farther from the line.
 - (a) Calculate the new best fit line with that one point changed; write the equation of that line along with the coordinates of the new point.
 - (b) How much did the slope of the best fit line change when you changed that point?
13. Graph the scatter plot from problem 11 and change one point as you did in the previous problem.
 - (a) Calculate the new best fit line with that one point changed; write the equation of that line along with the coordinates of the new point.
 - (b) Did changing that one point seem to affect the slope of the best fit line more or less than it did in the previous problem? What might account for this difference?
14. Shiva is trying to beat the samosa-eating record. The current record is 53.5 samosas in 12 minutes. Each day he practices and the following table shows how many samosas he eats each day for the first week of his training.

Table 3.3:

| Day | No. of samosas |
|-----|----------------|
| 1 | 30 |
| 2 | 34 |
| 3 | 36 |
| 4 | 36 |
| 5 | 40 |
| 6 | 43 |
| 7 | 45 |

- (a) Draw a scatter plot and find an equation to fit the data.
 - (b) Will he be ready for the contest if it occurs two weeks from the day he started training?
 - (c) What are the meanings of the slope and the y -intercept in this problem?
15. Anne is trying to find the elasticity coefficient of a Superball. She drops the ball from different heights and measures the maximum height of the ball after the bounce. The table below shows the data she collected.

Table 3.4:

| Initial height (cm) | Bounce height (cm) |
|---------------------|--------------------|
| 30 | 22 |
| 35 | 26 |
| 40 | 29 |
| 45 | 34 |
| 50 | 38 |
| 55 | 40 |
| 60 | 45 |
| 65 | 50 |
| 70 | 52 |

- (a) Draw a scatter plot and find the equation.
- (b) What height would she have to drop the ball from for it to bounce 65 cm?
- (c) What are the meanings of the slope and the y-intercept in this problem?
- (d) Does the y-intercept make sense? Why isn't it (0, 0)?
16. The following table shows the median California family income from 1995 to 2002 as reported by the US Census Bureau.

Table 3.5:

| Year | Income |
|------|--------|
| 1995 | 53,807 |
| 1996 | 55,217 |
| 1997 | 55,209 |
| 1998 | 55,415 |
| 1999 | 63,100 |
| 2000 | 63,206 |
| 2001 | 63,761 |
| 2002 | 65,766 |

- (a) Draw a scatter plot and find the equation.
- (b) What would you expect the median annual income of a Californian family to be in year 2010?
- (c) What are the meanings of the slope and the y-intercept in this problem?
- (d) Inflation in the U.S. is measured by the Consumer Price Index, which increased by 20% between 1995 and 2002. Did the median income of California families keep up with inflation over that time period? (In other words, did it increase by at least 20%?)

3.4 Predicting with Linear Models

Learning Objectives

- Interpolate using an equation.
- Extrapolate using an equation.
- Predict using an equation.

Introduction

Katja's sales figures were trending downward quickly at first, and she used a line of best fit to describe the numbers. But now they seem to be decreasing more slowly, and fitting the line less and less accurately. How can she make a more accurate prediction of what next week's sales will be?

In the last lesson we saw how to find the equation of a line of best fit and how to use this equation to make predictions. The line of “best fit” is a good method if the relationship between the dependent and the independent variables is linear. In this section you will learn other methods that are useful even when the relationship isn't linear.

Linear Interpolation

We use linear interpolation to fill in gaps in our data—that is, to estimate values that fall in between the values we already know. To do this, we use a straight line to connect the known data points on either side of the unknown point, and use the equation of that line to estimate the value we are looking for.

Example 1

The following table shows the median ages of first marriage for men and women, as gathered by the U.S. Census Bureau.

Table 3.6:

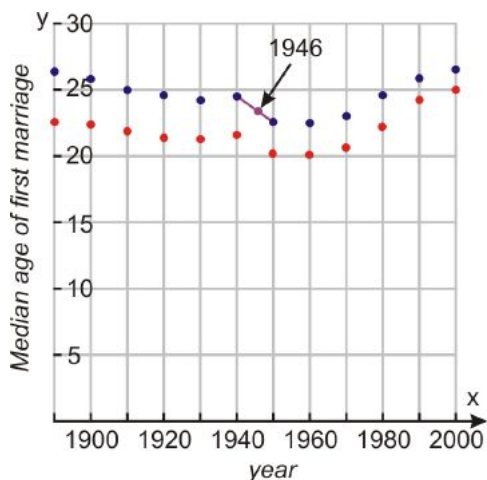
| Year | Median age of males | Median age of females |
|------|---------------------|-----------------------|
| 1890 | 26.1 | 22.0 |
| 1900 | 25.9 | 21.9 |
| 1910 | 25.1 | 21.6 |
| 1920 | 24.6 | 21.2 |
| 1930 | 24.3 | 21.3 |
| 1940 | 24.3 | 21.5 |
| 1950 | 22.8 | 20.3 |
| 1960 | 22.8 | 20.3 |
| 1970 | 23.2 | 20.8 |
| 1980 | 24.7 | 22.0 |
| 1990 | 26.1 | 23.9 |
| 2000 | 26.8 | 25.1 |

Estimate the median age for the first marriage of a male in the year 1946.

Solution

We connect the two points on either side of 1946 with a straight line and find its equation. Here's how

that looks on a scatter plot:



We find the equation by plugging in the two data points:

$$m = \frac{22.8 - 24.3}{1950 - 1940} = \frac{-1.5}{10} = -0.15$$

$$y = -0.15x + b$$

$$24.3 = -0.15(1940) + b$$

$$b = 315.3$$

Our equation is $y = -0.15x + 315.3$.

To estimate the median age of marriage of males in the year 1946, we plug $x = 1946$ into the equation we just found:

$$y = -0.15(1946) + 315.3 = 23.4 \text{ years old}$$

Example 2

The Center for Disease Control collects information about the health of the American people and behaviors that might lead to bad health. The following table shows the percent of women who smoke during pregnancy.

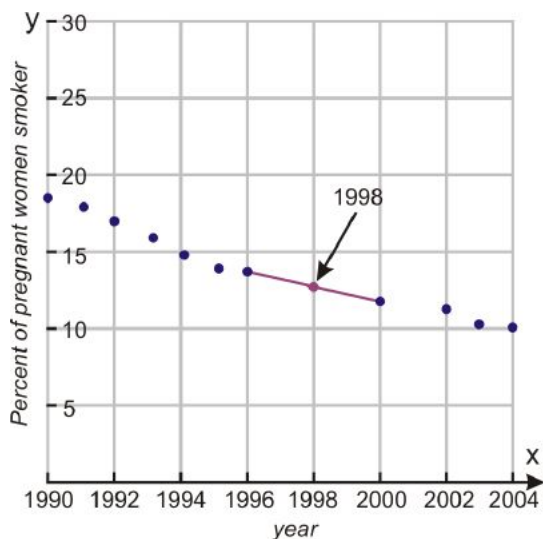
Table 3.7:

| Year | Percent of pregnant women smokers |
|------|-----------------------------------|
| 1990 | 18.4 |
| 1991 | 17.7 |
| 1992 | 16.9 |
| 1993 | 15.8 |
| 1994 | 14.6 |
| 1995 | 13.9 |
| 1996 | 13.6 |
| 2000 | 12.2 |
| 2002 | 11.4 |
| 2003 | 10.4 |
| 2004 | 10.2 |

Estimate the percentage of pregnant women that were smoking in the year 1998.

Solution

We connect the two points on either side of 1998 with a straight line and find its equation. Here's how that looks on a scatter plot:



We find the equation by plugging in the two data points:

$$\begin{aligned}m &= \frac{12.2 - 13.6}{2000 - 1996} = \frac{-1.4}{4} = -0.35 \\y &= -0.35x + b \\12.2 &= -0.35(2000) + b \\b &= 712.2\end{aligned}$$

Our equation is $y = -0.35x + 712.2$.

To estimate the percentage of pregnant women who smoked in the year 1998, we plug $x = 1998$ into the equation we just found:

$$y = -0.35(1998) + 712.2 = 12.9\%$$

For non-linear data, linear interpolation is often not accurate enough for our purposes. If the points in the data set change by a large amount in the interval you're interested in, then linear interpolation may not give a good estimate. In that case, it can be replaced by **polynomial interpolation**, which uses a curve instead of a straight line to estimate values between points. But that's beyond the scope of this lesson.

Linear Extrapolation

Linear extrapolation can help us estimate values that are outside the range of our data set. The strategy is similar to linear interpolation: we pick the two data points that are closest to the one we're looking for, find the equation of the line between them, and use that equation to estimate the coordinates of the missing point.

Example 3

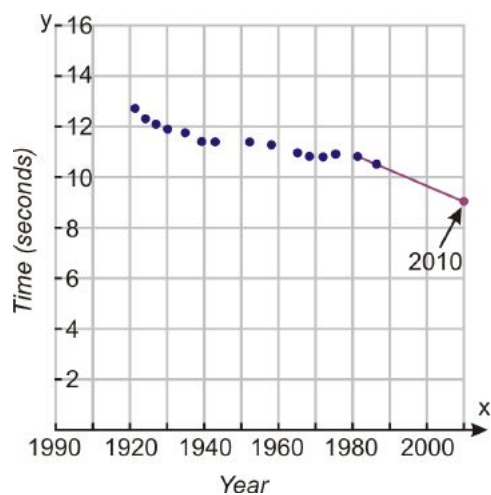
The winning times for the women's 100 meter race are given in the following table. Estimate the winning time in the year 2010. Is this a good estimate?

Table 3.8:

| Winner | Country | Year | Time (seconds) |
|--------------------------|--------------|------|----------------|
| Mary Lines | UK | 1922 | 12.8 |
| Leni Schmidt | Germany | 1925 | 12.4 |
| Gerturd Glasitsch | Germany | 1927 | 12.1 |
| Tollien Schuurman | Netherlands | 1930 | 12.0 |
| Helen Stephens | USA | 1935 | 11.8 |
| Lulu Mae Hymes | USA | 1939 | 11.5 |
| Fanny Blankers-Koen | Netherlands | 1943 | 11.5 |
| Marjorie Jackson | Australia | 1952 | 11.4 |
| Vera Krepkina | Soviet Union | 1958 | 11.3 |
| Wyomia Tyus | USA | 1964 | 11.2 |
| Barbara Ferrell | USA | 1968 | 11.1 |
| Ellen Strophal | East Germany | 1972 | 11.0 |
| Inge Helten | West Germany | 1976 | 11.0 |
| Marlies Gohr | East Germany | 1982 | 10.9 |
| Florence Griffith Joyner | USA | 1988 | 10.5 |

Solution

We start by making a scatter plot of the data; then we connect the last two points on the graph and find the equation of the line.



$$m = \frac{10.5 - 10.9}{1988 - 1982} = \frac{-0.4}{6} = -0.067$$

$$y = -0.067x + b$$

$$10.5 = -0.067(1988) + b$$

$$b = 143.7$$

Our equation is $y = -0.067x + 143.7$.

The winning time in year 2010 is estimated to be:

$$y = -0.067(2010) + 143.7 = 9.03 \text{ seconds.}$$

Unfortunately, this estimate actually isn't very accurate. This example demonstrates the weakness of linear extrapolation; it uses only a couple of points, instead of using *all* the points like the best fit line method, so

it doesn't give as accurate results when the data points follow a linear pattern. In this particular example, the last data point clearly doesn't fit in with the general trend of the data, so the slope of the extrapolation line is much steeper than it would be if we'd used a line of best fit. (As a historical note, the last data point corresponds to the winning time for Florence Griffith Joyner in 1988. After her race she was accused of using performance-enhancing drugs, but this fact was never proven. In addition, there was a question about the accuracy of the timing: some officials said that tail-wind was not accounted for in this race, even though all the other races of the day were affected by a strong wind.)

Here's an example of a problem where linear extrapolation does work better than the line of best fit method.

Example 4

A cylinder is filled with water to a height of 73 centimeters. The water is drained through a hole in the bottom of the cylinder and measurements are taken at 2 second intervals. The following table shows the height of the water level in the cylinder at different times.

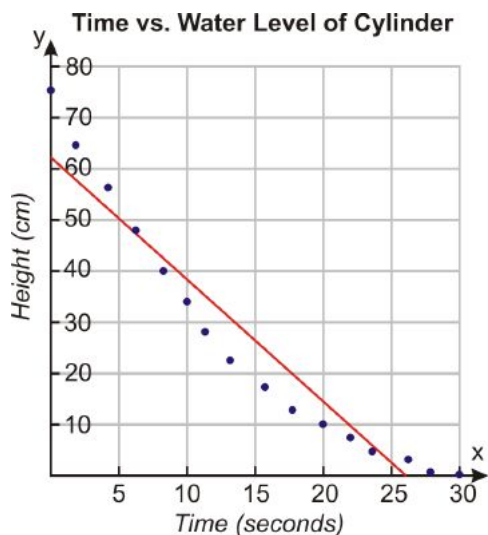
Table 3.9:

| Time (seconds) | Water level (cm) |
|----------------|------------------|
| 0.0 | 73 |
| 2.0 | 63.9 |
| 4.0 | 55.5 |
| 6.0 | 47.2 |
| 8.0 | 40.0 |
| 10.0 | 33.4 |
| 12.0 | 27.4 |
| 14.0 | 21.9 |
| 16.0 | 17.1 |
| 18.0 | 12.9 |
| 20.0 | 9.4 |
| 22.0 | 6.3 |
| 24.0 | 3.9 |
| 26.0 | 2.0 |
| 28.0 | 0.7 |
| 30.0 | 0.1 |

- a) *Find the water level at time 15 seconds.*
- b) *Find the water level at time 27 seconds*
- c) *What would be the original height of the water in the cylinder if the water takes 5 extra seconds to drain? (Find the height at time of -5 seconds.)*

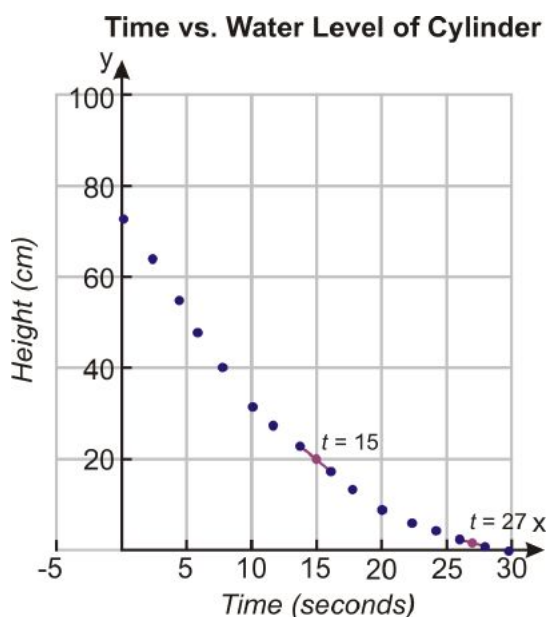
Solution

Here's what the line of best fit would look like for this data set:



Notice that the data points don't really make a line, and so the line of best fit still isn't a terribly good fit. Just a glance tells us that we'd estimate the water level at 15 seconds to be about 27 cm, which is *more* than the water level at 14 seconds. That's clearly not possible! Similarly, at 27 seconds we'd estimate the water to have all drained out, which it clearly hasn't yet.

So let's see what happens if we use linear interpolation and extrapolation instead. First, here are the lines we'd use to interpolate between 14 and 16 seconds, and between 26 and 28 seconds.



a) The slope of the line between points (14, 21.9) and (16, 17.1) is $m = \frac{17.1-21.9}{16-14} = \frac{-4.8}{2} = -2.4$. So $y = -2.4x + b \Rightarrow 21.9 = -2.4(14) + b \Rightarrow b = 55.5$, and the equation is $y = -2.4x + 55.5$.

Plugging in $x = 15$ gives us $y = -2.4(15) + 55.5 = 19.5$ cm.

b) The slope of the line between points (26, 2) and (28, 0.7) is $m = \frac{0.7-2}{28-26} = \frac{-1.3}{2} = -0.65$, so $y = -0.65x + b \Rightarrow 2 = -0.65(26) + b \Rightarrow b = 18.9$, and the equation is $y = -0.65x + 18.9$.

Plugging in $x = 27$, we get $y = -0.65(27) + 18.9 = 1.35$ cm.

c) Finally, we can use extrapolation to estimate the height of the water at -5 seconds. The slope of the line between points (0, 73) and (2, 63.9) is $m = \frac{63.9-73}{2-0} = \frac{-9.1}{2} = -4.55$, so the equation of the line is

$$y = -4.55x + 73.$$

Plugging in $x = -5$ gives us $y = -4.55(-5) + 73 = 95.75$ cm.

To make linear interpolation easier in the future, you might want to use the calculator at http://www.ajdesigner.com/php_interpolation_equation.php. Plug in the coordinates of the first known data point in the blanks labeled x_1 and y_1 , and the coordinates of the second point in the blanks labeled x_3 and y_3 ; then enter the x -coordinate of the point in between in the blank labeled x_2 , and the y -coordinate will be displayed below when you click “Calculate.”

Review Questions

1. Use the data from Example 1 (*Median age at first marriage*) to estimate the age at marriage for females in 1946. Fit a line, by hand, to the data before 1970.
2. Use the data from Example 1 (*Median age at first marriage*) to estimate the age at marriage for females in 1984. Fit a line, by hand, to the data from 1970 on in order to estimate this accurately.
3. Use the data from Example 1 (*Median age at first marriage*) to estimate the age at marriage for males in 1995. Use linear interpolation between the 1990 and 2000 data points.
4. Use the data from Example 2 (*Pregnant women and smoking*) to estimate the percentage of pregnant smokers in 1997. Use linear interpolation between the 1996 and 2000 data points.
5. Use the data from Example 2 (*Pregnant women and smoking*) to estimate the percentage of pregnant smokers in 2006. Use linear extrapolation with the final two data points.
6. Use the data from Example 3 (*Winning times*) to estimate the winning time for the female 100-meter race in 1920. Use linear extrapolation because the first two or three data points have a different slope than the rest of the data.
7. The table below shows the highest temperature vs. the hours of daylight for the 15th day of each month in the year 2006 in San Diego, California.

Table 3.10:

| Hours of daylight | High temperature (F) |
|-------------------|----------------------|
| 10.25 | 60 |
| 11.0 | 62 |
| 12 | 62 |
| 13 | 66 |
| 13.8 | 68 |
| 14.3 | 73 |
| 14 | 86 |
| 13.4 | 75 |
| 12.4 | 71 |
| 11.4 | 66 |
| 10.5 | 73 |
| 10 | 61 |

- (a) What would be a better way to organize this table if you want to make the relationship between daylight hours and temperature easier to see?
- (b) Estimate the high temperature for a day with 13.2 hours of daylight using linear interpolation.
- (c) Estimate the high temperature for a day with 9 hours of daylight using linear extrapolation. Is the

prediction accurate?

(d) Estimate the high temperature for a day with 9 hours of daylight using a line of best fit.

The table below lists expected life expectancies based on year of birth (US Census Bureau). Use it to answer questions 8-15.

Table 3.11:

| Birth year | Life expectancy in years |
|------------|--------------------------|
| 1930 | 59.7 |
| 1940 | 62.9 |
| 1950 | 68.2 |
| 1960 | 69.7 |
| 1970 | 70.8 |
| 1980 | 73.7 |
| 1990 | 75.4 |
| 2000 | 77 |

8. Make a scatter plot of the data.
9. Use a line of best fit to estimate the life expectancy of a person born in 1955.
10. Use linear interpolation to estimate the life expectancy of a person born in 1955.
11. Use a line of best fit to estimate the life expectancy of a person born in 1976.
12. Use linear interpolation to estimate the life expectancy of a person born in 1976.
13. Use a line of best fit to estimate the life expectancy of a person born in 2012.
14. Use linear extrapolation to estimate the life expectancy of a person born in 2012.
15. Which method gives better estimates for this data set? Why?

The table below lists the high temperature for the first day of the month for the year 2006 in San Diego, California (Weather Underground). Use it to answer questions 16-21.

Table 3.12:

| Month number | Temperature (F) |
|--------------|-----------------|
| 1 | 63 |
| 2 | 66 |
| 3 | 61 |
| 4 | 64 |
| 5 | 71 |
| 6 | 78 |
| 7 | 88 |
| 8 | 78 |
| 9 | 81 |
| 10 | 75 |
| 11 | 68 |
| 12 | 69 |

16. Draw a scatter plot of the data.
17. Use a line of best fit to estimate the temperature in the middle of the 4th month (month 4.5).

18. Use linear interpolation to estimate the temperature in the middle of the 4th month (month 4.5).
19. Use a line of best fit to estimate the temperature for month 13 (January 2007).
20. Use linear extrapolation to estimate the temperature for month 13 (January 2007).
21. Which method gives better estimates for this data set? Why?
22. Name a real-world situation where you might want to make predictions based on available data. Would linear extrapolation/interpolation or the best fit method be better to use in that situation? Why?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9615>.

Chapter 4

Exponential Functions

4.1 Exponent Properties Involving Products

Learning Objectives

- Use the product of a power property.
- Use the power of a product property.
- Simplify expressions involving product properties of exponents.

Introduction

Back in chapter 1, we briefly covered expressions involving exponents, like 3^5 or x^3 . In these expressions, the number on the bottom is called the **base** and the number on top is the **power** or **exponent**. The whole expression is equal to the base multiplied by itself a number of times equal to the exponent; in other words, the exponent tells us how many copies of the base number to multiply together.

Example 1

Write in exponential form.

- a) $2 \cdot 2$
- b) $(-3)(-3)(-3)$
- c) $y \cdot y \cdot y \cdot y \cdot y$
- d) $(3a)(3a)(3a)(3a)$

Solution

- a) $2 \cdot 2 = 2^2$ because we have 2 factors of 2
- b) $(-3)(-3)(-3) = (-3)^3$ because we have 3 factors of (-3)
- c) $y \cdot y \cdot y \cdot y \cdot y = y^5$ because we have 5 factors of y
- d) $(3a)(3a)(3a)(3a) = (3a)^4$ because we have 4 factors of $3a$

When the base is a variable, it's convenient to leave the expression in exponential form; if we didn't write x^7 , we'd have to write $x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$ instead. But when the base is a number, we can simplify the expression further than that; for example, 2^7 equals $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, but we can multiply all those 2's to get 128.

Let's simplify the expressions from Example 1.

Example 2

Simplify.

a) 2^2

b) $(-3)^3$

c) y^5

d) $(3a)^4$

Solution

a) $2^2 = 2 \cdot 2 = 4$

b) $(-3)^3 = (-3)(-3)(-3) = -27$

c) y^5 is already simplified

d) $(3a)^4 = (3a)(3a)(3a)(3a) = 3 \cdot 3 \cdot 3 \cdot 3 \cdot a \cdot a \cdot a \cdot a = 81a^4$

Be careful when taking powers of negative numbers. Remember these rules:

$$(\text{negative number}) \cdot (\text{positive number}) = \text{negative number}$$

$$(\text{negative number}) \cdot (\text{negative number}) = \text{positive number}$$

So **even powers of negative numbers** are always positive. Since there are an even number of factors, we pair up the negative numbers and all the negatives cancel out.

$$(-2)^6 = (-2)(-2)(-2)(-2)(-2)(-2) = \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4} = +64$$

And **odd powers of negative numbers** are always negative. Since there are an odd number of factors, we can still pair up negative numbers to get positive numbers, but there will always be one negative factor left over, so the answer is negative:

$$(-2)^5 = (-2)(-2)(-2)(-2)(-2) = \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)}_{-2} = -32$$

Use the Product of Powers Property

So what happens when we multiply one power of x by another? Let's see what happens when we multiply x **to the power of 5** by x **cubed**. To illustrate better, we'll use the full factored form for each:

$$\underbrace{(x \cdot x \cdot x \cdot x \cdot x)}_{x^5} \cdot \underbrace{(x \cdot x \cdot x)}_{x^3} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^8}$$

So $x^5 \times x^3 = x^8$. You may already see the pattern to multiplying powers, but let's confirm it with another example. We'll multiply x **squared** by x **to the power of 4**:

$$\underbrace{(x \cdot x)}_{x^2} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^6}$$

So $x^2 \times x^4 = x^6$. Look carefully at the powers and how many factors there are in each calculation. 5 x 's times 3 x 's equals $(5 + 3) = 8$ x 's. 2 x 's times 4 x 's equals $(2 + 4) = 6$ x 's.

You should see that when we take the product of two powers of x , the number of x 's in the answer is the total number of x 's in all the terms you are multiplying. In other words, the exponent in the answer is the sum of the exponents in the product.

Product Rule for Exponents: $x^n \cdot x^m = x^{(n+m)}$

There are some easy mistakes you can make with this rule, however. Let's see how to avoid them.

Example 3

Multiply $2^2 \cdot 2^3$.

Solution

$$2^2 \cdot 2^3 = 2^5 = 32$$

Note that when you use the product rule you **don't multiply the bases**. In other words, you must avoid the common error of writing $2^2 \cdot 2^3 = 4^5$. You can see this is true if you multiply out each expression: 4 times 8 is definitely 32, not 1024.

Example 4

Multiply $2^2 \cdot 3^3$.

Solution

$$2^2 \cdot 3^3 = 4 \cdot 27 = 108$$

In this case, we can't actually use the product rule at all, because it only applies to terms that have the *same base*. In a case like this, where the bases are different, we just have to multiply out the numbers by hand—the answer is *not* 2^5 or 3^5 or 6^5 or anything simple like that.

Use the Power of a Product Property

What happens when we raise a whole expression to a power? Let's take x *to the power of 4* and *cube it*. Again we'll use the full factored form for each expression:

$$(x^4)^3 = x^4 \times x^4 \times x^4 \quad 3 \text{ factors of } \{x \text{ to the power } 4\}$$

$$(x \cdot x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x = x^{12}$$

So $(x^4)^3 = x^{12}$. You can see that when we raise a power of x to a new power, the powers multiply.

Power Rule for Exponents: $(x^n)^m = x^{(n \cdot m)}$

If we have a product of more than one term inside the parentheses, then we have to distribute the exponent over all the factors, like distributing multiplication over addition. For example:

$$(x^2y)^4 = (x^2)^4 \cdot (y)^4 = x^8y^4.$$

Or, writing it out the long way:

$$(x^2y)^4 = (x^2y)(x^2y)(x^2y)(x^2y) = (x \cdot x \cdot y)(x \cdot x \cdot y)(x \cdot x \cdot y)(x \cdot x \cdot y)$$

$$= x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y = x^8y^4$$

Note that this does NOT work if you have a sum or difference inside the parentheses! For example, $(x+y)^2 \neq x^2+y^2$. This is an easy mistake to make, but you can avoid it if you remember what an exponent means: if you multiply out $(x+y)^2$ it becomes $(x+y)(x+y)$, and that's not the same as x^2+y^2 . We'll learn how we can simplify this expression in a later chapter.

The following video from YourTeacher.com may make it clearer how the power rule works for a variety of exponential expressions:

http://www.youtube.com/watch?v=Mm4y_I8-hoU

Example 5

Simplify the following expressions.

a) $3^5 \cdot 3^7$

b) $2^6 \cdot 2$

c) $(4^2)^3$

Solution

When we're just working with numbers instead of variables, we can use the product rule and the power rule, or we can just do the multiplication and then simplify.

a) We can use the product rule first and then evaluate the result: $3^5 \cdot 3^7 = 3^{12} = 531441$.

OR we can evaluate each part separately and then multiply them: $3^5 \cdot 3^7 = 243 \cdot 2187 = 531441$.

b) We can use the product rule first and then evaluate the result: $2^6 \cdot 2 = 2^7 = 128$.

OR we can evaluate each part separately and then multiply them: $2^6 \cdot 2 = 64 \cdot 2 = 128$.

c) We can use the power rule first and then evaluate the result: $(4^2)^3 = 4^6 = 4096$.

OR we can evaluate the expression inside the parentheses first, and then apply the exponent outside the parentheses: $(4^2)^3 = (16)^3 = 4096$.

Example 6

Simplify the following expressions.

a) $x^2 \cdot x^7$

b) $(y^3)^5$

Solution

When we're just working with variables, all we can do is simplify as much as possible using the product and power rules.

a) $x^2 \cdot x^7 = x^{2+7} = x^9$

b) $(y^3)^5 = y^{3 \times 5} = y^{15}$

Example 7

Simplify the following expressions.

a) $(3x^2y^3) \cdot (4xy^2)$

b) $(4xyz) \cdot (x^2y^3) \cdot (2yz^4)$

c) $(2a^3b^3)^2$

Solution

When we have a mix of numbers and variables, we apply the rules to each number and variable separately.

a) First we group like terms together: $(3x^2y^3) \cdot (4xy^2) = (3 \cdot 4) \cdot (x^2 \cdot x) \cdot (y^3 \cdot y^2)$

Then we multiply the numbers or apply the product rule on each grouping: $= 12x^3y^5$

b) Group like terms together: $(4xyz) \cdot (x^2y^3) \cdot (2yz^4) = (4 \cdot 2) \cdot (x \cdot x^2) \cdot (y \cdot y^3 \cdot y) \cdot (z \cdot z^4)$

Multiply the numbers or apply the product rule on each grouping: $= 8x^3y^5z^5$

c) Apply the power rule for each separate term in the parentheses: $(2a^3b^3)^2 = 2^2 \cdot (a^3)^2 \cdot (b^3)^2$
Multiply the numbers or apply the power rule for each term $= 4a^6b^6$

Example 8

Simplify the following expressions.

a) $(x^2)^2 \cdot x^3$

b) $(2x^2y) \cdot (3xy^2)^3$

c) $(4a^2b^3)^2 \cdot (2ab^4)^3$

Solution

In problems where we need to apply the product and power rules together, we must keep in mind the order of operations. Exponent operations take precedence over multiplication.

a) We apply the power rule first: $(x^2)^2 \cdot x^3 = x^4 \cdot x^3$

Then apply the product rule to combine the two terms: $x^4 \cdot x^3 = x^7$

b) Apply the power rule first: $(2x^2y) \cdot (3xy^2)^3 = (2x^2y) \cdot (27x^3y^6)$

Then apply the product rule to combine the two terms: $(2x^2y) \cdot (27x^3y^6) = 54x^5y^7$

c) Apply the power rule on each of the terms separately: $(4a^2b^3)^2 \cdot (2ab^4)^3 = (16a^4b^6) \cdot (8a^3b^{12})$

Then apply the product rule to combine the two terms: $(16a^4b^6) \cdot (8a^3b^{12}) = 128a^7b^{18}$

Review Questions

Write in exponential notation:

1. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

2. $3x \cdot 3x \cdot 3x$

3. $(-2a)(-2a)(-2a)(-2a)$

4. $6 \cdot 6 \cdot 6 \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y$

5. $2 \cdot x \cdot y \cdot 2 \cdot 2 \cdot y \cdot x$

Find each number.

6. 5^4

7. $(-2)^6$

8. $(0.1)^5$

9. $(-0.6)^3$

10. $(1.2)^2 + 5^3$

11. $3^2 \cdot (0.2)^3$

Multiply and simplify:

12. $6^3 \cdot 6^6$

13. $2^2 \cdot 2^4 \cdot 2^6$

14. $3^2 \cdot 4^3$

15. $x^2 \cdot x^4$

16. $(-2y^4)(-3y)$

17. $(4a^2)(-3a)(-5a^4)$

Simplify:

18. $(a^3)^4$

19. $(xy)^2$

20. $(3a^2b^3)^4$

21. $(-2xy^4z^2)^5$

22. $(-8x)^3(5x)^2$

23. $(4a^2)(-2a^3)^4$

24. $(12xy)(12xy)^2$

25. $(2xy^2)(-x^2y)^2(3x^2y^2)$

4.2 Exponent Properties Involving Quotients

Learning Objectives

- Use the quotient of powers property.
- Use the power of a quotient property.
- Simplify expressions involving quotient properties of exponents.

Use the Quotient of Powers Property

The rules for simplifying quotients of exponents are a lot like the rules for simplifying products. Let's look at what happens when we divide x^7 by x^4 :

$$\frac{x^7}{x^4} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = \frac{x \cdot x \cdot x}{1} = x^3$$

You can see that when we divide two powers of x , the number of x 's in the solution is the number of x 's in the top of the fraction minus the number of x 's in the bottom. In other words, when dividing expressions with the same base, we keep the same base and simply subtract the exponent in the denominator from the exponent in the numerator.

Quotient Rule for Exponents: $\frac{x^n}{x^m} = x^{(n-m)}$

When we have expressions with more than one base, we apply the quotient rule separately for each base:

$$\frac{x^5y^3}{x^3y^2} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}} \cdot \frac{\cancel{y} \cdot \cancel{y} \cdot y}{\cancel{y} \cdot \cancel{y}} = \frac{x \cdot x}{1} \cdot \frac{y}{1} = x^2y \quad \text{OR} \quad \frac{x^5y^3}{x^3y^2} = x^{5-3} \cdot y^{3-2} = x^2y$$

Example 1

Simplify each of the following expressions using the quotient rule.

a) $\frac{x^{10}}{x^5}$

b) $\frac{a^6}{a}$

c) $\frac{a^5b^4}{a^3b^2}$

Solution

a) $\frac{x^{10}}{x^5} = x^{10-5} = x^5$

$$\text{b) } \frac{a^6}{a} = a^{6-1} = a^5$$

$$\text{c) } \frac{a^5 b^4}{a^3 b^2} = a^{5-3} \cdot b^{4-2} = a^2 b^2$$

Now let's see what happens if the exponent in the denominator is bigger than the exponent in the numerator. For example, what happens when we apply the quotient rule to $\frac{x^4}{x^7}$?

The quotient rule tells us to subtract the exponents. 4 minus 7 is -3, so our answer is x^{-3} . A negative exponent! What does that mean?

Well, let's look at what we get when we do the division longhand by writing each term in factored form:

$$\frac{x^4}{x^7} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x} = \frac{1}{x \cdot x \cdot x} = \frac{1}{x^3}$$

Even when the exponent in the denominator is bigger than the exponent in the numerator, we can still subtract the powers. The x 's that are left over after the others have been canceled out just end up in the denominator instead of the numerator. Just as $\frac{x^7}{x^4}$ would be equal to $\frac{x^3}{1}$ (or simply x^3), $\frac{x^4}{x^7}$ is equal to $\frac{1}{x^3}$. And you can also see that $\frac{1}{x^3}$ is equal to x^{-3} . We'll learn more about negative exponents shortly.

Example 2

Simplify the following expressions, leaving all exponents positive.

$$\text{a) } \frac{x^2}{x^6}$$

$$\text{b) } \frac{a^2 b^6}{a^5 b}$$

Solution

a) Subtract the exponent in the numerator from the exponent in the denominator and leave the x 's in the denominator: $\frac{x^2}{x^6} = \frac{1}{x^{6-2}} = \frac{1}{x^4}$

b) Apply the rule to each variable separately: $\frac{a^2 b^6}{a^5 b} = \frac{1}{a^{5-2}} \cdot \frac{b^{6-1}}{1} = \frac{b^5}{a^3}$

The Power of a Quotient Property

When we raise a whole quotient to a power, another special rule applies. Here is an example:

$$\left(\frac{x^3}{y^2}\right)^4 = \left(\frac{x^3}{y^2}\right) \cdot \left(\frac{x^3}{y^2}\right) \cdot \left(\frac{x^3}{y^2}\right) \cdot \left(\frac{x^3}{y^2}\right) = \frac{(x \cdot x \cdot x) \cdot (x \cdot x \cdot x) \cdot (x \cdot x \cdot x) \cdot (x \cdot x \cdot x)}{(y \cdot y) \cdot (y \cdot y) \cdot (y \cdot y) \cdot (y \cdot y)} = \frac{x^{12}}{y^8}$$

Notice that the exponent outside the parentheses is multiplied by the exponent in the numerator and the exponent in the denominator, separately. This is called the power of a quotient rule:

$$\textbf{Power Rule for Quotients: } \left(\frac{x^n}{y^m}\right)^p = \frac{x^{n \cdot p}}{y^{m \cdot p}}$$

Let's apply these new rules to a few examples.

Example 3

Simplify the following expressions.

$$\text{a) } \frac{4^5}{4^2}$$

$$\text{b) } \frac{5^3}{5^7}$$

$$\text{c) } \left(\frac{3^4}{5^2}\right)^2$$

Solution

Since there are just numbers and no variables, we can evaluate the expressions and get rid of the exponents completely.

a) We can use the quotient rule first and then evaluate the result: $\frac{4^5}{4^2} = 4^{5-2} = 4^3 = 64$

OR we can evaluate each part separately and then divide: $\frac{4^5}{4^2} = \frac{1024}{16} = 64$

b) Use the quotient rule first and then evaluate the result: $\frac{5^3}{5^7} = \frac{1}{5^4} = \frac{1}{625}$

OR evaluate each part separately and then reduce: $\frac{5^3}{5^7} = \frac{125}{78125} = \frac{1}{625}$

Notice that it makes more sense to apply the quotient rule first for examples (a) and (b). Applying the exponent rules to simplify the expression *before* plugging in actual numbers means that we end up with smaller, easier numbers to work with.

c) Use the power rule for quotients first and then evaluate the result: $\left(\frac{3^4}{5^2}\right)^2 = \frac{3^8}{5^4} = \frac{6561}{625}$

OR evaluate inside the parentheses first and then apply the exponent: $\left(\frac{3^4}{5^2}\right)^2 = \left(\frac{81}{25}\right)^2 = \frac{6561}{625}$

Example 4

Simplify the following expressions:

a) $\frac{x^{12}}{x^5}$

b) $\left(\frac{x^4}{x}\right)^5$

Solution

a) Use the quotient rule: $\frac{x^{12}}{x^5} = x^{12-5} = x^7$

b) Use the power rule for quotients and then the quotient rule: $\left(\frac{x^4}{x}\right)^5 = \frac{x^{20}}{x^5} = x^{15}$

OR use the quotient rule inside the parentheses first, then apply the power rule: $\left(\frac{x^4}{x}\right)^5 = (x^3)^5 = x^{15}$

Example 5

Simplify the following expressions.

a) $\frac{6x^2y^3}{2xy^2}$

b) $\left(\frac{2a^3b^3}{8a^7b}\right)^2$

Solution

When we have a mix of numbers and variables, we apply the rules to each number or each variable separately.

a) Group like terms together: $\frac{6x^2y^3}{2xy^2} = \frac{6}{2} \cdot \frac{x^2}{x} \cdot \frac{y^3}{y^2}$

Then reduce the numbers and apply the quotient rule on each fraction to get $3xy$.

b) Apply the quotient rule inside the parentheses first: $\left(\frac{2a^3b^3}{8a^7b}\right)^2 = \left(\frac{b^2}{4a^4}\right)^2$

Then apply the power rule for quotients: $\left(\frac{b^2}{4a^4}\right)^2 = \frac{b^4}{16a^8}$

Example 6

Simplify the following expressions.

a) $(x^2)^2 \cdot \frac{x^6}{x^4}$

b) $\left(\frac{16a^2}{4b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$

Solution

In problems where we need to apply several rules together, we must keep the order of operations in mind.

a) We apply the power rule first on the first term:

$$(x^2)^2 \cdot \frac{x^6}{x^4} = x^4 \cdot \frac{x^6}{x^4}$$

Then apply the quotient rule to simplify the fraction:

$$x^4 \cdot \frac{x^6}{x^4} = x^4 \cdot x^2$$

And finally simplify with the product rule:

$$x^4 \cdot x^2 = x^6$$

b) $\left(\frac{16a^2}{4b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$

Simplify inside the parentheses by reducing the numbers:

$$\left(\frac{4a^2}{b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$$

Then apply the power rule to the first fraction:

$$\left(\frac{4a^2}{b^5}\right)^3 \cdot \frac{b^2}{a^{16}} = \frac{64a^6}{b^{15}} \cdot \frac{b^2}{a^{16}}$$

Group like terms together:

$$\frac{64a^6}{b^{15}} \cdot \frac{b^2}{a^{16}} = 64 \cdot \frac{a^6}{a^{16}} \cdot \frac{b^2}{b^{15}}$$

And apply the quotient rule to each fraction:

$$64 \cdot \frac{a^6}{a^{16}} \cdot \frac{b^2}{b^{15}} = \frac{64}{a^{10}b^{13}}$$

Review Questions

Evaluate the following expressions.

1. $\frac{5^6}{5^2}$
2. $\frac{6^7}{6^3}$
3. $\frac{3^4}{3^{10}}$
4. $\left(\frac{3}{8}\right)^2$
5. $\left(\frac{2^2}{3^3}\right)^3$
6. $\frac{2^2 \cdot 3^2}{5^2}$
7. $\frac{3^3 \cdot 5^2}{3^7}$
8. $\left(\frac{2^3 \cdot 4^2}{2^4}\right)^2$

Simplify the following expressions.

9. $\frac{a^3}{a^2}$
10. $\frac{x^5}{x^9}$
11. $\left(\frac{a^3b^4}{a^2b}\right)^3$
12. $\frac{x^6y^2}{x^2y^5}$
13. $\frac{6a^3}{2a^2}$
14. $\frac{15x^5}{5x}$
15. $\left(\frac{18a^4}{15a^{10}}\right)^4$
16. $\frac{25yx^6}{20y^5x^2}$
17. $\left(\frac{x^6y^2}{x^4y^4}\right)^3$
18. $\left(\frac{6a^2}{4b^4}\right)^2 \cdot \frac{5b}{3a}$
19. $\frac{(3ab)^2(4a^3b^4)^3}{(6a^2b)^4}$
20. $\frac{(2a^2bc^2)(6abc^3)}{4ab^2c}$
21. $\frac{(2a^2bc^2)(6abc^3)}{4ab^2c}$ for $a = 2, b = 1$, and $c = 3$
22. $\left(\frac{3x^2y}{2z}\right)^3 \cdot \frac{z^2}{x}$ for $x = 1, y = 2$, and $z = -1$
23. $\frac{2x^3}{xy^2} \cdot \left(\frac{x}{2y}\right)^2$ for $x = 2, y = -3$
24. $\frac{2x^3}{xy^2} \cdot \left(\frac{x}{2y}\right)^2$ for $x = 0, y = 6$
25. If $a = 2$ and $b = 3$, simplify $\frac{(a^2b)(bc)^3}{a^3c^2}$ as much as possible.

4.3 Zero, Negative, and Fractional Exponents

Learning Objectives

- Simplify expressions with negative exponents.
- Simplify expressions with zero exponents.
- Simplify expression with fractional exponents.
- Evaluate exponential expressions.

Introduction

The product and quotient rules for exponents lead to many interesting concepts. For example, so far we've mostly just considered positive, whole numbers as exponents, but you might be wondering what happens when the exponent isn't a positive whole number. What does it mean to raise something to the power of zero, or -1, or $\frac{1}{2}$? In this lesson, we'll find out.

Simplify Expressions With Negative Exponents

When we learned the quotient rule for exponents $\left(\frac{x^n}{x^m} = x^{(n-m)}\right)$, we saw that it applies even when the exponent in the denominator is bigger than the one in the numerator. Canceling out the factors in the numerator and denominator leaves the leftover factors in the denominator, and subtracting the exponents leaves a

negative number. So negative exponents simply represent fractions with exponents in the denominator. This can be summarized in a rule:

Negative Power Rule for Exponents: $x^{-n} = \frac{1}{x^n}$, where $x \neq 0$

Negative exponents can be applied to products and quotients also. Here's an example of a negative exponent being applied to a product:

$$\begin{aligned}(x^3y)^{-2} &= x^{-6}y^{-2} && \text{using the power rule} \\ x^{-6}y^{-2} &= \frac{1}{x^6} \cdot \frac{1}{y^2} = \frac{1}{x^6y^2} && \text{using the negative power rule separately on each variable}\end{aligned}$$

And here's one applied to a quotient:

$$\begin{aligned}\left(\frac{a}{b}\right)^{-3} &= \frac{a^{-3}}{b^{-3}} && \text{using the power rule for quotients} \\ \frac{a^{-3}}{b^{-3}} &= \frac{a^{-3}}{1} \cdot \frac{1}{b^{-3}} = \frac{1}{a^3} \cdot \frac{b^3}{1} && \text{using the negative power rule on each variable separately} \\ \frac{1}{a^3} \cdot \frac{b^3}{1} &= \frac{b^3}{a^3} && \text{simplifying the division of fractions} \\ \frac{b^3}{a^3} &= \left(\frac{b}{a}\right)^3 && \text{using the power rule for quotients in reverse.}\end{aligned}$$

That last step wasn't really necessary, but putting the answer in that form shows us something useful: $\left(\frac{a}{b}\right)^{-3}$ is equal to $\left(\frac{b}{a}\right)^3$. This is an example of a rule we can apply more generally:

Negative Power Rule for Fractions: $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$, where $x \neq 0, y \neq 0$

This rule can be useful when you want to write out an expression without using fractions.

Example 1

Write the following expressions without fractions.

- a) $\frac{1}{x}$
- b) $\frac{2}{x^2}$
- c) $\frac{x^2}{y^3}$
- d) $\frac{3}{xy}$

Solution

- a) $\frac{1}{x} = x^{-1}$
- b) $\frac{2}{x^2} = 2x^{-2}$
- c) $\frac{x^2}{y^3} = x^2y^{-3}$
- d) $\frac{3}{xy} = 3x^{-1}y^{-1}$

Example 2

Simplify the following expressions and write them without fractions.

- a) $\frac{4a^2b^3}{2a^5b}$
- b) $\left(\frac{x}{3y^2}\right)^3 \cdot \frac{x^2y}{4}$

Solution

a) Reduce the numbers and apply the quotient rule to each variable separately:

$$\frac{4a^2b^3}{2a^5b} = 2 \cdot a^{2-5} \cdot b^{3-1} = 2a^{-3}b^2$$

b) Apply the power rule for quotients first:

$$\left(\frac{2x}{y^2}\right)^3 \cdot \frac{x^2y}{4} = \frac{8x^3}{y^6} \cdot \frac{x^2y}{4}$$

Then simplify the numbers, and use the product rule on the x 's and the quotient rule on the y 's:

$$\frac{8x^3}{y^6} \cdot \frac{x^2y}{4} = 2 \cdot x^{3+2} \cdot y^{1-6} = 2x^5y^{-5}$$

You can also use the negative power rule the other way around if you want to write an expression without negative exponents.

Example 3

Write the following expressions without negative exponents.

a) $3x^{-3}$

b) $a^2b^{-3}c^{-1}$

c) $4x^{-1}y^3$

d) $\frac{2x^{-2}}{y^{-3}}$

Solution

a) $3x^{-3} = \frac{3}{x^3}$

b) $a^2b^{-3}c^{-1} = \frac{a^2}{b^3c}$

c) $4x^{-1}y^3 = \frac{4y^3}{x}$

d) $\frac{2x^{-2}}{y^{-3}} = \frac{2y^3}{x^2}$

Example 4

Simplify the following expressions and write the answers without negative powers.

a) $\left(\frac{ab^{-2}}{b^3}\right)^2$

b) $\frac{x^{-3}y^2}{x^2y^{-2}}$

Solution

a) Apply the quotient rule inside the parentheses: $\left(\frac{ab^{-2}}{b^3}\right)^2 = (ab^{-5})^2$

Then apply the power rule: $(ab^{-5})^2 = a^2b^{-10} = \frac{a^2}{b^{10}}$

b) Apply the quotient rule to each variable separately: $\frac{x^{-3}y^2}{x^2y^{-2}} = x^{-3-2}y^{2-(-2)} = x^{-5}y^4 = \frac{y^4}{x^5}$

Simplify Expressions with Exponents of Zero

Let's look again at the quotient rule for exponents $\left(\frac{x^n}{x^m} = x^{(n-m)}\right)$ and consider what happens when $n = m$. For example, what happens when we divide x^4 by x^4 ? Applying the quotient rule tells us that $\frac{x^4}{x^4} = x^{(4-4)} = x^0$ —so what does that zero mean?

Well, we first discovered the quotient rule by considering how the factors of x cancel in such a fraction. Let's do that again with our example of x^4 divided by x^4 :

$$\frac{x^4}{x^4} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = 1$$

So $x^0 = 1$! You can see that this works for any value of the exponent, not just 4:

$$\frac{x^n}{x^n} = x^{(n-n)} = x^0$$

Since there is the same number of x 's in the numerator as in the denominator, they cancel each other out and we get $x^0 = 1$. This rule applies for all expressions:

Zero Rule for Exponents: $x^0 = 1$, where $x \neq 0$

For more on zero and negative exponents, watch the following video at squidoo.com: http://www.google.com/url?sa=t&2ae0dMWsa2wJ_CwzckXNA&cad=rja.

Simplify Expressions With Fractional Exponents

So far we've only looked at expressions where the exponents are positive and negative integers. The rules we've learned work exactly the same if the powers are fractions or irrational numbers—but what does a fractional exponent even mean? Let's see if we can figure that out by using the rules we already know.

Suppose we have an expression like $9^{\frac{1}{2}}$ —how can we relate this expression to one that we already know how to work with? For example, how could we turn it into an expression that doesn't have any fractional exponents?

Well, the power rule tells us that if we raise an exponential expression to a power, we can multiply the exponents. For example, if we raise $9^{\frac{1}{2}}$ to the power of 2, we get $\left(9^{\frac{1}{2}}\right)^2 = 9^{2 \cdot \frac{1}{2}} = 9^1 = 9$.

So if $9^{\frac{1}{2}}$ squared equals 9, what does $9^{\frac{1}{2}}$ itself equal? Well, 3 is the number whose square is 9 (that is, it's the square root of 9), so $9^{\frac{1}{2}}$ must equal 3. And that's true for all numbers and variables: a number raised to the power of $\frac{1}{2}$ is just the square root of the number. We can write that as $\sqrt{x} = x^{\frac{1}{2}}$, and then we can see that's true because $\left(\sqrt{x}\right)^2 = x$ just as $\left(x^{\frac{1}{2}}\right)^2 = x$.

Similarly, a number to the power of $\frac{1}{3}$ is just the cube root of the number, and so on. In general, $x^{\frac{1}{n}} = \sqrt[n]{x}$. And when we raise a number to a power and then take the root of it, we still get a fractional exponent; for example, $\sqrt[3]{x^4} = \left(x^4\right)^{\frac{1}{3}} = x^{\frac{4}{3}}$. In general, the rule is as follows:

Rule for Fractional Exponents: $\sqrt[n]{a^n} = a^{\frac{n}{n}}$ and $\left(\sqrt[n]{a}\right)^n = a^{\frac{n}{n}}$

We'll examine roots and radicals in detail in a later chapter. In this section, we'll focus on how exponent rules apply to fractional exponents.

Example 5

Simplify the following expressions.

a) $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}}$

b) $\left(a^{\frac{1}{3}}\right)^2$

c) $\frac{a^{\frac{5}{2}}}{a^{\frac{1}{2}}}$

d) $\left(\frac{x^2}{y^3}\right)^{\frac{1}{3}}$

Solution

- a) Apply the product rule: $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{2} + \frac{1}{3}} = a^{\frac{5}{6}}$
- b) Apply the power rule: $(a^{\frac{1}{3}})^2 = a^{\frac{2}{3}}$
- c) Apply the quotient rule: $\frac{a^{\frac{5}{2}}}{a^{\frac{1}{2}}} = a^{\frac{5}{2} - \frac{1}{2}} = a^{\frac{4}{2}} = a^2$
- d) Apply the power rule for quotients: $(\frac{x^2}{y^3})^{\frac{1}{3}} = \frac{x^{\frac{2}{3}}}{y}$

Evaluate Exponential Expressions

When evaluating expressions we must keep in mind the order of operations. You must remember **PEMDAS**:

1. Evaluate inside the **P**arentheses.
2. Evaluate **E**xponents.
3. Perform **M**ultiplication and **D**ivision operations from left to right.
4. Perform **A**ddition and **S**ubtraction operations from left to right.

Example 6

Evaluate the following expressions.

- a) 5^0
- b) $(\frac{2}{3})^3$
- c) $16^{\frac{1}{2}}$
- d) $8^{-\frac{1}{3}}$

Solution

- a) $5^0 = 1$ A number raised to the power 0 is always 1.
- b) $(\frac{2}{3})^3 = \frac{2^3}{3^3} = \frac{8}{27}$
- c) $16^{\frac{1}{2}} = \sqrt{16} = 4$ Remember that an exponent of $\frac{1}{2}$ means taking the square root.
- d) $8^{-\frac{1}{3}} = \frac{1}{8^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$ Remember that an exponent of $\frac{1}{3}$ means taking the cube root.

Example 7

Evaluate the following expressions.

- a) $3 \cdot 5^2 - 10 \cdot 5 + 1$
- b) $\frac{2 \cdot 4^2 - 3 \cdot 5^2}{3^2 - 2^2}$
- c) $(\frac{3^3}{2^2})^{-2} \cdot \frac{3}{4}$

Solution

- a) Evaluate the exponent: $3 \cdot 5^2 - 10 \cdot 5 + 1 = 3 \cdot 25 - 10 \cdot 5 + 1$

Perform multiplications from left to right: $3 \cdot 25 - 10 \cdot 5 + 1 = 75 - 50 + 1$

Perform additions and subtractions from left to right: $75 - 50 + 1 = 26$

b) Treat the expressions in the numerator and denominator of the fraction like they are in parentheses:

$$\frac{(2 \cdot 4^2 - 3 \cdot 5^2)}{(3^2 - 2^2)} = \frac{(2 \cdot 16 - 3 \cdot 25)}{(9 - 4)} = \frac{(32 - 75)}{5} = \frac{-43}{5}$$

$$c) \left(\frac{3^3}{2^2}\right)^{-2} \cdot \frac{3}{4} = \left(\frac{2^2}{3^3}\right)^2 \cdot \frac{3}{4} = \frac{2^4}{3^6} \cdot \frac{3}{4} = \frac{2^4}{3^6} \cdot \frac{3}{2^2} = \frac{2^2}{3^5} = \frac{4}{243}$$

Example 8

Evaluate the following expressions for $x = 2, y = -1, z = 3$.

a) $2x^2 - 3y^3 + 4z$

b) $(x^2 - y^2)^2$

c) $\left(\frac{3x^2y^5}{4z}\right)^{-2}$

Solution

a) $2x^2 - 3y^3 + 4z = 2 \cdot 2^2 - 3 \cdot (-1)^3 + 4 \cdot 3 = 2 \cdot 4 - 3 \cdot (-1) + 4 \cdot 3 = 8 + 3 + 12 = 23$

b) $(x^2 - y^2)^2 = (2^2 - (-1)^2)^2 = (4 - 1)^2 = 3^2 = 9$

c) $\left(\frac{3x^2y^5}{4z}\right)^{-2} = \left(\frac{3 \cdot 2^2 \cdot (-1)^5}{4 \cdot 3}\right)^{-2} = \left(\frac{3 \cdot 4 \cdot (-1)}{12}\right)^{-2} = \left(\frac{-12}{12}\right)^{-2} = \left(\frac{-1}{1}\right)^{-2} = \left(\frac{1}{-1}\right)^2 = (-1)^2 = 1$

Review Questions

Simplify the following expressions in such a way that there aren't any negative exponents in the answer.

1. $x^{-1}y^2$

2. x^{-4}

3. $\frac{x^{-3}}{x^{-7}}$

4. $\frac{x^{-3}y^{-5}}{z^{-7}}$

5. $(x^{\frac{1}{2}}y^{\frac{-2}{3}})(x^2y^{\frac{1}{3}})$

6. $\left(\frac{a}{b}\right)^{-2}$

7. $(3a^{-2}b^2c^3)^3$

8. $x^{-3} \cdot x^3$

Simplify the following expressions in such a way that there aren't any fractions in the answer.

9. $\frac{a^{-3}(a^5)}{a^{-6}}$

10. $\frac{5x^6y^2}{x^8y}$

11. $\frac{(4ab^6)^3}{(ab)^5}$

12. $\left(\frac{3x}{y^{\frac{1}{3}}}\right)^3$

13. $\frac{3x^2y^{\frac{3}{2}}}{xy^{\frac{1}{2}}}$

14. $\frac{(3x^3)(4x^4)}{(2y)^2}$

15. $\frac{a^{-2}b^{-3}}{c^{-1}}$

16. $\frac{x^{\frac{1}{2}}y^{\frac{5}{2}}}{x^{\frac{3}{2}}y^{\frac{3}{2}}}$

Evaluate the following expressions to a single number.

17. 3^{-2}
18. $(6.2)^0$
19. $8^{-4} \cdot 8^6$
20. $(16^{\frac{1}{2}})^3$
21. $x^2 \cdot 4x^3 \cdot y^4 \cdot 4y^2$, if $x = 2$ and $y = -1$
22. $a^4(b^2)^3 + 2ab$, if $a = -2$ and $b = 1$
23. $5x^2 - 2y^3 + 3z$, if $x = 3, y = 2$, and $z = 4$
24. $(\frac{a^2}{b^3})^{-2}$, if $a = 5$ and $b = 3$
25. $(\frac{x^{-2}}{y^4})^{\frac{1}{2}}$, if $x = -3$ and $y = 2$

4.4 Scientific Notation

Learning Objectives

- Write numbers in scientific notation.
- Evaluate expressions in scientific notation.
- Evaluate expressions in scientific notation using a graphing calculator.

Introduction

Consider the number six hundred and forty three thousand, two hundred and ninety seven. We write it as 643,297 and each digit's position has a "value" assigned to it. You may have seen a table like this before:

| hundred-thousands | ten-thousands | thousands | hundreds | tens | units |
|-------------------|---------------|-----------|----------|------|-------|
| 6 | 4 | 3 | 2 | 9 | 7 |

We've seen that when we write an exponent above a number, it means that we have to multiply a certain number of copies of that number together. We've also seen that a zero exponent always gives us 1, and negative exponents give us fractional answers.

Look carefully at the table above. Do you notice that all the column headings are powers of ten? Here they are listed:

$$\begin{aligned}
 100,000 &= 10^5 \\
 10,000 &= 10^4 \\
 1,000 &= 10^3 \\
 100 &= 10^2 \\
 10 &= 10^1
 \end{aligned}$$

Even the "units" column is really just a power of ten. **Unit** means 1, and 1 is 10^0 .

If we divide 643,297 by 100,000 we get 6.43297; if we multiply 6.43297 by 100,000 we get 643,297. But we have just seen that 100,000 is the same as 10^5 , so if we multiply 6.43297 by 10^5 we should also get 643,297. In other words,

$$643,297 = 6.43297 \times 10^5$$

Writing Numbers in Scientific Notation

In scientific notation, numbers are always written in the form $a \times 10^b$, where b is an integer and a is between 1 and 10 (that is, it has exactly 1 nonzero digit before the decimal). This notation is especially useful for numbers that are either very small or very large.

Here's a set of examples:

$$1.07 \times 10^4 = 10,700$$

$$1.07 \times 10^3 = 1,070$$

$$1.07 \times 10^2 = 107$$

$$1.07 \times 10^1 = 10.7$$

$$1.07 \times 10^0 = 1.07$$

$$1.07 \times 10^{-1} = 0.107$$

$$1.07 \times 10^{-2} = 0.0107$$

$$1.07 \times 10^{-3} = 0.00107$$

$$1.07 \times 10^{-4} = 0.000107$$

Look at the first example and notice where the decimal point is in both expressions.

$$1.07 \times 10^4 = 1.07 \times 1000 = 10,700.0$$

decimal point after 1st digit 4 zeros 4 decimal places difference

So the exponent on the ten acts to move the decimal point over to the right. An exponent of 4 moves it 4 places and an exponent of 3 would move it 3 places.

$$1.07 \times 10^3 = 1,070.0$$

3 decimal places difference

$$1.07 \times 10^2 = 107.0$$

2 decimal places difference

This makes sense because each time you multiply by 10, you move the decimal point one place to the right. 1.07 times 10 is 10.7, times 10 again is 107.0, and so on.

Similarly, if you look at the later examples in the table, you can see that a negative exponent on the 10 means the decimal point moves that many places to the left. This is because multiplying by 10^{-1} is the same as multiplying by $\frac{1}{10}$, which is like dividing by 10. So instead of moving the decimal point one place to the right for every multiple of 10, we move it one place to the left for every multiple of $\frac{1}{10}$.

That's how to convert numbers from scientific notation to standard form. When we're converting numbers *to* scientific notation, however, we have to apply the whole process backwards. First we move the decimal point until it's immediately after the first nonzero digit; then we count how many places we moved it. If we moved it to the *left*, the exponent on the 10 is positive; if we moved it to the *right*, it's negative.

So, for example, to write 0.000032 in scientific notation, we'd first move the decimal five places to the right to get 3.2; then, since we moved it right, the exponent on the 10 should be *negative* five, so the number in scientific notation is 3.2×10^{-5} .

You can double-check whether you've got the right direction by comparing the number in scientific notation with the number in standard form, and thinking "Does this represent a *big* number or a *small* number?" A positive exponent on the 10 represents a number bigger than 10 and a negative exponent represents a number smaller than 10, and you can easily tell if the number in standard form is bigger or smaller than 10 just by looking at it.

For more practice, try the online tool at http://hotmath.com/util/hm_flash_movie.html?movie=/learning_activities/interactivities/sciNotation.swf. Click the arrow buttons to move the decimal point until the number in the middle is written in proper scientific notation, and see how the exponent changes as you move the decimal point.

Example 1

Write the following numbers in scientific notation.

- a) 63
- b) 9,654
- c) 653,937,000
- d) 0.003
- e) 0.000056
- f) 0.00005007

Solution

- a) $63 = 6.3 \times 10 = 6.3 \times 10^1$
- b) $9,654 = 9.654 \times 1,000 = 9.654 \times 10^3$
- c) $653,937,000 = 6.53937000 \times 100,000,000 = 6.53937 \times 10^8$
- d) $0.003 = 3 \times \frac{1}{1000} = 3 \times 10^{-3}$
- e) $0.000056 = 5.6 \times \frac{1}{100,000} = 5.6 \times 10^{-5}$
- f) $0.00005007 = 5.007 \times \frac{1}{100,000} = 5.007 \times 10^{-5}$

Evaluating Expressions in Scientific Notation

When we are faced with products and quotients involving scientific notation, we need to remember the rules for exponents that we learned earlier. It's relatively straightforward to work with scientific notation problems if you remember to combine all the powers of 10 together. The following examples illustrate this.

Example 2

Evaluate the following expressions and write your answer in scientific notation.

- a) $(3.2 \times 10^6) \cdot (8.7 \times 10^{11})$
- b) $(5.2 \times 10^{-4}) \cdot (3.8 \times 10^{-19})$
- c) $(1.7 \times 10^6) \cdot (2.7 \times 10^{-11})$

Solution

The key to evaluating expressions involving scientific notation is to group the powers of 10 together and deal with them separately.

- a) $(3.2 \times 10^6)(8.7 \times 10^{11}) = \underbrace{3.2 \times 8.7}_{27.84} \times \underbrace{10^6 \times 10^{11}}_{10^{17}} = 27.84 \times 10^{17}$. But 27.84×10^{17} isn't in proper scientific

notation, because it has more than one digit before the decimal point. We need to move the decimal point one more place to the left and add 1 to the exponent, which gives us 2.784×10^{18} .

$$\text{b) } (5.2 \times 10^{-4})(3.8 \times 10^{-19}) = \underbrace{5.2 \times 3.8}_{19.76} \times \underbrace{10^{-4} \times 10^{-19}}_{10^{-23}} = 19.76 \times 10^{-23} = 1.976 \times 10^{-22}$$

$$\text{c) } (1.7 \times 10^6)(2.7 \times 10^{-11}) = \underbrace{1.7 \times 2.7}_{4.59} \times \underbrace{10^6 \times 10^{-11}}_{10^{-5}} = 4.59 \times 10^{-5}$$

When we use scientific notation in the real world, we often round off our calculations. Since we're often dealing with very big or very small numbers, it can be easier to round off so that we don't have to keep track of as many digits—and scientific notation helps us with that by saving us from writing out all the extra zeros. For example, if we round off 4,227, 457,903 to 4,200,000,000, we can then write it in scientific notation as simply 4.2×10^9 .

When rounding, we often talk of **significant figures** or **significant digits**. Significant figures include

- all nonzero digits
- all zeros that come *before* a nonzero digit and *after* either a decimal point or another nonzero digit

For example, the number 4000 has one significant digit; the zeros don't count because there's no nonzero digit after them. But the number 4000.5 has five significant digits: the 4, the 5, and all the zeros in between. And the number 0.003 has three significant digits: the 3 and the two zeros that come between the 3 and the decimal point.

Example 3

Evaluate the following expressions. Round to 3 significant figures and write your answer in scientific notation.

- a) $(3.2 \times 10^6) \div (8.7 \times 10^{11})$
 b) $(5.2 \times 10^{-4}) \div (3.8 \times 10^{-19})$
 c) $(1.7 \times 10^6) \div (2.7 \times 10^{-11})$

Solution

It's easier if we convert to fractions and THEN separate out the powers of 10.

a)

$$\begin{aligned} (3.2 \times 10^6) \div (8.7 \times 10^{11}) &= \frac{3.2 \times 10^6}{8.7 \times 10^{11}} && \text{— separate out the powers of 10 :} \\ &= \frac{3.2}{8.7} \times \frac{10^6}{10^{11}} && \text{— evaluate each fraction (round to 3 s.f.) :} \\ &= 0.368 \times 10^{(6-11)} \\ &= 0.368 \times 10^{-5} && \text{— remember how to write scientific notation!} \\ &= 3.68 \times 10^{-6} \end{aligned}$$

b)

$$\begin{aligned} (5.2 \times 10^{-4}) \div (3.8 \times 10^{-19}) &= \frac{5.2 \times 10^{-4}}{3.8 \times 10^{-19}} && \text{— separate the powers of 10 :} \\ &= \frac{5.2}{3.8} \times \frac{10^{-4}}{10^{-19}} && \text{— evaluate each fraction (round to 3 s.f.)} \\ &= 1.37 \times 10^{((-4)-(-19))} \\ &= 1.37 \times 10^{15} \end{aligned}$$

c)

$$\begin{aligned}
 (1.7 \times 10^6) \div (2.7 \times 10^{-11}) &= \frac{1.7 \times 10^6}{2.7 \times 10^{-11}} \\
 &= \frac{1.7}{2.7} \times \frac{10^6}{10^{-11}} \\
 &= 0.630 \times 10^{6-(-11)} \\
 &= 0.630 \times 10^{17} \\
 &= 6.30 \times 10^{16}
 \end{aligned}$$

– next we separate the powers of 10 :

– evaluate each fraction (round to 3 s.f.)

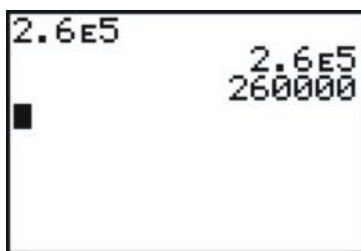
Note that we have to leave in the final zero to indicate that the result has been rounded.

Evaluate Expressions in Scientific Notation Using a Graphing Calculator

All scientific and graphing calculators can use scientific notation, and it's very useful to know how.

To insert a number in scientific notation, use the **[EE]** button. This is **[2nd] [,]** on some TI models.

For example, to enter 2.6×10^5 , enter 2.6 **[EE]** 5. When you hit **[ENTER]** the calculator displays 2.6E5 if it's set in **Scientific** mode, or 260000 if it's set in **Normal** mode.



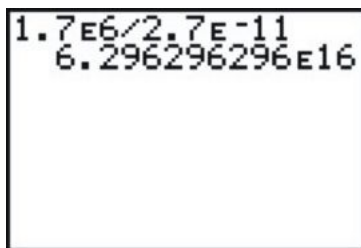
(To change the mode, press the 'Mode' key.)

Example 4

Evaluate $(2.3 \times 10^6) \times (4.9 \times 10^{-10})$ using a graphing calculator.

Solution

Enter 2.3 **[EE]** 6 \times 4.9 **[EE]** - 10 and press **[ENTER]**.



The calculator displays 6.296296296E16 whether it's in Normal mode or Scientific mode. That's because the number is so big that even in Normal mode it won't fit on the screen. The answer displayed instead isn't the precisely correct answer; it's rounded off to 10 significant figures.

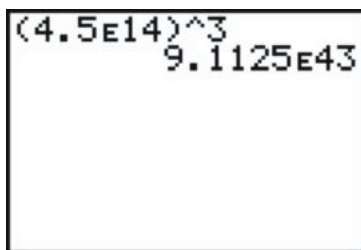
Since it's a repeating decimal, though, we can write it more efficiently *and* more precisely as $6.\overline{296} \times 10^{16}$.

Example 5

Evaluate $(4.5 \times 10^{14})^3$ using a graphing calculator.

Solution

Enter $(4.5 \text{ [EE] } 14)^3$ and press **[ENTER]**.



The calculator displays 9.1125E43. The answer is 9.1125×10^{43} .

Solve Real-World Problems Using Scientific Notation

Example 6

The mass of a single lithium atom is approximately one percent of one millionth of one billionth of one billionth of one kilogram. Express this mass in scientific notation.

Solution

We know that a *percent* is $\frac{1}{100}$, and so our calculation for the mass (in kg) is:

$$\frac{1}{100} \times \frac{1}{1,000,000} \times \frac{1}{1,000,000,000} \times \frac{1}{1,000,000,000} = 10^{-2} \times 10^{-6} \times 10^{-9} \times 10^{-9}$$

Next we use the product of powers rule we learned earlier:

$$10^{-2} \times 10^{-6} \times 10^{-9} \times 10^{-9} = 10^{((-2)+(-6)+(-9)+(-9))} = 10^{-26} \text{ kg.}$$

The mass of one lithium atom is approximately $1 \times 10^{-26} \text{ kg}$.

Example 7

*You could fit about 3 million *E. coli* bacteria on the head of a pin. If the size of the pin head in question is $1.2 \times 10^{-5} \text{ m}^2$, calculate the area taken up by one *E. coli* bacterium. Express your answer in scientific notation*

Solution

Since we need our answer in scientific notation, it makes sense to convert 3 million to that format first:

$$3,000,000 = 3 \times 10^6$$

Next we need an expression involving our unknown, the area taken up by one bacterium. Call this A .

$$3 \times 10^6 \cdot A = 1.2 \times 10^{-5} \quad \text{— since 3 million of them make up the area of the pin — head}$$

Isolate A :

$$\begin{aligned} A &= \frac{1}{3 \times 10^6} \cdot 1.2 \times 10^{-5} && \text{— rearranging the terms gives :} \\ A &= \frac{1.2}{3} \cdot \frac{1}{10^6} \times 10^{-5} && \text{— then using the definition of a negative exponent :} \\ A &= \frac{1.2}{3} \cdot 10^{-6} \times 10^{-5} && \text{— evaluate \& combine exponents using the product rule :} \\ A &= 0.4 \times 10^{-11} && \text{— but we can't leave our answer like this, so ...} \end{aligned}$$

The area of one bacterium is $4.0 \times 10^{-12} \text{ m}^2$.

(Notice that we had to move the decimal point over one place to the right, subtracting 1 from the exponent on the 10.)

Review Questions

Write the numerical value of the following.

1. 3.102×10^2
2. 7.4×10^4
3. 1.75×10^{-3}
4. 2.9×10^{-5}
5. 9.99×10^{-9}

Write the following numbers in scientific notation.

6. 120,000
7. 1,765,244
8. 12
9. 0.00281
10. 0.000000027

How many significant digits are in each of the following?

11. 38553000
12. 2754000.23
13. 0.0000222
14. 0.0002000079

Round each of the following to two significant digits.

15. 3.0132
16. 82.9913

Perform the following operations and write your answer in scientific notation.

17. $(3.5 \times 10^4) \cdot (2.2 \times 10^7)$
18. $\frac{2.1 \times 10^9}{3 \times 10^2}$
19. $(3.1 \times 10^{-3}) \cdot (1.2 \times 10^{-5})$
20. $\frac{7.4 \times 10^{-5}}{3.7 \times 10^{-2}}$
21. $12,000,000 \times 400,000$
22. $3,000,000 \times 0.00000000022$
23. $\frac{17,000}{680,000,000}$
24. $\frac{25,000,000}{0.000000005}$
25. $\frac{0.000000000042}{0.00014}$
26. The moon is approximately a sphere with radius $r = 1.08 \times 10^3 \text{ miles}$. Use the formula Surface Area $= 4\pi r^2$ to determine the surface area of the moon, in square miles. Express your answer in scientific notation, rounded to two significant figures.

27. The charge on one electron is approximately 1.60×10^{19} coulombs. One **Faraday** is equal to the total charge on 6.02×10^{23} electrons. What, in coulombs, is the charge on one Faraday?
28. Proxima Centauri, the next closest star to our Sun, is approximately 2.5×10^{13} miles away. If light from Proxima Centauri takes 3.7×10^4 hours to reach us from there, calculate the speed of light in miles per hour. Express your answer in scientific notation, rounded to 2 significant figures.

4.5 Geometric Sequences

Learning Objectives

- Identify a geometric sequence
- Graph a geometric sequence.
- Solve real-world problems involving geometric sequences.

Introduction

Consider the following question:

Which would you prefer, being given one million dollars, or one penny the first day, double that penny the next day, and then double the previous day's pennies and so on for a month?

At first glance it's easy to say "Give me the million!" But why don't we do a few calculations to see how the other choice stacks up?

You start with a penny the first day and keep doubling each day. Doubling means that we keep multiplying by 2 each day for one month (30 days).

On the first day, you get 1 penny, or 2^0 pennies.

On the second day, you get 2 pennies, or 2^1 pennies.

On the third day, you get 4 pennies, or 2^2 pennies. Do you see the pattern yet?

On the fourth day, you get 8 pennies, or 2^3 pennies. Each day, the exponent is one less than the number of that day.

So on the thirtieth day, you get 2^{29} pennies, which is 536,870,912 pennies, or \$5,368,709.12. That's a lot more than a million dollars, even just counting the amount you get on that one day!

This problem is an example of a geometric sequence. In this section, we'll find out what a geometric sequence is and how to solve problems involving geometric sequences.

Identify a Geometric Sequence

A **geometric sequence** is a sequence of numbers in which each number in the sequence is found by multiplying the previous number by a fixed amount called the **common ratio**. In other words, the ratio between any term and the term before it is always the same. In the previous example the common ratio was 2, as the number of pennies doubled each day.

The common ratio, r , in any geometric sequence can be found by dividing any term by the preceding term. Here are some examples of geometric sequences and their common ratios.

| | | |
|--|--------------------|---|
| 4, 16, 64, 256, ... | $r = 4$ | (divide 16 by 4 to get 4) |
| 15, 30, 60, 120, ... | $r = 2$ | (divide 30 by 15 to get 2) |
| $11, \frac{11}{2}, \frac{11}{4}, \frac{11}{8}, \frac{11}{16}, \dots$ | $r = \frac{1}{2}$ | (divide $\frac{11}{2}$ by 11 to get $\frac{1}{2}$) |
| $25, -5, 1, -\frac{1}{5}, \frac{1}{25}, \dots$ | $r = -\frac{1}{5}$ | (divide 1 by -5 to get $-\frac{1}{5}$) |

If we know the common ratio r , we can find the next term in the sequence just by multiplying the last term by r . Also, if there are any terms missing in the sequence, we can find them by multiplying the term before each missing term by the common ratio.

Example 1

Fill in the missing terms in each geometric sequence.

a) 1, _____, 25, 125, _____

b) 20, _____, 5, _____, 1.25

Solution

a) First we can find the common ratio by dividing 125 by 25 to obtain $r = 5$.

To find the first missing term, we multiply 1 by the common ratio: $1 \cdot 5 = 5$

To find the second missing term, we multiply 125 by the common ratio: $125 \cdot 5 = 625$

Sequence (a) becomes: **1, 5, 25, 125, 625,...**

b) We need to find the common ratio first, but how do we do that when we have no terms next to each other that we can divide?

Well, we know that to get from 20 to 5 in the sequence we must multiply 20 by the common ratio *twice*: once to get to the second term in the sequence, and again to get to five. So we can say $20 \cdot r \cdot r = 5$, or $20 \cdot r^2 = 5$.

Dividing both sides by 20, we get $r^2 = \frac{5}{20} = \frac{1}{4}$, or $r = \frac{1}{2}$ (because $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$).

To get the first missing term, we multiply 20 by $\frac{1}{2}$ and get 10.

To get the second missing term, we multiply 5 by $\frac{1}{2}$ and get 2.5.

Sequence (b) becomes: **20, 10, 5, 2.5, 1.25,...**

You can see that if we keep multiplying by the common ratio, we can find any term in the sequence that we want—the tenth term, the fiftieth term, the thousandth term.... However, it would be awfully tedious to keep multiplying over and over again in order to find a term that is a long way from the start. What could we do instead of just multiplying repeatedly?

Let's look at a geometric sequence that starts with the number 7 and has common ratio of 2.

| | | | |
|---|-------------------------------------|----|---------------|
| The 1 st term is: | 7 | or | $7 \cdot 2^0$ |
| We obtain the 2 nd term by multiplying by 2 : | $7 \cdot 2$ | or | $7 \cdot 2^1$ |
| We obtain the 3 rd term by multiplying by 2 again: | $7 \cdot 2 \cdot 2$ | or | $7 \cdot 2^2$ |
| We obtain the 4 th term by multiplying by 2 again: | $7 \cdot 2 \cdot 2 \cdot 2$ | or | $7 \cdot 2^3$ |
| We obtain the 5 th term by multiplying by 2 again: | $7 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ | or | $7 \cdot 2^4$ |
| The nth term would be: | $7 \cdot 2^{n-1}$ | | |

The n th term is $7 \cdot 2^{n-1}$ because the 7 is multiplied by 2 once for the 2^{nd} term, twice for the third term, and so on—for each term, one less time than the term's place in the sequence. In general, we write a geometric sequence with n terms like this:

$$a, ar, ar^2, ar^3, \dots, ar^{n-1}$$

The formula for finding a specific term in a geometric sequence is:

n^{th} term in a geometric sequence: $a_n = a_1 r^{n-1}$

(a_1 = first term, r = common ratio)

Example 2

For each of these geometric sequences, find the eighth term in the sequence.

a) 1, 2, 4,...

b) 16, -8, 4, -2, 1,...

Solution

a) First we need to find the common ratio: $r = \frac{2}{1} = 2$.

The eighth term is given by the formula $a_8 = a_1 r^7 = 1 \cdot 2^7 = 128$

In other words, to get the eighth term we start with the first term, which is 1, and then multiply by 2 seven times.

b) The common ratio is $r = \frac{-8}{16} = \frac{-1}{2}$

The eighth term in the sequence is $a_8 = a_1 r^7 = 16 \cdot \left(\frac{-1}{2}\right)^7 = 16 \cdot \frac{(-1)^7}{2^7} = 16 \cdot \frac{-1}{2^7} = \frac{-16}{128} = -\frac{1}{8}$

Let's take another look at the terms in that second sequence. Notice that they alternate **positive, negative, positive, negative** all the way down the list. When you see this pattern, you know the common ratio is negative; multiplying by a negative number each time means that the sign of each term is opposite the sign of the previous term.

Solve Real-World Problems Involving Geometric Sequences

Let's solve two application problems involving geometric sequences.

Example 3

A courtier presented the Indian king with a beautiful, hand-made chessboard. The king asked what he would like in return for his gift and the courtier surprised the king by asking for one grain of rice on the first square, two grains on the second, four grains on the third, etc. The king readily agreed and asked for the rice to be brought. (From Meadows et al. 1972, via Porritt 2005) How many grains of rice does the king have to put on the last square?

Solution

A chessboard is an 8×8 square grid, so it contains a total of 64 squares.

The courtier asked for one grain of rice on the first square, 2 grains of rice on the second square, 4 grains of rice on the third square and so on. We can write this as a geometric sequence:

1, 2, 4,...

The numbers double each time, so the common ratio is $r = 2$.

The problem asks how many grains of rice the king needs to put on the last square, so we need to find the 64^{th} term in the sequence. Let's use our formula:

$a_n = a_1 r^{n-1}$, where a_n is the n th term, a_1 is the first term and r is the common ratio.

$a_{64} = 1 \cdot 2^{63} = 9,223,372,036,854,775,808$ **grains of rice.**

The problem we just solved has real applications in business and technology. In technology strategy, the **Second Half of the Chessboard** is a phrase, coined by a man named Ray Kurzweil, in reference to the point where an exponentially growing factor begins to have a significant economic impact on an organization's overall business strategy.

The total number of grains of rice on the **first half** of the chessboard is $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024 \dots + 2,147,483,648$, for a total of exactly 4,294,967,295 grains of rice, or about 100,000 kg of rice (the mass of one grain of rice being roughly 25 mg). This total amount is about $\frac{1}{1,000,000^{th}}$ of total rice production in India in the year 2005 and is an amount the king could surely have afforded.

The total number of grains of rice on the **second half** of the chessboard is $2^{32} + 2^{33} + 2^{34} \dots + 2^{63}$, for a total of 18,446,744,069,414,584,320 grains of rice. This is about 460 billion tons, or 6 times the entire weight of all living matter on Earth. The king didn't realize what he was agreeing to—perhaps he should have studied algebra! [Wikipedia; GNU-FDL]

Example 4

A super-ball has a 75% rebound ratio—that is, when it bounces repeatedly, each bounce is 75% as high as the previous bounce. When you drop it from a height of 20 feet:

- a) *how high does the ball bounce after it strikes the ground for the third time?*
- b) *how high does the ball bounce after it strikes the ground for the seventeenth time?*

Solution

We can write a geometric sequence that gives the height of each bounce with the common ratio of $r = \frac{3}{4}$:

$$20, 20 \cdot \frac{3}{4}, 20 \cdot \left(\frac{3}{4}\right)^2, 20 \cdot \left(\frac{3}{4}\right)^3 \dots$$

- a) The ball starts at a height of 20 feet; after the first bounce it reaches a height of $20 \cdot \frac{3}{4} = 15$ *feet*.

After the second bounce it reaches a height of $20 \cdot \left(\frac{3}{4}\right)^2 = 11.25$ *feet*.

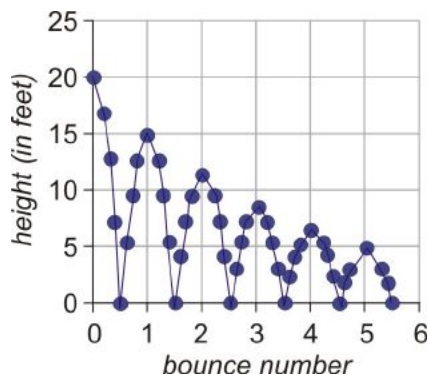
After the third bounce it reaches a height of $20 \cdot \left(\frac{3}{4}\right)^3 = 8.44$ *feet*.

- b) Notice that the height after the first bounce corresponds to the second term in the sequence, the height after the second bounce corresponds to the third term in the sequence and so on.

This means that the height after the seventeenth bounce corresponds to the 18th term in the sequence. You can find the height by using the formula for the 18th term:

$$a_{18} = 20 \cdot \left(\frac{3}{4}\right)^{17} = 0.15 \text{ feet.}$$

Here is a graph that represents this information. (The heights at points other than the top of each bounce are just approximations.)



For more practice finding the terms in geometric sequences, try the browser game at [http://www.netsoc.tcd.ie/~jgilbert, site/applets/sequences_and_series/geometric_sequences.html](http://www.netsoc.tcd.ie/~jgilbert/site/applets/sequences_and_series/geometric_sequences.html).

Review Questions

Determine the first five terms of each geometric sequence.

1. $a_1 = 2, r = 3$
2. $a_1 = 90, r = -\frac{1}{3}$
3. $a_1 = 6, r = -2$
4. $a_1 = 1, r = 5$
5. $a_1 = 5, r = 5$
6. $a_1 = 25, r = 5$
7. What do you notice about the last three sequences?

Find the missing terms in each geometric sequence.

8. 3, ____, 48, 192, ____
9. 81, ____, ____, ____, 1
10. $\frac{9}{4}$, ____, ____, $\frac{2}{3}$, ____
11. 2, ____, ____, -54, 162

Find the indicated term of each geometric sequence.

12. $a_1 = 4, r = 2$; find a_6
13. $a_1 = -7, r = -\frac{3}{4}$; find a_4
14. $a_1 = -10, r = -3$; find a_{10}
15. In a geometric sequence, $a_3 = 28$ and $a_5 = 112$; find r and a_1 .
16. In a geometric sequence, $a_2 = 28$ and $a_5 = 112$; find r and a_1 .
17. As you can see from the previous two questions, the same terms can show up in sequences with different ratios.
 - (a) Write a geometric sequence that has 1 and 9 as two of the terms (not necessarily the first two).
 - (b) Write a different geometric sequence that also has 1 and 9 as two of the terms.
 - (c) Write a geometric sequence that has 6 and 24 as two of the terms.
 - (d) Write a different geometric sequence that also has 6 and 24 as two of the terms.
 - (e) What is the common ratio of the sequence whose first three terms are 2, 6, 18?
 - (f) What is the common ratio of the sequence whose first three terms are 18, 6, 2?

- (g) What is the relationship between those ratios?
18. Anne goes bungee jumping off a bridge above water. On the initial jump, the bungee cord stretches by 120 feet. On the next bounce the stretch is 60% of the original jump and each additional bounce the rope stretches by 60% of the previous stretch.
- (a) What will the rope stretch be on the third bounce?
- (b) What will be the rope stretch be on the 12th bounce?

4.6 Exponential Functions

Learning Objectives

- Graph an exponential function.
- Compare graphs of exponential functions.
- Analyze the properties of exponential functions.

Introduction

A colony of bacteria has a population of three thousand at noon on Monday. During the next week, the colony's population doubles every day. What is the population of the bacteria colony just before midnight on Saturday?

At first glance, this seems like a problem you could solve using a geometric sequence. And you could, if the bacteria population doubled all at once every day; since it doubled every day for five days, the final population would be 3000 times 2^5 .

But bacteria don't reproduce all at once; their population grows slowly over the course of an entire day. So how do we figure out the population after five *and a half* days?

Exponential Functions

Exponential functions are a lot like geometrical sequences. The main difference between them is that a geometric sequence is **discrete** while an exponential function is **continuous**.

Discrete means that the sequence has values only at distinct points (the 1st term, 2nd term, etc.)

Continuous means that the function has values for all possible values of x . The integers are included, but also all the numbers in between.

The problem with the bacteria is an example of a continuous function. Here's an example of a discrete function:

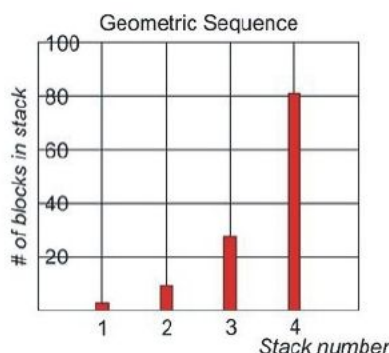
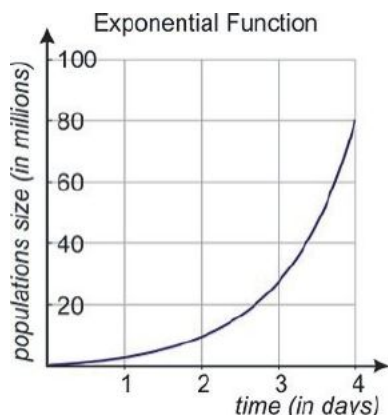
An ant walks past several stacks of Lego blocks. There is one block in the first stack, 3 blocks in the 2nd stack and 9 blocks in the 3rd stack. In fact, in each successive stack there are triple the number of blocks than in the previous stack.

In this example, each stack has a distinct number of blocks and the next stack is made by adding a certain number of whole pieces all at once. More importantly, however, there are no values of the sequence **between** the stacks. You can't ask how high the stack is between the 2nd and 3rd stack, as no stack exists at that position!

As a result of this difference, we use a geometric series to describe quantities that have values at discrete points, and we use exponential functions to describe quantities that have values that change continuously.

When we graph an exponential function, we draw the graph with a solid curve to show that the function has values at any time during the day. On the other hand, when we graph a geometric sequence, we draw discrete points to signify that the sequence only has value at those points but not in between.

Here are graphs for the two examples above:



The formula for an exponential function is similar to the formula for finding the terms in a geometric sequence. An exponential function takes the form

$$y = A \cdot b^x$$

where A is the starting amount and b is the amount by which the total is multiplied every time. For example, the bacteria problem above would have the equation $y = 3000 \cdot 2^x$.

Compare Graphs of Exponential Functions

Let's graph a few exponential functions and see what happens as we change the constants in the formula. The basic shape of the exponential function should stay the same—but it may become steeper or shallower depending on the constants we are using.

First, let's see what happens when we change the value of A .

Example 1

Compare the graphs of $y = 2^x$ and $y = 3 \cdot 2^x$.

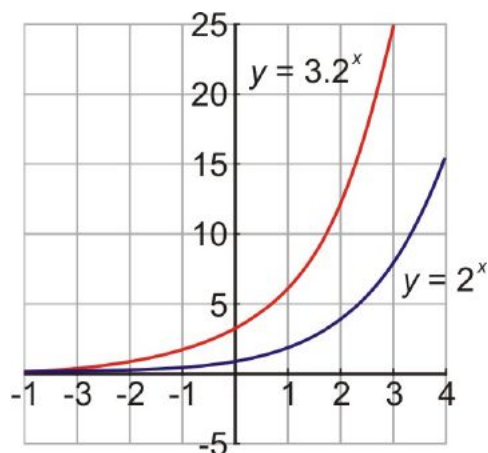
Solution

Let's make a table of values for both functions.

Table 4.1:

| x | $y = 2^x$ | $y = 3 \cdot 2^x$ |
|-----|---------------|--|
| -3 | $\frac{1}{8}$ | $y = 3 \cdot 2^{-3} = 3 \cdot \frac{1}{2^3} = \frac{3}{8}$ |
| -2 | $\frac{1}{4}$ | $y = 3 \cdot 2^{-2} = 3 \cdot \frac{1}{2^2} = \frac{3}{4}$ |
| -1 | $\frac{1}{2}$ | $y = 3 \cdot 2^{-1} = 3 \cdot \frac{1}{2^1} = \frac{3}{2}$ |
| 0 | 1 | $y = 3 \cdot 2^0 = 3$ |
| 1 | 2 | $y = 3 \cdot 2^1 = 6$ |
| 2 | 4 | $y = 3 \cdot 2^2 = 3 \cdot 4 = 12$ |
| 3 | 8 | $y = 3 \cdot 2^3 = 3 \cdot 8 = 24$ |

Now let's use this table to graph the functions.



We can see that the function $y = 3 \cdot 2^x$ is bigger than the function $y = 2^x$. In both functions, the value of y doubles every time x increases by one. However, $y = 3 \cdot 2^x$ “starts” with a value of 3, while $y = 2^x$ “starts” with a value of 1, so it makes sense that $y = 3 \cdot 2^x$ would be bigger as its values of y keep getting doubled.

Similarly, if the starting value of A is smaller, the values of the entire function will be smaller.

Example 2

Compare the graphs of $y = 2^x$ and $y = \frac{1}{3} \cdot 2^x$.

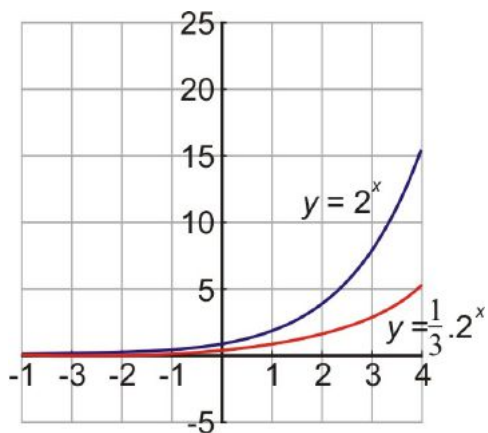
Solution

Let's make a table of values for both functions.

Table 4.2:

| x | $y = 2^x$ | $y = \frac{1}{3} \cdot 2^x$ |
|-----|---------------|---|
| -3 | $\frac{1}{8}$ | $y = \frac{1}{3} \cdot 2^{-3} = \frac{1}{3} \cdot \frac{1}{2^3} = \frac{1}{24}$ |
| -2 | $\frac{1}{4}$ | $y = \frac{1}{3} \cdot 2^{-2} = \frac{1}{3} \cdot \frac{1}{2^2} = \frac{1}{12}$ |
| -1 | $\frac{1}{2}$ | $y = \frac{1}{3} \cdot 2^{-1} = \frac{1}{3} \cdot \frac{1}{2^1} = \frac{1}{6}$ |
| 0 | 1 | $y = \frac{1}{3} \cdot 2^0 = \frac{1}{3}$ |
| 1 | 2 | $y = \frac{1}{3} \cdot 2^1 = \frac{2}{3}$ |
| 2 | 4 | $y = \frac{1}{3} \cdot 2^2 = \frac{1}{3} \cdot 4 = \frac{4}{3}$ |
| 3 | 8 | $y = \frac{1}{3} \cdot 2^3 = \frac{1}{3} \cdot 8 = \frac{8}{3}$ |

Now let's use this table to graph the functions.



As we expected, the exponential function $y = \frac{1}{3} \cdot 2^x$ is smaller than the exponential function $y = 2^x$. So what happens if the starting value of A is negative? Let's find out.

Example 3

Graph the exponential function $y = -5 \cdot 2^x$.

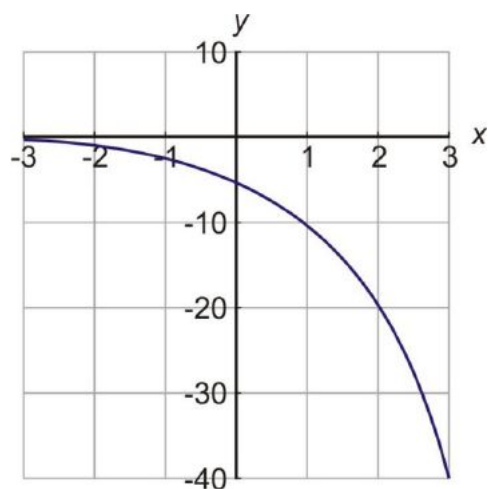
Solution

Let's make a table of values:

Table 4.3:

| x | $y = -5 \cdot 2^x$ |
|-----|--------------------|
| -2 | $-\frac{5}{4}$ |
| -1 | $-\frac{5}{2}$ |
| 0 | -5 |
| 1 | -10 |
| 2 | -20 |
| 3 | -40 |

Now let's graph the function:



This result shouldn't be unexpected. Since the starting value is negative and keeps doubling over time, it makes sense that the value of y gets farther from zero, but in a negative direction. The graph is basically

just like the graph of $y = 5 \cdot 2^x$, only mirror-reversed about the x -axis.

Now, let's compare exponential functions whose bases (b) are different.

Example 4

Graph the following exponential functions on the same graph: $y = 2^x$, $y = 3^x$, $y = 5^x$, $y = 10^x$.

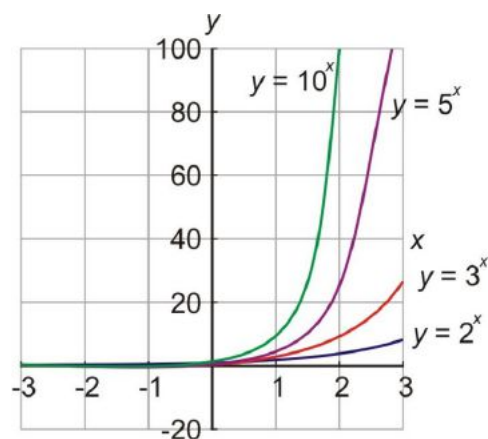
Solution

First we'll make a table of values for all four functions.

Table 4.4:

| x | $y = 2^x$ | $y = 3^x$ | $y = 5^x$ | $y = 10^x$ |
|-----|---------------|---------------|----------------|-----------------|
| -2 | $\frac{1}{4}$ | $\frac{1}{9}$ | $\frac{1}{25}$ | $\frac{1}{100}$ |
| -1 | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{5}$ | $\frac{1}{10}$ |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 2 | 3 | 5 | 10 |
| 2 | 4 | 9 | 25 | 100 |
| 3 | 8 | 27 | 125 | 1000 |

Now let's graph the functions.



Notice that for $x = 0$, all four functions equal 1. They all “start out” at the same point, but the ones with higher values for b grow faster when x is positive—and also shrink faster when x is negative.

Finally, let's explore what happens for values of b that are less than 1.

Example 5

Graph the exponential function $y = 5 \cdot \left(\frac{1}{2}\right)^x$.

Solution

Let's start by making a table of values. (Remember that a fraction to a negative power is equivalent to its reciprocal to the same positive power.)

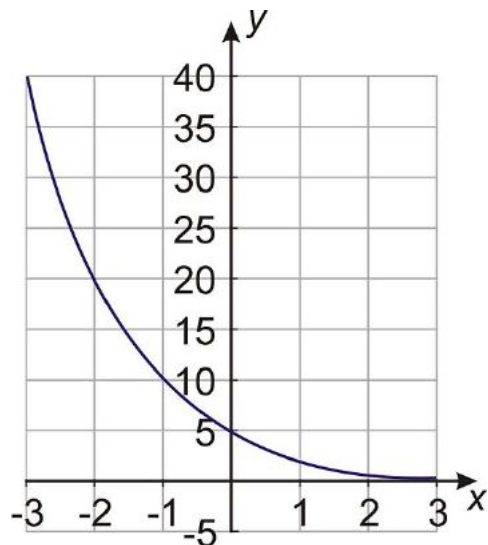
Table 4.5:

| x | $y = 5 \cdot \left(\frac{1}{2}\right)^x$ |
|-----|--|
| -3 | $y = 5 \cdot \left(\frac{1}{2}\right)^{-3} = 5 \cdot 2^3 = 40$ |
| -2 | $y = 5 \cdot \left(\frac{1}{2}\right)^{-2} = 5 \cdot 2^2 = 20$ |

Table 4.5: (continued)

| x | $y = 5 \cdot \left(\frac{1}{2}\right)^x$ |
|-----|--|
| -1 | $y = 5 \cdot \left(\frac{1}{2}\right)^{-1} = 5 \cdot 2^1 = 10$ |
| 0 | $y = 5 \cdot \left(\frac{1}{2}\right)^0 = 5 \cdot 1 = 5$ |
| 1 | $y = 5 \cdot \left(\frac{1}{2}\right)^1 = \frac{5}{2}$ |
| 2 | $y = 5 \cdot \left(\frac{1}{2}\right)^2 = \frac{5}{4}$ |

Now let's graph the function.



This graph looks very different than the graphs from the previous example! What's going on here?

When we raise a number greater than 1 to the power of x , it gets bigger as x gets bigger. But when we raise a number smaller than 1 to the power of x , it gets *smaller* as x gets bigger—as you can see from the table of values above. This makes sense because multiplying any number by a quantity less than 1 always makes it smaller.

So, when the base b of an exponential function is between 0 and 1, the graph is like an ordinary exponential graph, only decreasing instead of increasing. Graphs like this represent **exponential decay** instead of **exponential growth**. Exponential decay functions are used to describe quantities that decrease over a period of time.

When b can be written as a fraction, we can use the Property of Negative Exponents to write the function in a different form. For instance, $y = 5 \cdot \left(\frac{1}{2}\right)^x$ is equivalent to $5 \cdot 2^{-x}$. These two forms are both commonly used, so it's important to know that they are equivalent.

Example 6

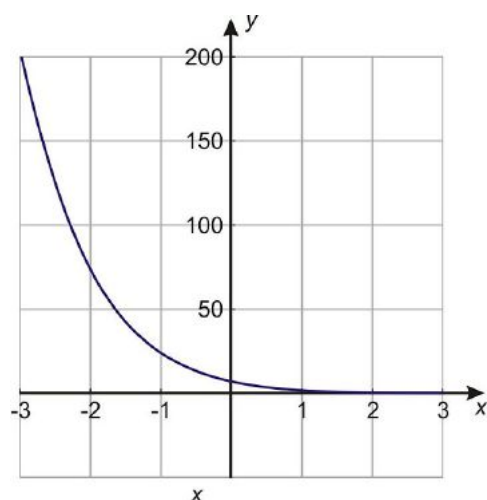
Graph the exponential function $y = 8 \cdot 3^{-x}$.

Solution

Here is our table of values and the graph of the function.

Table 4.6:

| x | $y = 8 \cdot 3^{-x}$ |
|-----|---|
| -3 | $y = 8 \cdot 3^{-(-3)} = 8 \cdot 3^3 = 216$ |
| -2 | $y = 8 \cdot 3^{-(-2)} = 8 \cdot 3^2 = 72$ |
| -1 | $y = 8 \cdot 3^{-(-1)} = 8 \cdot 3^1 = 24$ |
| 0 | $y = 8 \cdot 3^0 = 8$ |
| 1 | $y = 8 \cdot 3^{-1} = \frac{8}{3}$ |
| 2 | $y = 8 \cdot 3^{-2} = \frac{8}{9}$ |

**Example 7**

Graph the functions $y = 4^x$ and $y = 4^{-x}$ on the same coordinate axes.

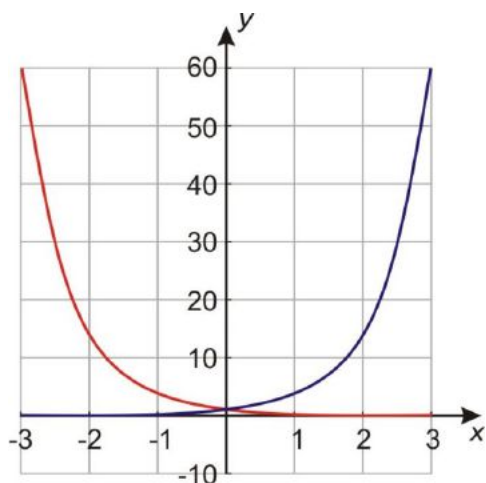
Solution

Here is the table of values for the two functions. Looking at the values in the table, we can see that the two functions are “backwards” of each other, in the sense that the values for the two functions are reciprocals.

Table 4.7:

| x | $y = 4^x$ | $y = 4^{-x}$ |
|-----|-----------------------------|-----------------------------|
| -3 | $y = 4^{-3} = \frac{1}{64}$ | $y = 4^{-(-3)} = 64$ |
| -2 | $y = 4^{-2} = \frac{1}{16}$ | $y = 4^{-(-2)} = 16$ |
| -1 | $y = 4^{-1} = \frac{1}{4}$ | $y = 4^{-(-1)} = 4$ |
| 0 | $y = 4^0 = 1$ | $y = 4^0 = 1$ |
| 1 | $y = 4^1 = 4$ | $y = 4^{-1} = \frac{1}{4}$ |
| 2 | $y = 4^2 = 16$ | $y = 4^{-2} = \frac{1}{16}$ |
| 3 | $y = 4^3 = \frac{1}{64}$ | $y = 4^{-3} = \frac{1}{64}$ |

Here is the graph of the two functions. Notice that the two functions are mirror images of each other if the mirror is placed vertically on the y -axis.



In the next lesson, you'll see how exponential growth and decay functions can be used to represent situations in the real world.

Review Questions

Graph the following exponential functions by making a table of values.

1. $y = 3^x$
2. $y = 5 \cdot 3^x$
3. $y = 40 \cdot 4^x$
4. $y = 3 \cdot 10^x$

Graph the following exponential functions.

5. $y = \left(\frac{1}{5}\right)^x$
6. $y = 4 \cdot \left(\frac{2}{3}\right)^x$
7. $y = 3^{-x}$
8. $y = \frac{3}{4} \cdot 6^{-x}$
9. Which two of the eight graphs above are mirror images of each other?
10. What function would produce a graph that is the mirror image of the one in problem 4?
11. How else might you write the exponential function in problem 5?
12. How else might you write the function in problem 6?

Solve the following problems.

13. A chain letter is sent out to 10 people telling everyone to make 10 copies of the letter and send each one to a new person.
 - (a) Assume that everyone who receives the letter sends it to ten new people and that each cycle takes a week. How many people receive the letter on the sixth week?
 - (b) What if everyone only sends the letter to 9 new people? How many people will then get letters on the sixth week?
14. Nadia received \$200 for her 10th birthday. If she saves it in a bank account with 7.5% interest compounded yearly, how much money will she have in the bank by her 21st birthday?

4.7 Applications of Exponential Functions

Learning Objectives

- Apply the problem-solving plan to problems involving exponential functions.
- Solve real-world problems involving exponential growth.
- Solve real-world problems involving exponential decay.

Introduction

For her eighth birthday, Shelley's grandmother gave her a full bag of candy. Shelley counted her candy and found out that there were 160 pieces in the bag. As you might suspect, Shelley loves candy, so she ate half the candy on the first day. Then her mother told her that if she eats it at that rate, the candy will only last one more day—so Shelley devised a clever plan. She will always eat half of the candy that is left in the bag each day. She thinks that this way she can eat candy every day and never run out.

How much candy does Shelley have at the end of the week? Will the candy really last forever?

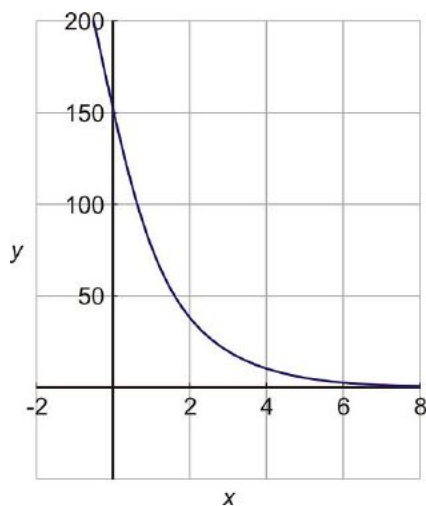
Let's make a table of values for this problem.

| | | | | | | | | |
|--------------|-----|----|----|----|----|---|-----|------|
| Day | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| # of candies | 160 | 80 | 40 | 20 | 10 | 5 | 2.5 | 1.25 |

You can see that if Shelley eats half the candies each day, then by the end of the week she only has 1.25 candies left in her bag.

Let's write an equation for this exponential function. Using the formula $y = A \cdot b^x$, we can see that A is 160 (the number of candies she starts out with and b is $\frac{1}{2}$, so our equation is $y = 160 \cdot \left(\frac{1}{2}\right)^x$.)

Now let's graph this function. The resulting graph is shown below.



So, will Shelley's candy last forever? We saw that by the end of the week she has 1.25 candies left, so there doesn't seem to be much hope for that. But if you look at the graph, you'll see that the graph never really gets to zero. Theoretically there will always be *some* candy left, but Shelley will be eating very tiny fractions of a candy every day after the first week!

This is a fundamental feature of an exponential decay function. Its values get smaller and smaller but never quite reach zero. In mathematics, we say that the function has an **asymptote** at $y = 0$; in other

words, it gets closer and closer to the line $y = 0$ but never quite meets it.

Problem-Solving Strategies

Remember our problem-solving plan from earlier?

1. Understand the problem.
2. Devise a plan – Translate.
3. Carry out the plan – Solve.
4. Look – Check and Interpret.

We can use this plan to solve application problems involving exponential functions. Compound interest, loudness of sound, population increase, population decrease or radioactive decay are all applications of exponential functions. In these problems, we'll use the methods of constructing a table and identifying a pattern to help us devise a plan for solving the problems.

Example 1

Suppose \$4000 is invested at 6% interest compounded annually. How much money will there be in the bank at the end of 5 years? At the end of 20 years?

Solution

Step 1: Read the problem and summarize the information.

\$4000 is invested at 6% interest compounded annually; we want to know how much money we have in five years.

Assign variables:

Let x = time in years

Let y = amount of money in investment account

Step 2: Look for a pattern.

We start with \$4000 and each year we add 6% interest to the amount in the bank.

| | |
|-----------------------|--|
| Start: | \$4000 |
| 1 st year: | Interest = $4000 \times (0.06) = \$240$ This is added to the previous amount: $\$4000 + \$4000 \times (0.06)$ $= \$4000(1 + 0.06)$ $= \$4000(1.06)$ $= \$4240$ |
| 2 nd year | Previous amount + interest on the previous amount $= \$4240(1 + 0.06)$ $= \$4240(1.06)$ $= \$4494.40$ |

The pattern is that each year we multiply the previous amount by the factor of 1.06.

Let's fill in a table of values:

| | | | | | | |
|------------------------|------|------|--------|---------|---------|--------|
| Time (years) | 0 | 1 | 2 | 3 | 4 | 5 |
| Investments amount(\$) | 4000 | 4240 | 4494.4 | 4764.06 | 5049.90 | 5352.9 |

We see that **at the end of five years we have \$5352.90 in the investment account.**

Step 3: Find a formula.

We were able to find the amount after 5 years just by following the pattern, but rather than follow that pattern for another 15 years, it's easier to use it to find a general formula. Since the original investment is multiplied by 1.06 each year, we can use exponential notation. Our formula is $y = 4000 \cdot (1.06)^x$, where x is the number of years since the investment.

To find the amount after 5 years we plug $x = 5$ into the equation:

$$y = 4000 \cdot (1.06)^5 = \$5352.90$$

To find the amount after 20 years we plug $x = 20$ into the equation:

$$y = 4000 \cdot (1.06)^{20} = \$12828.54$$

Step 4: Check.

Looking back over the solution, we see that we obtained the answers to the questions we were asked and the answers make sense.

To check our answers, we can plug some low values of x into the formula to see if they match the values in the table:

$$x = 0 : y = 4000 \cdot (1.06)^0 = 4000$$

$$x = 1 : y = 4000 \cdot (1.06)^1 = 4240$$

$$x = 2 : y = 4000 \cdot (1.06)^2 = 4494.4$$

The answers match the values we found earlier. The amount of increase gets larger each year, and that makes sense because the interest is 6% of an amount that is larger every year.

Example 2

In 2002 the population of schoolchildren in a city was 90,000. This population decreases at a rate of 5% each year. What will be the population of school children in year 2010?

Solution

Step 1: Read the problem and summarize the information.

The population is 90,000; the rate of decrease is 5% each year; we want the population after 8 years.

Assign variables:

Let x = time since 2002 (in years)

Let y = population of school children

Step 2: Look for a pattern.

Let's start in 2002, when the population is 90,000.

The rate of decrease is 5% each year, so the amount in 2003 is 90,000 minus 5% of 90,000, or 95% of 90,000.

$$\text{In 2003 :} \quad \text{Population} = 90,000 \times 0.95$$

$$\text{In 2004 :} \quad \text{Population} = 90,000 \times 0.95 \times 0.95$$

The pattern is that for each year we multiply by a factor of 0.95

Let's fill in a table of values:

| | | | | | | |
|------------|--------|--------|--------|--------|--------|--------|
| Year | 2002 | 2003 | 2004 | 2005 | 2006 | 2007 |
| Population | 90,000 | 85,500 | 81,225 | 77,164 | 73,306 | 69,640 |

Step 3: Find a formula.

Since we multiply by 0.95 every year, our exponential formula is $y = 90000 \cdot (0.95)^x$, where x is the number of years since 2002. To find the population in 2010 (8 years after 2002), we plug in $x = 8$:

$$y = 90000 \cdot (0.95)^8 = 59,708 \text{ schoolchildren.}$$

Step 4: Check.

Looking back over the solution, we see that we answered the question we were asked and that it makes sense. The answer makes sense because the numbers decrease each year as we expected. We can check that the formula is correct by plugging in the values of x from the table to see if the values match those given by the formula.

| | |
|----------------------|--|
| Year 2002, $x = 0$: | Population = $y = 90000 \cdot (0.95)^0 = 90,000$ |
| Year 2003, $x = 1$: | Population = $y = 90000 \cdot (0.95)^1 = 85,500$ |
| Year 2004, $x = 2$: | Population = $y = 90000 \cdot (0.95)^2 = 81,225$ |

Solve Real-World Problems Involving Exponential Growth

Now we'll look at some more real-world problems involving exponential functions. We'll start with situations involving exponential growth.

Example 3

The population of a town is estimated to increase by 15% per year. The population today is 20 thousand. Make a graph of the population function and find out what the population will be ten years from now.

Solution

First, we need to write a function that describes the population of the town.

The general form of an exponential function is $y = A \cdot b^x$.

Define y as the population of the town.

Define x as the number of years from now.

A is the initial population, so $A = 20$ (thousand).

Finally we must find what b is. We are told that the population increases by 15% each year. To calculate percents we have to change them into decimals: 15% is equivalent to 0.15. So each year, the population increases by 15% of A , or $0.15A$.

To find the total population for the following year, we must add the *current* population to the *increase* in population. In other words, $A + 0.15A = 1.15A$. So the population must be multiplied by a factor of 1.15 each year. This means that the base of the exponential is $b = 1.15$.

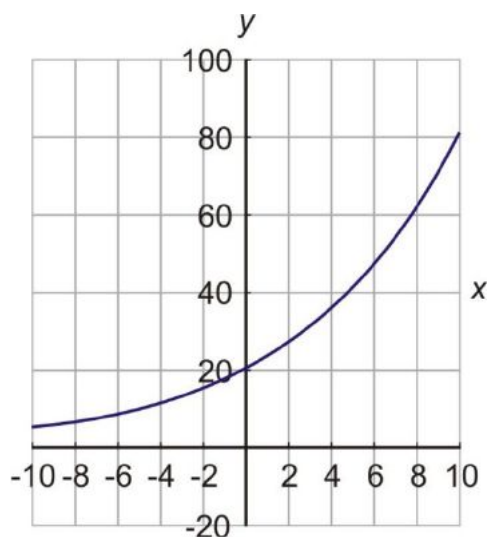
The formula that describes this problem is $y = 20 \cdot (1.15)^x$.

Now let's make a table of values.

Table 4.8:

| x | $y = 20 \cdot (1.15)^x$ |
|-----|-------------------------|
| -10 | 4.9 |
| -5 | 9.9 |
| 0 | 20 |
| 5 | 40.2 |
| 10 | 80.9 |

Now we can graph the function.



Notice that we used negative values of x in our table of values. Does it make sense to think of negative time? Yes; negative time can represent time in the past. For example, $x = -5$ in this problem represents the population from five years ago.

The question asked in the problem was: *what will be the population of the town ten years from now?* To find that number, we plug $x = 10$ into the equation we found: $y = 20 \cdot (1.15)^{10} = 80,911$.

The town will have 80,911 people ten years from now.

Example 4

Peter earned \$1500 last summer. If he deposited the money in a bank account that earns 5% interest compounded yearly, how much money will he have after five years?

Solution

This problem deals with interest which is compounded yearly. This means that each year the interest is calculated on the amount of money you have in the bank. That interest is added to the original amount and next year the interest is calculated on this new amount, so you get paid interest on the interest.

Let's write a function that describes the amount of money in the bank.

The general form of an exponential function is $y = A \cdot b^x$.

Define y as the amount of money in the bank.

Define x as the number of years from now.

A is the initial amount, so $A = 1500$.

Now we have to find what b is.

We're told that the interest is 5% each year, which is 0.05 in decimal form. When we add $0.05A$ to A , we get $1.05A$, so that is the factor we multiply by each year. The base of the exponential is $b = 1.05$.

The formula that describes this problem is $y = 1500 \cdot 1.05^x$. To find the total amount of money in the bank at the end of five years, we simply plug in $x = 5$.

$$y = 1500 \cdot (1.05)^5 = \$1914.42$$

Solve Real-World Problems Involving Exponential Decay

Exponential decay problems appear in several application problems. Some examples of these are **half-life problems** and **depreciation problems**. Let's solve an example of each of these problems.

Example 5

A radioactive substance has a half-life of one week. In other words, at the end of every week the level of radioactivity is half of its value at the beginning of the week. The initial level of radioactivity is 20 counts per second.

Draw the graph of the amount of radioactivity against time in weeks.

Find the formula that gives the radioactivity in terms of time.

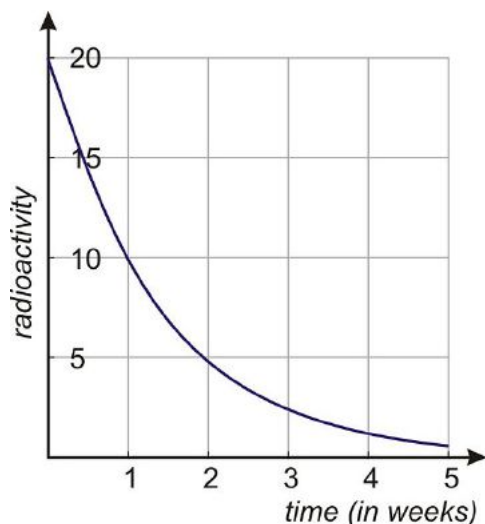
Find the radioactivity left after three weeks.

Solution

Let's start by making a table of values and then draw the graph.

Table 4.9:

| Time | Radioactivity |
|------|---------------|
| 0 | 20 |
| 1 | 10 |
| 2 | 5 |
| 3 | 2.5 |
| 4 | 1.25 |
| 5 | 0.625 |



Exponential decay fits the general formula $y = A \cdot b^x$. In this case:

y is the amount of radioactivity

x is the time in weeks

$A = 20$ is the starting amount

$b = \frac{1}{2}$ since the substance loses half its value each week

The formula for this problem is $y = 20 \cdot \left(\frac{1}{2}\right)^x$ or $y = 20 \cdot 2^{-x}$. To find out how much radioactivity is left after three weeks, we plug $x = 3$ into this formula.

$$y = 20 \cdot \left(\frac{1}{2}\right)^3 = 20 \cdot \left(\frac{1}{8}\right) = 2.5$$

Example 6

The cost of a new car is \$32,000. It depreciates at a rate of 15% per year. This means that it loses 15% of each value each year.

Draw the graph of the car's value against time in year.

Find the formula that gives the value of the car in terms of time.

Find the value of the car when it is four years old.

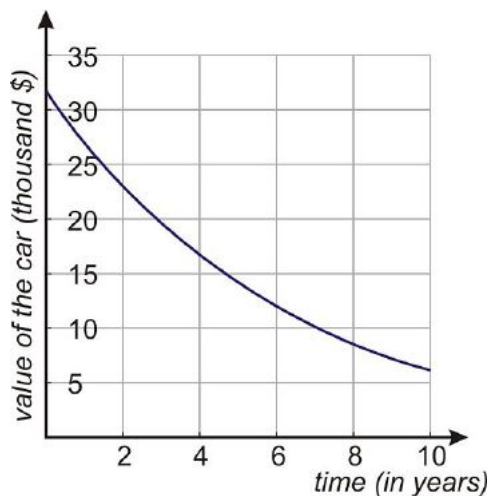
Solution

Let's start by making a table of values. To fill in the values we start with 32,000 at time $t = 0$. Then we multiply the value of the car by 85% for each passing year. (Since the car loses 15% of its value, that means it keeps 85% of its value). Remember that 85% means that we multiply by the decimal .85.

Table 4.10:

| Time | Value (thousands) |
|------|-------------------|
| 0 | 32 |
| 1 | 27.2 |
| 2 | 23.1 |
| 3 | 19.7 |
| 4 | 16.7 |
| 5 | 14.2 |

Now draw the graph:



Let's start with the general formula $y = A \cdot b^x$

In this case:

y is the value of the car,

x is the time in years,

$A = 32$ is the starting amount in thousands,

$b = 0.85$ since we multiply the amount by this factor to get the value of the car next year

The formula for this problem is $y = 32 \cdot (0.85)^x$.

Finally, to find the value of the car when it is four years old, we plug $x = 4$ into that formula: $y = 32 \cdot (0.85)^4 = 16.7$ thousand dollars, or **\$16,704** if we don't round.

Review Questions

Solve the following application problems.

- Half-life:** Suppose a radioactive substance decays at a rate of 3.5% per hour.
 - What percent of the substance is left after 6 hours?
 - What percent is left after 12 hours?
 - The substance is safe to handle when at least 50% of it has decayed. Make a guess as to how many hours this will take.
 - Test your guess. How close were you?
- Population decrease:** In 1990 a rural area has 1200 bird species.
 - If species of birds are becoming extinct at the rate of 1.5% per decade (ten years), how many bird species will be left in the year 2020?
 - At that same rate, how many were there in 1980?
- Growth:** Janine owns a chain of fast food restaurants that operated 200 stores in 1999. If the rate of increase is 8% annually, how many stores does the restaurant operate in 2007?
- Investment:** Paul invests \$360 in an account that pays 7.25% compounded annually.
 - What is the total amount in the account after 12 years?
 - If Paul invests an equal amount in an account that pays 5% compounded quarterly (four times a year), what will be the amount in that account after 12 years?

- (c) Which is the better investment?
5. The cost of a new ATV (all-terrain vehicle) is \$7200. It depreciates at 18% per year.
- (a) Draw the graph of the vehicle's value against time in years.
 - (b) Find the formula that gives the value of the ATV in terms of time.
 - (c) Find the value of the ATV when it is ten years old.
6. A person is infected by a certain bacterial infection. When he goes to the doctor the population of bacteria is 2 million. The doctor prescribes an antibiotic that reduces the bacteria population to $\frac{1}{4}$ of its size each day.
- (a) Draw the graph of the size of the bacteria population against time in days.
 - (b) Find the formula that gives the size of the bacteria population in terms of time.
 - (c) Find the size of the bacteria population ten days after the drug was first taken.
 - (d) Find the size of the bacteria population after 2 weeks (14 days).

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9618>.

Chapter 5

Logarithms

5.1 Introduction to Logarithms

1. On day 0, you have 1 penny. Every day, you double.
 - (a) How many pennies do you have on day 10?
 - (b) How many pennies do you have on day n ?
 - (c) On *what day* do you have 32 pennies? Before you answer, *express this question as an equation, where x is the variable you want to solve for.*
 - (d) Now, what is x ?
2. A radioactive substance is decaying. There is currently 100g of the substance.
 - (a) How much substance will there be after 3 half-lives?
 - (b) How much substance will there be after n half-lives?
 - (c) After *how many half-lives* will there be 1g of the substance left? Before you answer, *express this question as an equation, where x is the variable you want to solve for.*
 - (d) Now, what is x ? (Your answer will be approximate.)

In both of the problems above, part (d) required you to *invert* the normal exponential function. Instead of going *from time to amount*, it asked you to go *from amount to time*. (This is what an inverse function does—it goes the other way—remember?)

So let's go ahead and talk formally about an inverse exponential function. Remember that an inverse function goes *backward*. If $f(x) = 2^x$ turns a 3 into an 8, then $f^{-1}(x)$ must turn an 8 into a 3.

So, fill in the following table (on the left) with a bunch of x and y values for the mysterious *inverse function* of 2^x . Pick x -values that will make for easy y -values. See if you can find a few x -values that make y be 0 or negative numbers!

On the right, fill in x and y values for the *inverse function* of 10^x .

Table 5.1:

| Inverse of 2^x | Inverse of 2^x |
|------------------|------------------|
| x | y |
| 8 | 3 |

Table 5.1: (continued)

| Inverse of 2^x | Inverse of 2^x |
|------------------|------------------|
| | |
| | |

Table 5.2:

| Inverse of 10^x | Inverse of 10^x |
|-------------------|-------------------|
| x | y |
| | |
| | |

Now, let's see if we can get a bit of a handle on this type of function.

In some ways, it's like a square root. \sqrt{x} is the inverse of x^2 . When you see \sqrt{x} you are really seeing a mathematical question: "What number, squared, gives me x ?"

Now, we have the inverse of 2^x (which is quite different from x^2 of course). But this new function is also a question: see if you can figure out what it is. That is, try to write a question that will reliably get me from the left-hand column to the right-hand column in the first table above.

Do the same for the second table above.

Now, come up with a word problem of your own, similar to the first two in this exercise, but related to compound interest.

Homework—Logs

$\log_2 8 =$ asks the question: "2 to what power is 8?" Based on that, you can answer the following questions:

- $\log_2 8 =$
- $\log_3 9 =$
- $\log_{10} 10 =$
- $\log_{10} 100 =$
- $\log_{10} 1000 =$
- $\log_{10} 1,000,000 =$
- Looking at your answers to problems 3 – 6, what does the \log_{10} tell you about a number?
- Multiple choice: which of the following is closest to $\log_{10} 500$?
 - 1
 - $1\frac{1}{2}$
 - 2
 - $2\frac{1}{2}$

E. 3

9. $\log_{10} 1 =$

10. $\log_{10} \frac{1}{10} =$

11. $\log_{10} \frac{1}{100} =$

12. $\log_{10}(0.01) =$

13. $\log_{10} 0 =$

14. $\log_{10}(-1) =$

15. $\log_9 81 =$

16. $\log_9 \frac{1}{9} =$

17. $\log_9 3 =$

18. $\log_9 \frac{1}{81} =$

19. $\log_9 \frac{1}{3} =$

20. $\log_5(5^4) =$

21. $5^{\log_5 4} =$

OK. When I say, $\sqrt{36} = 6$, that's the *same thing* as saying $6^2 = 36$. Why? Because $\sqrt{36}$ asks a question: "What squared equals 36?" So the equation $\sqrt{36} = 6$ is providing an answer: "*six* squared equals 36."

You can look at logs in a similar way. If I say $\log_2 32 = 5$ I'm asking a question: "2 to what power is 32?" And I'm answering: "5. 2 to the *fifth* power is 32." So saying $\log_2 32 = 5$ is the *same thing* as saying $2^5 = 32$.

Based on this kind of reasoning, *rewrite the following logarithm statements as exponent statements.*

22. $\log_2 8 = 3$

23. $\log_3(\frac{1}{3}) = -1$

24. $\log_x(1) = 0$

25. $\log_a x = y$

Now do the same thing backward: *rewrite the following exponent statements as logarithm statements.*

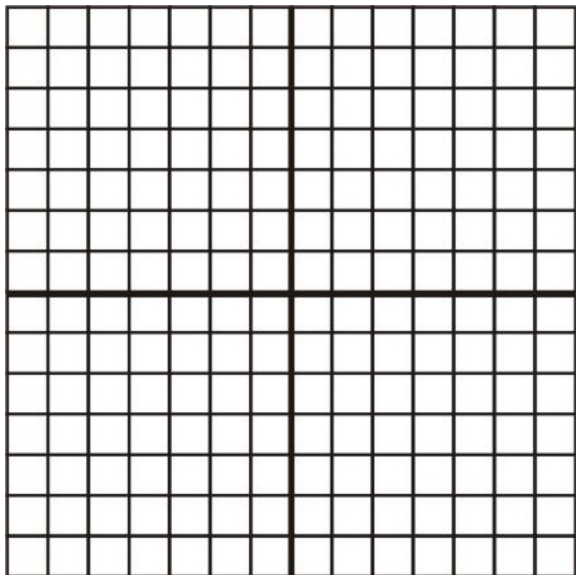
26. $4^3 = 64$

27. $8^{-\frac{2}{3}} = \frac{1}{4}$

28. $a^b = c$

Finally...you don't understand a function until you graph it...

29. a. Draw a graph of $y = \log_2 x$. Plot at least 5 points to draw the graph.



b. What are the domain and range of the graph? What does that tell you about this function?

5.2 Properties of Logarithms

1. $\log_2(2) =$
2. $\log_2(2 \times 2) =$
3. $\log_2(2 \times 2 \times 2) =$
4. $\log_2(2 \times 2 \times 2 \times 2) =$
5. $\log_2(2 \times 2 \times 2 \times 2 \times 2) =$
6. Based on numbers 1 – 5, finish this sentence in words: when you take \log_2 of a number, you find:
7. $\log_2(8) =$
8. $\log_2(16) =$
9. $\log_2(8 \times 16) =$
10. $\log_3(9) =$
11. $\log_3(27) =$
12. $\log_3(9 \times 27) =$
13. Based on numbers 7 – 12, write an algebraic generalization about logs.
14. Now, let's dig more deeply into that one. Rewrite problems 7 – 9 so they look like problems 1 – 5: that is, so the thing you are taking the log of is written as a power of 2.
 - (a) #7:
 - (b) #8:
 - (c) #9:
 - (d) Based on this rewriting, can you explain *why* your generalization from #13 works?
15. $\log_5(25) =$
16. $\log_5\left(\frac{1}{25}\right) =$
17. $\log_2(32) =$
18. $\log_2\left(\frac{1}{32}\right) =$
19. Based on numbers 15 – 18, write an algebraic generalization about logs.
20. $\log_3(81) =$
21. $\log_3(81 \times 81) =$
22. $\log_3(81)^2 =$

23. $\log_3(81 \times 81 \times 81) =$
24. $\log_3(81)^3 =$
25. $\log_3(81 \times 81 \times 81 \times 81) =$
26. $\log_3(81)^4 =$
27. Based on numbers 20 – 26 write an algebraic generalization about logs.

Homework—Properties of Logarithms

Memorise these three rules

$$\log_x(ab) = \log_x a + \log_x b$$

$$\log_x\left(\frac{a}{b}\right) = \log_x a - \log_x b$$

$$\log_x(a^b) = b \log_x a$$

1. In class, we demonstrated the *first* and *third* rules above. For instance, for the first rule:

$$\log_2 8 = \log_2(2 \times 2 \times 2) = 3$$

$$\log_2 16 = \log_2(2 \times 2 \times 2 \times 2) = 4$$

$$\log_2(8 \times 16) = \log_2[(2 \times 2 \times 2)(2 \times 2 \times 2 \times 2)] = 7$$

This demonstrates that when you multiply two numbers, their logs *add*.

Now, you come up with a similar demonstration of the second rule of logs, that shows why when you *divide* two numbers, their logs *subtract*.

Now we're going to practice *applying* those three rules. *Take my word for these two facts.* (You don't have to memorize them, but you will be using them for this homework.)

- $\log_5 8 = 1.29$
- $\log_5 60 = 2.54$

Now, use those facts to answer the following questions.

2. $\log_5 480 =$

(*Hint:* $480 = 8 \times 60$. So this is $\log_5(8 \times 60)$. Which rule above helps you rewrite this?)

3. How can you use your calculator to test your answer to #2? (I'm assuming here that you can't find $\log_5 480$ on your calculator, but you can do exponents.) Run the test—did it work?

4. $\log_5\left(\frac{2}{15}\right) =$

5. $\log_5\left(\frac{15}{2}\right) =$

6. $\log_5 64 =$

7. $\log_5(5)^{23} =$

8. $5(\log_5 23) =$

Simplify, using the $\log(xy)$ property:

9. $\log_a(x \cdot x \cdot x \cdot x)$

10. $\log_a(x \cdot 1)$

Below bracket are different size

Simplify, using the $\log\left(\frac{x}{y}\right)$ property:

11. $\log_a\left(\frac{1}{x}\right)$

12. $\log_a\left(\frac{x}{1}\right)$

13. $\log_a\left(\frac{x}{x}\right)$

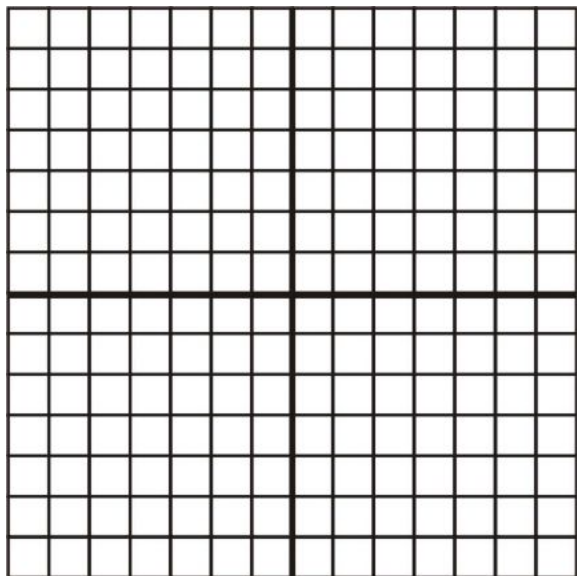
Simplify, using the $\log(x)^b$ property:

14. $\log_a(x)^4$

15. $\log_a(x)^0$

16. $\log_a(x)^{-1}$

17. a. Draw a graph of $y = \log_{\frac{1}{2}} x$. Plot at least 5 points to draw the graph.



b. What are the domain and range of the graph? What does that tell you about this function?

Using the Laws of Logarithms

$$\log_x(ab) = \log_x a + \log_x b$$

$$\log_x\left(\frac{a}{b}\right) = \log_x a - \log_x b$$

$$\log_x(a^b) = b \log_x a$$

1. Simplify: $\log_3(x^2) - \log_3(x)$

2. Simplify: $\log_3(9x) - \log_3(x)$

3. Simplify: $\frac{\log(x^2)}{\log(x)}$

4. Solve for x .

$$\log(2x + 5) = \log(8 - x)$$

5. Solve for x :

$$\log(3) + \log(x + 2) = \log(12)$$

6. Solve for x :

$$\ln(x) + \ln(x - 5) = \ln(14)$$

7. Solve for y in terms of x :

$$\log(x) = \log(5y) - \log(3y - 7)$$

5.3 So What Are Logarithms Good For, Anyway?

1. **Compound Interest.** Andy invests \$1,000 in a bank that pays out 7% interest, compounded annually. Note that your answers to parts (a) and (c) will be numbers, but your answers to parts (b) and (d) will be formulae.

- After 3 years, how much money does Andy have?
- After t years, how much money m does Andy have? $m(t) =$
- After how many years does Andy have exactly \$14,198.57?
- After how many years t does Andy have \$ m ? $t(m) =$

2. **Sound Intensity.** Sound is a wave in the air—the loudness of the sound is related to the intensity of the wave. The intensity of a whisper is approximately 100; the intensity of a normal conversation is approximately 1,000,000. Assuming that a person starts whispering at time $t = 0$, and gradually raises his voice to a normal conversational level by time $t = 10$, show a possible graph of the intensity of his voice. (*You can't get the graph exactly, since you only know the beginning and the end, but show the general shape.)

3. That was pretty complicated, wasn't it? It's almost impossible to graph or visualize something going from a hundred to a million: the range is too big.

Fortunately, sound volume is usually not measured in intensity, but in *loudness*, which are defined by the formula: $L = 10 \log_{10} I$, where L is the loudness (measured in decibels), and I is the intensity.

- What is the loudness, in decibels, of a whisper?
- What is the loudness, in decibels, of a normal conversation?
- Now do the graph again—showing an evolution from whisper to conversation in 30 seconds—but this time, graph loudness instead of intensity.
- That was a heck of a lot nicer, wasn't it? (This one is sort of rhetorical.)
- The quietest sound a human being can hear is intensity 1. What is the loudness of that sound?
- The sound of a jet engine—which is roughly when things get so loud they are painful—is loudness 120 decibels. What is the intensity of that sound?
- The formula I gave above gives loudness as a function of intensity. Write the opposite function, that gives intensity as a function of loudness.
- If sound A is *twenty decibels higher* than sound B , *how much more intense* is it?

6. pH. In Chemistry, a very important quantity is the *concentration of Hydrogen ions*, written as $[H^+]$ —this is related to the acidity of a liquid. In a normal pond, the concentration of Hydrogen ions is around 10^{-6} moles/liter. (In other words, every liter of water has about 10^{-6} , or $\frac{1}{1,000,000}$ moles of Hydrogen ions.) Now, acid rain begins to fall on that pond, and the concentration of Hydrogen ions begins to go up, until the concentration is 10^{-4} moles/liter (every liter has $\frac{1}{10,000}$ moles of H^+).

- How much did the concentration go up by?
 - Acidity is usually not measured as concentration (because the numbers are very unmanageable, as you can see), but as pH, which is defined as $-\log_{10}[H^+]$. What is the *pH* of the normal pond?
 - What is the pH of the pond after the acid rain?
7. Based on numbers 2 – 5, write a brief description of *what kind of function* generally leads scientists to want to use a logarithmic scale.

Homework: What Are Logarithms Good For, Anyway?

- I invest \$300 in a bank that pays 5% interest, compounded annually. So after t years, I have $300(1.05)^t$ dollars in the bank. When I come back, I find that my account is worth \$1000. How many years has it been? *Your answer will not be a number—it will be a formula with a log in it.*
- The pH of a substance is given by the formula $\text{pH} = -\log_{10}[H^+]$, where $[H^+]$ is the concentration of Hydrogen ions.
 - If the Hydrogen concentration is $\frac{1}{10,000}$, what is the pH?
 - If the Hydrogen concentration is $\frac{1}{1,000,000}$, what is the pH?
 - What happens to the pH every time the Hydrogen concentration *divides* by 10?

You may have noticed that all our logarithmic functions use the base 10. Because this is so common, it is given a special name: the *common log*. When you see something like $\log(x)$ with no base written at all, that means the log is 10. (So $\log(x)$ is a shorthand way of writing $\log_{10}(x)$, just like \sqrt{x} is a shorthand way of writing $\sqrt[2]{x}$. With roots, if you don't see a little number there, you assume a 2. With logs, you assume a 10.)

- In the space below, write the question that $\log(x)$ asks.

Use the common log to answer the following questions.

- $\log 100$
- $\log 1,000$
- $\log 10,000$
- $\log (1 \text{ with } n \text{ 0s after it})$
- $\log 500$ (*use the log button on your calculator*)

OK, so the log button on your calculator does common logs, that is, logs base 10.

There is one other log button on your calculator. It is called the “natural log,” and it is written \ln (which sort of stands for “natural log” only backward—personally, I blame the French).

\ln means the log to the base e . What is e ? It's a long ugly number—kind of like π only different—it goes on forever and you can only approximate it, but it is somewhere around 2.7. Answer the following questions about the natural log.

- $\ln(e) =$
- $\ln(1) =$
- $\ln(0) =$

12. $\ln(e^5) =$

13. $\ln(3) =$ (**this is the only one that requires the \ln button on your calculator*)

Name: _____

Sample Test: Logarithms

1. $\log_3 3 =$

2. $\log_3 9 =$

3. $\log_3 27 =$

4. $\log_3 30 =$

(*approximately*)

5. $\log_3 1 =$

6. $\log_3(\frac{1}{3}) =$

7. $\log_3(\frac{1}{9}) =$

8. $\log_3(-3) =$

9. $\log_9 3 =$

10. $3^{\log_3 8} =$

11. $\log_{-3} 9 =$

12. $\log 100,000 =$

13. $\log \frac{1}{100,000} =$

14. $\ln e^3 =$

15. $\ln 4 =$

16. Rewrite as a logarithm equation (no exponents): $q^z = p$

17. Rewrite as an exponent equation (no logs): $\log_w g = j$

For questions 18 – 22, assume that...

$$\log_5 12 = 1.544$$

$$\log_5 20 = 1.861$$

18. $\log_5 240 =$

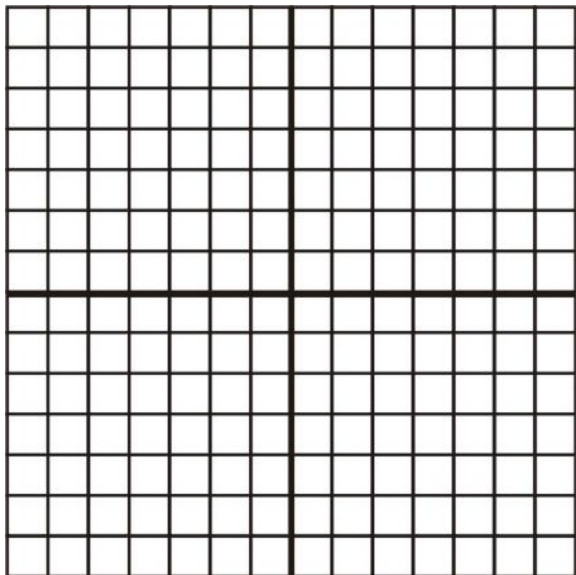
19. $\log_5(\frac{3}{5}) =$

20. $\log_5 1\frac{2}{3} =$

21. $\log_5(\frac{3}{5})^2 =$

22. $\log_5 400 =$

23. Graph $y = -\log_2 x + 2$.



24. What are the domain and range of the graph you drew in #23?
25. I invest \$200 in a bank that pays 4% interest, compounded annually. So after y years, I have $200(1.04)^y$ dollars in the bank. When I come back, I find that my account is worth \$1000. How many years has it been? *Your answer will be a formula with a log in it.*
26. The “loudness” of a sound is given by the formula $L = 10 \log I$, where L is the loudness (measured in decibels), and I is the intensity of the sound wave.
- If the sound wave intensity is 10, what is the loudness?
 - If the sound wave intensity is 10,000, what is the loudness?
 - If the sound wave intensity is 10,000,000, what is the loudness?
 - What happens to the loudness every time the sound wave intensity *multiplies* by 1,000?
27. Solve for x .

$$\ln(3) + \ln(x) = \ln(21)$$

28. solve for x .

$$\log_2(x) + \log_2(x + 10) = \log_2(11)$$

Extra credit: Solve for x . $e^x = (\text{cabin})$

Chapter 6

Polynomials

6.1 Addition and Subtraction of Polynomials

Learning Objectives

- Write a polynomial expression in standard form.
- Classify polynomial expression by degree.
- Add and subtract polynomials.
- Solve problems using addition and subtraction of polynomials.

Introduction

So far we've seen functions described by straight lines (linear functions) and functions where the variable appeared in the exponent (exponential functions). In this section we'll introduce polynomial functions. A **polynomial** is made up of different terms that contain **positive integer** powers of the variables. Here is an example of a polynomial:

$$4x^3 + 2x^2 - 3x + 1$$

Each part of the polynomial that is added or subtracted is called a **term** of the polynomial. The example above is a polynomial with *four terms*.

The numbers appearing in each term in front of the variable are called the **coefficients**. The number appearing all by itself without a variable is called a **constant**.

$4x^3 + 2x^2 - 3x + 1$

coefficients constant

In this case the coefficient of x^3 is **4**, the coefficient of x^2 is **2**, the coefficient of x is **-3** and the constant is **1**.

Degrees of Polynomials and Standard Form

Each term in the polynomial has a different **degree**. The degree of the term is the power of the variable in that term.

| | |
|--------|---|
| $4x^3$ | has degree 3 and is called a cubic term or 3^{rd} order term. |
| $2x^2$ | has degree 2 and is called a quadratic term or 2^{nd} order term. |
| $-3x$ | has degree 1 and is called a linear term or 1^{st} order term. |
| 1 | has degree 0 and is called the constant. |

By definition, **the degree of the polynomial** is the same as the degree of the term with the highest degree. This example is a polynomial of degree 3, which is also called a “cubic” polynomial. (Why do you think it is called a cubic?).

Polynomials can have more than one variable. Here is another example of a polynomial:

$$t^4 - 6s^3t^2 - 12st + 4s^4 - 5$$

This is a polynomial because all the exponents on the variables are positive integers. This polynomial has five terms. Let’s look at each term more closely.

Note: *The degree of a term is the sum of the powers on each variable in the term.* In other words, the degree of each term is the number of variables that are multiplied together in that term, whether those variables are the same or different.

| | |
|------------|---|
| t^4 | has a degree of 4, so it’s a 4^{th} order term |
| $-6s^3t^2$ | has a degree of 5, so it’s a 5^{th} order term. |
| $-12st$ | has a degree of 2, so it’s a 2^{nd} order term. |
| $4s^4$ | has a degree of 4, so it’s a 4^{th} order term. |
| -5 | is a constant, so its degree is 0. |

Since the highest degree of a term in this polynomial is 5, then this is polynomial of degree 5^{th} or a 5^{th} order polynomial.

A polynomial that has only one term has a special name. It is called a **monomial** (*mono* means one). A monomial can be a constant, a variable, or a product of a constant and one or more variables. You can see that each term in a polynomial is a monomial, so a polynomial is just the sum of several monomials. Here are some examples of monomials:

$$b^2 \quad -2ab^2 \quad 8 \quad \frac{1}{4}x^4 \quad -29xy$$

Example 1

For the following polynomials, identify the coefficient of each term, the constant, the degree of each term and the degree of the polynomial.

a) $x^5 - 3x^3 + 4x^2 - 5x + 7$

b) $x^4 - 3x^3y^2 + 8x - 12$

Solution

a) $x^5 - 3x^3 + 4x^2 - 5x + 7$

The coefficients of each term in order are 1, -3, 4, and -5 and the constant is 7.

The degrees of each term are 5, 3, 2, 1, and 0. Therefore the degree of the polynomial is 5.

b) $x^4 - 3x^3y^2 + 8x - 12$

The coefficients of each term in order are 1, -3, and 8 and the constant is -12.

The degrees of each term are 4, 5, 1, and 0. Therefore the degree of the polynomial is 5.

Example 2

Identify the following expressions as polynomials or non-polynomials.

a) $5x^5 - 2x$

b) $3x^2 - 2x^{-2}$

c) $x\sqrt{x} - 1$

d) $\frac{5}{x^3+1}$

e) $4x^{\frac{1}{3}}$

f) $4xy^2 - 2x^2y - 3 + y^3 - 3x^3$

Solution

a) This **is** a polynomial.

b) This is **not** a polynomial because it has a negative exponent.

c) This is **not** a polynomial because it has a radical.

d) This is **not** a polynomial because the power of x appears in the denominator of a fraction (and there is no way to rewrite it so that it does not).

e) This is **not** a polynomial because it has a fractional exponent.

f) This **is** a polynomial.

Often, we arrange the terms in a polynomial in order of decreasing power. This is called **standard form**.

The following polynomials are in standard form:

$$4x^4 - 3x^3 + 2x^2 - x + 1$$

$$a^4b^3 - 2a^3b^3 + 3a^4b - 5ab^2 + 2$$

The first term of a polynomial in standard form is called the **leading term**, and the coefficient of the leading term is called the **leading coefficient**.

The first polynomial above has the leading term $4x^4$, and the leading coefficient is 4.

The second polynomial above has the leading term a^4b^3 , and the leading coefficient is 1.

Example 3

Rearrange the terms in the following polynomials so that they are in standard form. Indicate the leading term and leading coefficient of each polynomial.

a) $7 - 3x^3 + 4x$

b) $ab - a^3 + 2b$

c) $-4b + 4 + b^2$

Solution

- a) $7 - 3x^3 + 4x$ becomes $-3x^3 + 4x + 7$. Leading term is $-3x^3$; leading coefficient is -3.
 b) $ab - a^3 + 2b$ becomes $-a^3 + ab + 2b$. Leading term is $-a^3$; leading coefficient is -1.
 c) $-4b + 4 + b^2$ becomes $b^2 - 4b + 4$. Leading term is b^2 ; leading coefficient is 1.

Simplifying Polynomials

A polynomial is simplified if it has no terms that are alike. **Like terms** are terms in the polynomial that have the same variable(s) with the same exponents, whether they have the same or different coefficients.

For example, $2x^2y$ and $5x^2y$ are like terms, but $6x^2y$ and $6xy^2$ are not like terms.

When a polynomial has like terms, we can simplify it by combining those terms.

$$x^2 + \frac{6xy}{\nearrow} - \frac{4xy}{\nwarrow} + y^2$$

Like terms

We can simplify this polynomial by combining the like terms $6xy$ and $-4xy$ into $(6 - 4)xy$, or $2xy$. The new polynomial is $x^2 + 2xy + y^2$.

Example 4

Simplify the following polynomials by collecting like terms and combining them.

- a) $2x - 4x^2 + 6 + x^2 - 4 + 4x$
 b) $a^3b^3 - 5ab^4 + 2a^3b - a^3b^3 + 3ab^4 - a^2b$

Solution

a) Rearrange the terms so that like terms are grouped together: $(-4x^2 + x^2) + (2x + 4x) + (6 - 4)$

Combine each set of like terms: $-3x^2 + 6x + 2$

b) Rearrange the terms so that like terms are grouped together: $(a^3b^3 - a^3b^3) + (-5ab^4 + 3ab^4) + 2a^3b - a^2b$

Combine each set of like terms: $0 - 2ab^4 + 2a^3b - a^2b = -2ab^4 + 2a^3b - a^2b$

Adding and Subtracting Polynomials

To add two or more polynomials, write their sum and then simplify by combining like terms.

Example 5

Add and simplify the resulting polynomials.

- a) Add $3x^2 - 4x + 7$ and $2x^3 - 4x^2 - 6x + 5$
 b) Add $x^2 - 2xy + y^2$ and $2y^2 - 3x^2$ and $10xy + y^3$

Solution

a)

$$(3x^2 - 4x + 7) + (2x^3 - 4x^2 - 6x + 5)$$

$$\text{Group like terms: } = 2x^3 + (3x^2 - 4x^2) + (-4x - 6x) + (7 + 5)$$

$$\text{Simplify: } = 2x^3 - x^2 - 10x + 12$$

b)

$$\begin{aligned}
 & (x^2 - 2xy + y^2) + (2y^2 - 3x^2) + (10xy + y^3) \\
 \text{Group like terms: } &= (x^2 - 3x^2) + (y^2 + 2y^2) + (-2xy + 10xy) + y^3 \\
 \text{Simplify: } &= -2x^2 + 3y^2 + 8xy + y^3
 \end{aligned}$$

To subtract one polynomial from another, add the opposite of each term of the polynomial you are subtracting.

Example 6

a) Subtract $4x^2 + 5x - 9$ from $x^3 - 3x^2 + 8x + 12$

b) Subtract $4a^2 - 8ab - 9b^2$ from $5b^2 - 2a^2$

Solution

a)

$$\begin{aligned}
 (4x^2 + 5x - 9) - (x^3 - 3x^2 + 8x + 12) &= (4x^2 + 5x - 9) + (-x^3 + 3x^2 - 8x - 12) \\
 \text{Group like terms: } &= -x^3 + (4x^2 + 3x^2) + (5x - 8x) + (-9 - 12) \\
 \text{Simplify: } &= -x^3 + 7x^2 - 3x - 21
 \end{aligned}$$

b)

$$\begin{aligned}
 (4a^2 - 8ab - 9b^2) - (5b^2 - 2a^2) &= (4a^2 - 8ab - 9b^2) + (-5b^2 + 2a^2) \\
 \text{Group like terms: } &= (4a^2 + 2a^2) + (-9b^2 - 5b^2) - 8ab \\
 \text{Simplify: } &= 6a^2 - 14b^2 - 8ab
 \end{aligned}$$

Note: An easy way to check your work after adding or subtracting polynomials is to substitute a convenient value in for the variable, and check that your answer and the problem both give the same value. For example, in part (b) above, if we let $a = 2$ and $b = 3$, then we can check as follows:

| Given | Solution |
|---|------------------------------|
| $(4a^2 - 8ab - 9b^2) - (5b^2 - 2a^2)$ | $6a^2 - 14b^2 - 8ab$ |
| $(4(2)^2 - 8(2)(3) - 9(3)^2) - (5(3)^2 - 2(2)^2)$ | $6(2)^2 - 14(3)^2 - 8(2)(3)$ |
| $(4(4) - 8(2)(3) - 9(9)) - (5(9) - 2(4))$ | $6(4) - 14(9) - 8(2)(3)$ |
| $(-113) - 37$ | $24 - 126 - 48$ |
| -150 | -150 |

Since both expressions evaluate to the same number when we substitute in arbitrary values for the variables, we can be reasonably sure that our answer is correct.

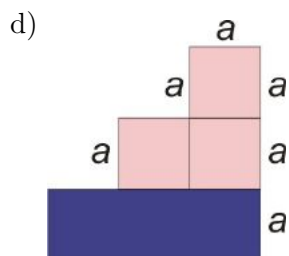
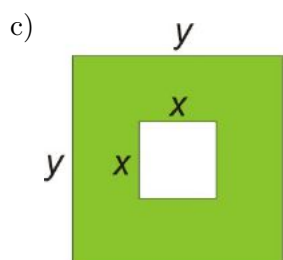
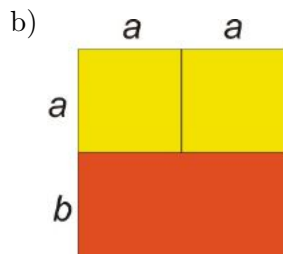
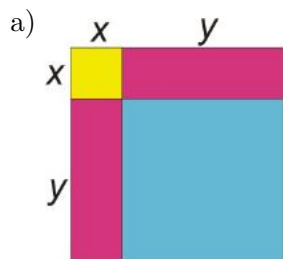
Note: When you use this method, do not choose 0 or 1 for checking since these can lead to common problems.

Problem Solving Using Addition or Subtraction of Polynomials

One way we can use polynomials is to find the area of a geometric figure.

Example 7

Write a polynomial that represents the area of each figure shown.



Solution

a) This shape is formed by two squares and two rectangles.

The blue square has area $y \times y = y^2$.

The yellow square has area $x \times x = x^2$.

The pink rectangles each have area $x \times y = xy$.

To find the total area of the figure we add all the separate areas:

$$\begin{aligned} \text{Total area} &= y^2 + x^2 + xy + xy \\ &= y^2 + x^2 + 2xy \end{aligned}$$

b) This shape is formed by two squares and one rectangle.

The yellow squares each have area $a \times a = a^2$.

The orange rectangle has area $2a \times b = 2ab$.

To find the total area of the figure we add all the separate areas:

$$\begin{aligned} \text{Total area} &= a^2 + a^2 + 2ab \\ &= 2a^2 + 2ab \end{aligned}$$

c) To find the area of the green region we find the area of the big square and subtract the area of the little square.

$$\begin{aligned} \text{The big square has area : } y \times y &= y^2. \\ \text{The little square has area : } x \times x &= x^2. \\ \text{Area of the green region} &= y^2 - x^2 \end{aligned}$$

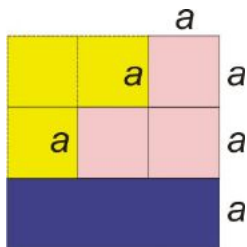
d) To find the area of the figure we can find the area of the big rectangle and add the areas of the pink squares.

$$\begin{aligned} \text{The pink squares each have area } a \times a &= a^2. \\ \text{The blue rectangle has area } 3a \times a &= 3a^2. \end{aligned}$$

To find the total area of the figure we add all the separate areas:

$$\text{Total area} = a^2 + a^2 + a^2 + 3a^2 = 6a^2$$

Another way to find this area is to find the area of the big square and subtract the areas of the three yellow squares:



$$\begin{aligned} \text{The big square has area } 3a \times 3a &= 9a^2. \\ \text{The yellow squares each have area } a \times a &= a^2. \end{aligned}$$

To find the total area of the figure we subtract:

$$\begin{aligned} \text{Area} &= 9a^2 - (a^2 + a^2 + a^2) \\ &= 9a^2 - 3a^2 \\ &= 6a^2 \end{aligned}$$

Further Practice

For more practice adding and subtracting polynomials, try playing the Battleship game at <http://www.quia.com/ba/28>. (The problems get harder as you play; watch out for trick questions!)

Review Questions

Indicate whether each expression is a polynomial.

1. $x^2 + 3x^{\frac{1}{2}}$
2. $\frac{1}{3}x^2y - 9y^2$
3. $3x^{-3}$
4. $\frac{2}{3}t^2 - \frac{1}{t^2}$
5. $\sqrt{x} - 2x$
6. $(x^{\frac{3}{2}})^2$

Express each polynomial in standard form. Give the degree of each polynomial.

7. $3 - 2x$
8. $8 - 4x + 3x^3$
9. $-5 + 2x - 5x^2 + 8x^3$
10. $x^2 - 9x^4 + 12$
11. $5x + 2x^2 - 3x$

Add and simplify.

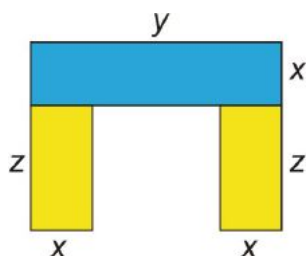
12. $(x + 8) + (-3x - 5)$
13. $(-2x^2 + 4x - 12) + (7x + x^2)$
14. $(2a^2b - 2a + 9) + (5a^2b - 4b + 5)$
15. $(6.9a^2 - 2.3b^2 + 2ab) + (3.1a - 2.5b^2 + b)$
16. $(\frac{3}{5}x^2 - \frac{1}{4}x + 4) + (\frac{1}{10}x^2 + \frac{1}{2}x - 2\frac{1}{5})$

Subtract and simplify.

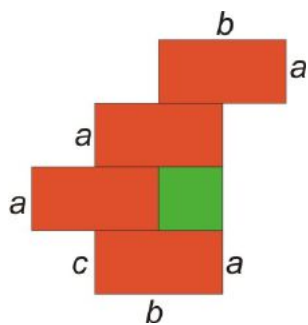
17. $(-t + 5t^2) - (5t^2 + 2t - 9)$
18. $(-y^2 + 4y - 5) - (5y^2 + 2y + 7)$
19. $(-5m^2 - m) - (3m^2 + 4m - 5)$
20. $(2a^2b - 3ab^2 + 5a^2b^2) - (2a^2b^2 + 4a^2b - 5b^2)$
21. $(3.5x^2y - 6xy + 4x) - (1.2x^2y - xy + 2y - 3)$

Find the area of the following figures.

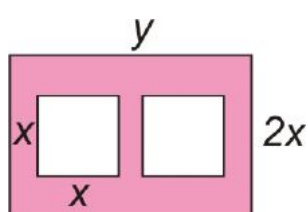
22.



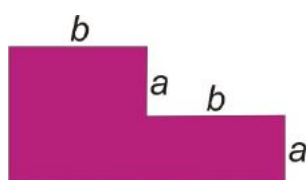
23.



24.



25.



6.2 Multiplication of Polynomials

Learning Objectives

- Multiply a polynomial by a monomial.
- Multiply a polynomial by a binomial.
- Solve problems using multiplication of polynomials.

Introduction

Just as we can add and subtract polynomials, we can also multiply them. The Distributive Property and the techniques you've learned for dealing with exponents will be useful here.

Multiplying a Polynomial by a Monomial

When multiplying polynomials, we must remember the exponent rules that we learned in the last chapter. Especially important is the product rule: $x^n \cdot x^m = x^{n+m}$.

If the expressions we are multiplying have coefficients and more than one variable, we multiply the coefficients just as we would any number and we apply the product rule on each variable separately.

Example 1

Multiply the following monomials.

- $(2x^2)(5x^3)$
- $(-3y^4)(2y^2)$
- $(3xy^5)(-6x^4y^2)$

d) $(-12a^2b^3c^4)(-3a^2b^2)$

Solution

a) $(2x^2)(5x^3) = (2 \cdot 5) \cdot (x^2 \cdot x^3) = 10x^{2+3} = 10x^5$

b) $(-3y^4)(2y^2) = (-3 \cdot 2) \cdot (y^4 \cdot y^2) = -6y^{4+2} = -6y^6$

c) $(3xy^5)(-6x^4y^2) = -18x^{1+4}y^{5+2} = -18x^5y^7$

d) $(-12a^2b^3c^4)(-3a^2b^2) = 36a^{2+2}b^{3+2}c^4 = 36a^4b^5c^4$

To multiply a polynomial by a monomial, we have to use the **Distributive Property**. Remember, that property says that $a(b + c) = ab + ac$.

Example 2

Multiply:

a) $3(x^2 + 3x - 5)$

b) $4x(3x^2 - 7)$

c) $-7y(4y^2 - 2y + 1)$

Solution

a) $3(x^2 + 3x - 5) = 3(x^2) + 3(3x) - 3(5) = 3x^2 + 9x - 15$

b) $4x(3x^2 - 7) = (4x)(3x^2) + (4x)(-7) = 12x^3 - 28x$

c) $-7y(4y^2 - 2y + 1) = (-7y)(4y^2) + (-7y)(-2y) + (-7y)(1) = -28y^3 + 14y^2 - 7y$

Notice that when we use the Distributive Property, the problem becomes a matter of just multiplying monomials by monomials and adding all the separate parts together.

Example 3

Multiply:

a) $2x^3(-3x^4 + 2x^3 - 10x^2 + 7x + 9)$

b) $-7a^2bc^3(5a^2 - 3b^2 - 9c^2)$

Solution

a)
$$\begin{aligned} 2x^3(-3x^4 + 2x^3 - 10x^2 + 7x + 9) &= (2x^3)(-3x^4) + (2x^3)(2x^3) + (2x^3)(-10x^2) + (2x^3)(7x) + (2x^3)(9) \\ &= -6x^7 + 4x^6 - 20x^5 + 14x^4 + 18x^3 \end{aligned}$$

b)
$$\begin{aligned} -7a^2bc^3(5a^2 - 3b^2 - 9c^2) &= (-7a^2bc^3)(5a^2) + (-7a^2bc^3)(-3b^2) + (-7a^2bc^3)(-9c^2) \\ &= -35a^4bc^3 + 21a^2b^3c^3 + 63a^2bc^5 \end{aligned}$$

Multiplying Two Polynomials

Let's start by multiplying two binomials together. A binomial is a polynomial with two terms, so a product of two binomials will take the form $(a + b)(c + d)$.

We can still use the Distributive Property here if we do it cleverly. First, let's think of the first set of parentheses as one term. The Distributive Property says that we can multiply that term by c , multiply it by d , and then add those two products together: $(a + b)(c + d) = (a + b) \cdot c + (a + b) \cdot d$.

We can rewrite this expression as $c(a + b) + d(a + b)$. Now let's look at each half separately. We can apply the distributive property again to each set of parentheses in turn, and that gives us $c(a + b) + d(a + b) = ca + cb + da + db$.

What you should notice is that when multiplying any two polynomials, *every term in one polynomial is multiplied by every term in the other polynomial*.

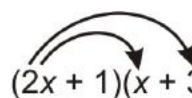
Example 4

Multiply and simplify: $(2x + 1)(x + 3)$


Solution

We must multiply each term in the first polynomial by each term in the second polynomial. Let's try to be systematic to make sure that we get all the products.

First, multiply the first term in the first set of parentheses by all the terms in the second set of parentheses.

$$(2x + 1)(x + 3) = (2x)(x) + (2x)(3) + \dots$$


Now we're done with the first term. Next we multiply the second term in the first parenthesis by all terms in the second parenthesis and add them to the previous terms.

$$(2x + 1)(x + 3) = (2x)(x) + (2x)(3) + (1)(x) + (1)(3)$$


Now we're done with the multiplication and we can simplify:

$$(2x)(x) + (2x)(3) + (1)(x) + (1)(3) = 2x^2 + 6x + x + 3 = 2x^2 + 7x + 3$$

This way of multiplying polynomials is called **in-line** multiplication or **horizontal** multiplication. Another method for multiplying polynomials is to use **vertical** multiplication, similar to the vertical multiplication you learned with regular numbers.

Example 5

Multiply and simplify:

- a) $(4x - 5)(x - 20)$
- b) $(3x - 2)(3x + 2)$
- c) $(3x^2 + 2x - 5)(2x - 3)$
- d) $(x^2 - 9)(4x^4 + 5x^2 - 2)$

Solution

a) With horizontal multiplication this would be

$$(4x - 5)(x - 20) = (4x)(x) + (4x)(-20) + (-5)(x) + (-5)(-20) = 4x^2 - 80x - 5x + 100 = 4x^2 - 85x + 100$$

To do vertical multiplication instead, we arrange the polynomials on top of each other with like terms in the same columns:

$$\begin{array}{r} 4x - 5 \\ x - 20 \\ \hline -80x + 100 \\ 4x^2 - 5x \\ \hline 4x^2 - 85x + 100 \end{array}$$

Both techniques result in the same answer: $4x^2 - 85x + 100$. We'll use vertical multiplication for the other problems.

b)

$$\begin{array}{r} 3x - 2 \\ 3x + 2 \\ \hline 6x - 4 \\ 9x^2 - 6x \\ \hline 9x^2 + 0x - 4 \end{array}$$

The answer is $9x^2 - 4$.

c) It's better to place the smaller polynomial on the bottom:

$$\begin{array}{r} 3x^2 + 2x - 5 \\ 2x - 3 \\ \hline -9x^2 - 6x + 15 \\ 6x^3 + 4x^2 - 10x \\ \hline 6x^3 - 5x^2 - 16x + 15 \end{array}$$

The answer is $6x^3 - 5x^2 - 16x + 15$.

d) Set up the multiplication vertically and leave gaps for missing powers of x :

$$\begin{array}{r} 4x^4 + 5x^2 - 2 \\ x^2 - 9 \\ \hline -36x^4 - 45x^2 + 18 \\ 4x^6 + 5x^4 - 2x^2 \\ \hline 4x^6 - 31x^4 - 47x^2 + 18 \end{array}$$

The answer is $4x^6 - 31x^4 - 47x^2 + 18$.

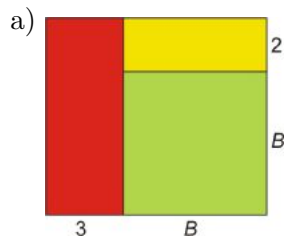
The Khan Academy video at <http://www.youtube.com/watch?v=Sc0e6xrRJYY> shows how multiplying two binomials together is related to the distributive property.

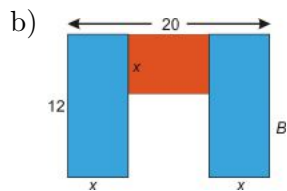
Solve Real-World Problems Using Multiplication of Polynomials

In this section, we'll see how multiplication of polynomials is applied to finding the areas and volumes of geometric shapes.

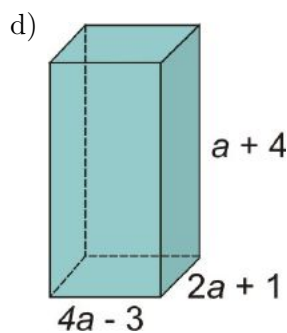
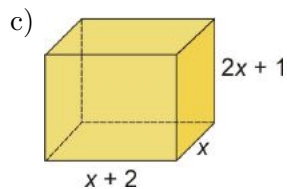
Example 6

Find the areas of the following figures:





Find the volumes of the following figures:



Solution

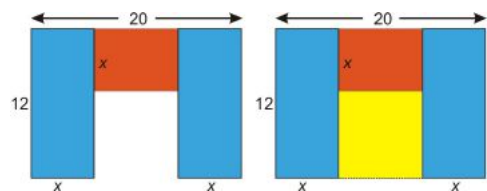
a) We use the formula for the area of a rectangle: Area = length \times width.

For the big rectangle:

$$\text{Length} = b + 3, \text{ Width} = b + 2$$

$$\begin{aligned} \text{Area} &= (b + 3)(b + 2) \\ &= b^2 + 2b + 3b + 6 \\ &= b^2 + 5b + 6 \end{aligned}$$

b) We could add up the areas of the blue and orange rectangles, but it's easier to just find the area of the whole *big* rectangle and subtract the area of the yellow rectangle.



$$\text{Area of big rectangle} = 20(12) = 240$$

$$\begin{aligned} \text{Area of yellow rectangle} &= (12 - x)(20 - 2x) \\ &= 240 - 24x - 20x + 2x^2 \\ &= 240 - 44x + 2x^2 \\ &= 2x^2 - 44x + 240 \end{aligned}$$

The desired area is the difference between the two:

$$\begin{aligned}
 \text{Area} &= 240 - (2x^2 - 44x + 240) \\
 &= 240 + (-2x^2 + 44x - 240) \\
 &= 240 - 2x^2 + 44x - 240 \\
 &= -2x^2 + 44x
 \end{aligned}$$

c) The volume of this shape = (area of the base)(height).

$$\begin{aligned}
 \text{Area of the base} &= x(x + 2) \\
 &= x^2 + 2x \\
 \text{Height} &= 2x + 1 \\
 \text{Volume} &= (x^2 + 2x)(2x + 1) \\
 &= 2x^3 + x^2 + 4x^2 + 2x \\
 &= 2x^3 + 5x^2 + 2x
 \end{aligned}$$

d) The volume of this shape = (area of the base)(height).

$$\begin{aligned}
 \text{Area of the base} &= (4a - 3)(2a + 1) \\
 &= 8a^2 + 4a - 6a - 3 \\
 &= 8a^2 - 2a - 3 \\
 \text{Height} &= a + 4 \\
 \text{Volume} &= (8a^2 - 2a - 3)(a + 4)
 \end{aligned}$$

Let's multiply using the vertical method:

$$\begin{array}{r}
 8a^2 - 2a - 3 \\
 \underline{ a + 4} \\
 32a^2 - 8a - 12 \\
 8a^3 - 2a^2 - 3a \\
 \hline
 8a^3 + 30a^2 - 11a - 12
 \end{array}$$

The volume is $8a^3 + 30a^2 - 11a - 12$.

Review Questions

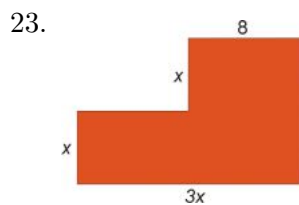
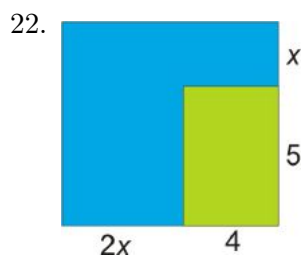
Multiply the following monomials.

1. $(2x)(-7x)$
2. $(10x)(3xy)$
3. $(4mn)(0.5nm^2)$
4. $(-5a^2b)(-12a^3b^3)$
5. $(3xy^2z^2)(15x^2yz^3)$

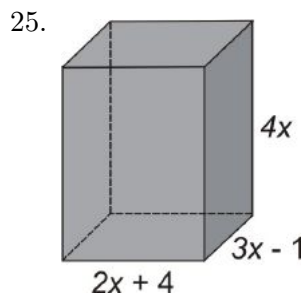
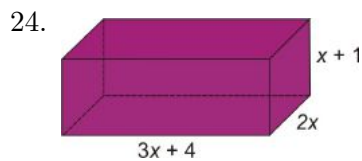
Multiply and simplify.

6. $17(8x - 10)$
7. $2x(4x - 5)$
8. $9x^3(3x^2 - 2x + 7)$
9. $3x(2y^2 + y - 5)$
10. $10q(3q^2r + 5r)$
11. $-3a^2b(9a^2 - 4b^2)$
12. $(x - 3)(x + 2)$
13. $(a + b)(a - 5)$
14. $(x + 2)(x^2 - 3)$
15. $(a^2 + 2)(3a^2 - 4)$
16. $(7x - 2)(9x - 5)$
17. $(2x - 1)(2x^2 - x + 3)$
18. $(3x + 2)(9x^2 - 6x + 4)$
19. $(a^2 + 2a - 3)(a^2 - 3a + 4)$
20. $3(x - 5)(2x + 7)$
21. $5x(x + 4)(2x - 3)$

Find the areas of the following figures.



Find the volumes of the following figures.



6.3 Special Products of Polynomials

Learning Objectives

- Find the square of a binomial
- Find the product of binomials using sum and difference formula
- Solve problems using special products of polynomials

Introduction

We saw that when we multiply two binomials we need to make sure to multiply each term in the first binomial with each term in the second binomial. Let's look at another example.

Multiply two linear binomials (binomials whose degree is 1):

$$(2x + 3)(x + 4)$$

When we multiply, we obtain a quadratic polynomial (one with degree 2) with four terms:

$$2x^2 + 8x + 3x + 12$$

The middle terms are like terms and we can combine them. We simplify and get $2x^2 + 11x + 12$. This is a quadratic, or second-degree, **trinomial** (polynomial with three terms).

You can see that every time we multiply two linear binomials with one variable, we will obtain a quadratic polynomial. In this section we'll talk about some special products of binomials.

Find the Square of a Binomial

One special binomial product is the **square of a binomial**. Consider the product $(x + 4)(x + 4)$.

Since we are multiplying the same expression by itself, that means we are squaring the expression. $(x + 4)(x + 4)$ is the same as $(x + 4)^2$.

When we multiply it out, we get $x^2 + 4x + 4x + 16$, which simplifies to $x^2 + 8x + 16$.

Notice that the two middle terms—the ones we added together to get $8x$ —were the same. Is this a coincidence? In order to find that out, let's square a general linear binomial.

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b) = a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2\end{aligned}$$

Sure enough, the middle terms are the same. How about if the expression we square is a difference instead of a sum?

$$\begin{aligned}(a - b)^2 &= (a - b)(a - b) = a^2 - ab - ab + b^2 \\ &= a^2 - 2ab + b^2\end{aligned}$$

It looks like the middle two terms are the same in general whenever we square a binomial. The general pattern is: to square a binomial, take the square of the first term, add or subtract twice the product of the terms, and add the square of the second term. You should remember these formulas:

$$(a + b)^2 = a^2 + 2ab + b^2$$

and

$$(a - b)^2 = a^2 - 2ab + b^2$$

Remember! Raising a polynomial to a power means that we multiply the polynomial by itself however many times the exponent indicates. For instance, $(a + b)^2 = (a + b)(a + b)$. **Don't make the common mistake of thinking that $(a + b)^2 = a^2 + b^2$!** To see why that's not true, try substituting numbers for a and b into the equation (for example, $a = 4$ and $b = 3$), and you will see that it is *not* a true statement. The middle term, $2ab$, is needed to make the equation work.

We can apply the formulas for squaring binomials to any number of problems.

Example 1

Square each binomial and simplify.

- a) $(x + 10)^2$
- b) $(2x - 3)^2$
- c) $(x^2 + 4)^2$
- d) $(5x - 2y)^2$

Solution

Let's use the square of a binomial formula to multiply each expression.

a) $(x + 10)^2$

If we let $a = x$ and $b = 10$, then our formula $(a + b)^2 = a^2 + 2ab + b^2$ becomes $(x + 10)^2 = x^2 + 2(x)(10) + 10^2$, which simplifies to $x^2 + 20x + 100$.

b) $(2x - 3)^2$

If we let $a = 2x$ and $b = 3$, then our formula $(a - b)^2 = a^2 - 2ab + b^2$ becomes $(2x - 3)^2 = (2x)^2 - 2(2x)(3) + (3)^2$, which simplifies to $4x^2 - 12x + 9$.

c) $(x^2 + 4)^2$

If we let $a = x^2$ and $b = 4$, then

$$\begin{aligned}(x^2 + 4)^2 &= (x^2)^2 + 2(x^2)(4) + (4)^2 \\ &= x^4 + 8x^2 + 16\end{aligned}$$

d) $(5x - 2y)^2$

If we let $a = 5x$ and $b = 2y$, then

$$\begin{aligned}(5x - 2y)^2 &= (5x)^2 - 2(5x)(2y) + (2y)^2 \\ &= 25x^2 - 20xy + 4y^2\end{aligned}$$

Find the Product of Binomials Using Sum and Difference Patterns

Another special binomial product is the product of a sum and a difference of terms. For example, let's multiply the following binomials.

$$\begin{aligned}(x + 4)(x - 4) &= x^2 - 4x + 4x - 16 \\ &= x^2 - 16\end{aligned}$$

Notice that the middle terms are opposites of each other, so they *cancel out* when we collect like terms. This is not a coincidence. This always happens when we multiply a sum and difference of the same terms. In general,

$$\begin{aligned}(a + b)(a - b) &= a^2 - ab + ab - b^2 \\ &= a^2 - b^2\end{aligned}$$

When multiplying a sum and difference of the same two terms, the middle terms cancel out. We get the square of the first term minus the square of the second term. You should remember this formula.

Sum and Difference Formula: $(a + b)(a - b) = a^2 - b^2$

Let's apply this formula to a few examples.

Example 2

Multiply the following binomials and simplify.

a) $(x + 3)(x - 3)$

b) $(5x + 9)(5x - 9)$

c) $(2x^3 + 7)(2x^3 - 7)$

d) $(4x + 5y)(4x - 5y)$

Solution

a) Let $a = x$ and $b = 3$, then:

$$\begin{aligned}(a + b)(a - b) &= a^2 - b^2 \\ (x + 3)(x - 3) &= x^2 - 3^2 \\ &= x^2 - 9\end{aligned}$$

b) Let $a = 5x$ and $b = 9$, then:

$$\begin{aligned}(a + b)(a - b) &= a^2 - b^2 \\ (5x + 9)(5x - 9) &= (5x)^2 - 9^2 \\ &= 25x^2 - 81\end{aligned}$$

c) Let $a = 2x^3$ and $b = 7$, then:

$$\begin{aligned}(2x^3 + 7)(2x^3 - 7) &= (2x^3)^2 - (7)^2 \\ &= 4x^6 - 49\end{aligned}$$

d) Let $a = 4x$ and $b = 5y$, then:

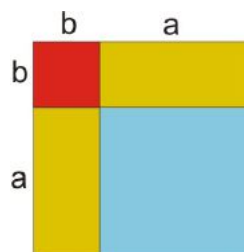
$$\begin{aligned}(4x + 5y)(4x - 5y) &= (4x)^2 - (5y)^2 \\ &= 16x^2 - 25y^2\end{aligned}$$

Solve Real-World Problems Using Special Products of Polynomials

Now let's see how special products of polynomials apply to geometry problems and to mental arithmetic.

Example 3

Find the area of the following square:



Solution

The length of each side is $(a + b)$, so the area is $(a + b)(a + b)$.

Notice that this gives a visual explanation of the square of a binomial. The blue square has area a^2 , the red square has area b^2 , and each rectangle has area ab , so added all together, the area $(a + b)(a + b)$ is equal to $a^2 + 2ab + b^2$.

The next example shows how you can use the special products to do fast mental calculations.

Example 4

Use the difference of squares and the binomial square formulas to find the products of the following numbers without using a calculator.

- a) 43×57
- b) 112×88
- c) 45^2
- d) 481×319

Solution

The key to these mental “tricks” is to rewrite each number as a sum or difference of numbers you know how to square easily.

- a) Rewrite 43 as $(50 - 7)$ and 57 as $(50 + 7)$.

$$\text{Then } 43 \times 57 = (50 - 7)(50 + 7) = (50)^2 - (7)^2 = 2500 - 49 = 2451$$

- b) Rewrite 112 as $(100 + 12)$ and 88 as $(100 - 12)$.

$$\text{Then } 112 \times 88 = (100 + 12)(100 - 12) = (100)^2 - (12)^2 = 10000 - 144 = 9856$$

- c) $45^2 = (40 + 5)^2 = (40)^2 + 2(40)(5) + (5)^2 = 1600 + 400 + 25 = 2025$

- d) Rewrite 481 as $(400 + 81)$ and 319 as $(400 - 81)$.

$$\text{Then } 481 \times 319 = (400 + 81)(400 - 81) = (400)^2 - (81)^2$$

$(400)^2$ is easy - it equals 160000.

$(81)^2$ is not easy to do mentally, so let's rewrite 81 as $80 + 1$.

$$(81)^2 = (80 + 1)^2 = (80)^2 + 2(80)(1) + (1)^2 = 6400 + 160 + 1 = 6561$$

Then $481 \times 319 = (400)^2 - (81)^2 = 160000 - 6561 = 153439$

Review Questions

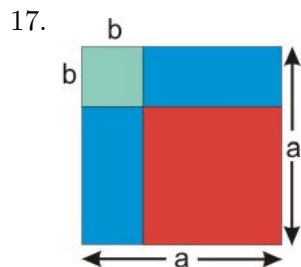
Use the special product rule for squaring binomials to multiply these expressions.

1. $(x + 9)^2$
2. $(3x - 7)^2$
3. $(5x - y)^2$
4. $(2x^3 - 3)^2$
5. $(4x^2 + y^2)^2$
6. $(8x - 3)^2$
7. $(2x + 5)(5 + 2x)$
8. $(xy - y)^2$

Use the special product of a sum and difference to multiply these expressions.

9. $(2x - 1)(2x + 1)$
10. $(x - 12)(x + 12)$
11. $(5a - 2b)(5a + 2b)$
12. $(ab - 1)(ab + 1)$
13. $(z^2 + y)(z^2 - y)$
14. $(2q^3 + r^2)(2q^3 - r^2)$
15. $(7s - t)(t + 7s)$
16. $(x^2y + xy^2)(x^2y - xy^2)$

Find the area of the lower right square in the following figure.



Multiply the following numbers using special products.

9. 45×55
10. 56^2
11. 1002×998
12. 36×44
13. 10.5×9.5
14. 100.2×9.8
15. -95×-105
16. 2×-2

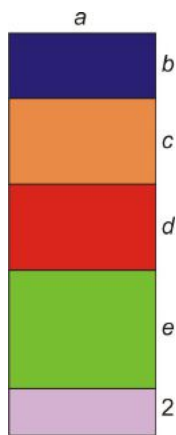
6.4 Polynomial Equations in Factored Form

Learning Objectives

- Use the zero-product property.
- Find greatest common monomial factors.
- Solve simple polynomial equations by factoring.

Introduction

In the last few sections, we learned how to multiply polynomials by using the Distributive Property. All the terms in one polynomial had to be multiplied by all the terms in the other polynomial. In this section, you'll start learning how to do this process in reverse. The reverse of distribution is called **factoring**.



The total area of the figure above can be found in two ways.

We could find the areas of all the small rectangles and add them: $ab + ac + ad + ae + 2a$.

Or, we could find the area of the big rectangle all at once. Its width is a and its length is $b + c + d + e + 2$, so its area is $a(b + c + d + e + 2)$.

Since the area of the rectangle is the same no matter what method we use, those two expressions must be equal.

$$ab + ac + ad + ae + 2a = a(b + c + d + e + 2)$$

To turn the right-hand side of this equation into the left-hand side, we would use the distributive property. To turn the left-hand side into the right-hand side, we would need to **factor** it. Since polynomials can be multiplied just like numbers, they can also be factored just like numbers—and we'll see later how this can help us solve problems.

Find the Greatest Common Monomial Factor

You will be learning several factoring methods in the next few sections. In most cases, factoring takes several steps to complete because we want to **factor completely**. That means that we factor until we can't factor any more.

Let's start with the simplest step: finding the greatest monomial factor. When we want to factor, we always look for common monomials first. Consider the following polynomial, written in expanded form:

$$ax + bx + cx + dx$$

A common factor is any factor that appears in all terms of the polynomial; it can be a number, a variable or a combination of numbers and variables. Notice that in our example, the factor x appears in all terms, so it is a common factor.

To factor out the x , we write it outside a set of parentheses. Inside the parentheses, we write what's left when we divide each term by x :

$$x(a + b + c + d)$$

Let's look at more examples.

Example 1

Factor:

a) $2x + 8$

b) $15x - 25$

c) $3a + 9b + 6$

Solution

a) We see that the factor 2 divides evenly into both terms: $2x + 8 = 2(x) + 2(4)$

We factor out the 2 by writing it in front of a parenthesis: $2()$

Inside the parenthesis we write what is left of each term when we divide by 2: $2(x + 4)$

b) We see that the factor of 5 divides evenly into all terms: $15x - 25 = 5(3x) - 5(5)$

Factor out the 5 to get: $5(3x - 5)$

c) We see that the factor of 3 divides evenly into all terms: $3a + 9b + 6 = 3(a) + 3(3b) + 3(2)$

Factor 3 to get: $3(a + 3b + 2)$

Example 2

Find the greatest common factor:

a) $a^3 - 3a^2 + 4a$

b) $12a^4 - 5a^3 + 7a^2$

Solution

a) Notice that the factor a appears in all terms of $a^3 - 3a^2 + 4a$, but each term has a raised to a different power. The greatest common factor of all the terms is simply a .

So first we rewrite $a^3 - 3a^2 + 4a$ as $a(a^2) + a(-3a) + a(4)$.

Then we factor out the a to get $a(a^2 - 3a + 4)$.

b) The factor a appears in all the terms, and it's always raised to at least the second power. So the greatest common factor of all the terms is a^2 .

We rewrite the expression $12a^4 - 5a^3 + 7a^2$ as $(12a^2 \cdot a^2) - (5a \cdot a^2) + (7 \cdot a^2)$

Factor out the a^2 to get $a^2(12a^2 - 5a + 7)$.

Example 3

Factor completely:

- a) $3ax + 9a$
- b) $x^3y + xy$
- c) $5x^3y - 15x^2y^2 + 25xy^3$

Solution

- a) Both terms have a common factor of 3, but they also have a common factor of a . It's simplest to factor these both out at once, which gives us $3a(x + 3)$.
- b) Both x and y are common factors. When we factor them both out at once, we get $xy(x^2 + 1)$.
- c) The common factors are 5, x , and y . Factoring out $5xy$ gives us $5xy(x^2 - 3xy + 5xy^2)$.

Use the Zero-Product Property

The most useful thing about factoring is that we can use it to help solve polynomial equations.

For example, consider an equation like $2x^2 + 5x - 42 = 0$. There's no good way to isolate x in this equation, so we can't solve it using any of the techniques we've already learned. But the left-hand side of the equation can be factored, making the equation $(x + 6)(2x - 7) = 0$.

How is this helpful? The answer lies in a useful property of multiplication: if two numbers multiply to zero, then at least one of those numbers must be zero. This is called the **Zero-Product Property**.

What does this mean for our polynomial equation? Since the product equals 0, then at least one of the factors on the left-hand side must equal zero. So we can find the two solutions by setting each factor equal to zero and solving each equation separately.

Setting the factors equal to zero gives us:

$$(x + 6) = 0 \qquad \text{OR} \qquad (2x - 7) = 0$$

Solving both of those equations gives us:

$$\begin{array}{ccc} x + 6 = 0 & & 2x - 7 = 0 \\ \underline{x = -6} & \text{OR} & 2x = 7 \\ & & x = \frac{7}{2} \end{array}$$

Notice that the solution is $x = -6$ **OR** $x = \frac{7}{2}$. The **OR** means that either of these values of x would make the product of the two factors equal to zero. Let's plug the solutions back into the equation and check that this is correct.

$$\begin{array}{ll} \text{Check : } x = -6; & \text{Check : } x = \frac{7}{2} \\ (x + 6)(2x - 7) = & (x + 6)(2x - 7) = \\ (-6 + 6)(2(-6) - 7) = & \left(\frac{7}{2} + 6\right)\left(2 \cdot \frac{7}{2} - 7\right) = \\ (0)(-19) = 0 & \left(\frac{19}{2}\right)(7 - 7) = \\ & \left(\frac{19}{2}\right)(0) = 0 \end{array}$$

Both solutions check out.

Factoring a polynomial is very useful because the Zero-Product Property allows us to break up the problem into simpler separate steps. When we can't factor a polynomial, the problem becomes harder and we must use other methods that you will learn later.

As a last note in this section, keep in mind that the Zero-Product Property only works when a product equals zero. For example, if you multiplied two numbers and the answer was nine, that wouldn't mean that one or both of the numbers must be nine. In order to use the property, the factored polynomial must be equal to zero.

Example 4

Solve each equation:

a) $(x - 9)(3x + 4) = 0$

b) $x(5x - 4) = 0$

c) $4x(x + 6)(4x - 9) = 0$

Solution

Since all the polynomials are in factored form, we can just set each factor equal to zero and solve the simpler equations separately

a) $(x - 9)(3x + 4) = 0$ can be split up into two linear equations:

$$x - 9 = 0$$

$$\underline{\underline{x = 9}}$$

or

$$3x + 4 = 0$$

$$3x = -4$$

$$\underline{\underline{x = -\frac{4}{3}}}$$

b) $x(5x - 4) = 0$ can be split up into two linear equations:

$$\underline{\underline{x = 0}}$$

or

$$5x - 4 = 0$$

$$5x = 4$$

$$\underline{\underline{x = \frac{4}{5}}}$$

c) $4x(x + 6)(4x - 9) = 0$ can be split up into three linear equations:

$$4x = 0$$

$$x = \frac{0}{4}$$

$$\underline{\underline{x = 0}}$$

or

$$x + 6 = 0$$

$$x = -6$$

or

$$4x - 9 = 0$$

$$4x = 9$$

$$\underline{\underline{x = \frac{9}{4}}}$$

Solve Simple Polynomial Equations by Factoring

Now that we know the basics of factoring, we can solve some simple polynomial equations. We already saw how we can use the Zero-Product Property to solve polynomials in factored form—now we can use that knowledge to solve polynomials by factoring them first. Here are the steps:

a) If necessary, **rewrite** the equation in standard form so that the right-hand side equals zero.

b) **Factor** the polynomial completely.

- c) Use the zero-product rule to **set each factor equal to zero**.
 d) **Solve** each equation from step 3.
 e) **Check** your answers by substituting your solutions into the original equation

Example 5

Solve the following polynomial equations.

- a) $x^2 - 2x = 0$
 b) $2x^2 = 5x$
 c) $9x^2y - 6xy = 0$

Solution

a) $x^2 - 2x = 0$

Rewrite: this is not necessary since the equation is in the correct form.

Factor: The common factor is x , so this factors as $x(x - 2) = 0$.

Set each factor equal to zero:

$$x = 0 \qquad \qquad \qquad \text{or} \qquad \qquad \qquad x - 2 = 0$$

Solve:

$$\underline{x = 0} \qquad \qquad \qquad \text{or} \qquad \qquad \qquad \underline{x = 2}$$

Check: Substitute each solution back into the original equation.

$$\begin{array}{ll} x = 0 \Rightarrow (0)^2 - 2(0) = 0 & \text{works out} \\ x = 2 \Rightarrow (2)^2 - 2(2) = 4 - 4 = 0 & \text{works out} \end{array}$$

Answer: $x = 0, x = 2$

b) $2x^2 = 5x$

Rewrite: $2x^2 = 5x \Rightarrow 2x^2 - 5x = 0$

Factor: The common factor is x , so this factors as $x(2x - 5) = 0$.

Set each factor equal to zero:

$$x = 0 \qquad \qquad \qquad \text{or} \qquad \qquad \qquad 2x - 5 = 0$$

Solve:

$$\underline{x = 0} \qquad \qquad \qquad \text{or} \qquad \qquad \qquad \begin{array}{l} 2x = 5 \\ x = \frac{5}{2} \\ \hline \end{array}$$

Check: Substitute each solution back into the original equation.

$$\begin{array}{ll} x = 0 \Rightarrow 2(0)^2 = 5(0) \Rightarrow 0 = 0 & \text{works out} \\ x = \frac{5}{2} \Rightarrow 2\left(\frac{5}{2}\right)^2 = 5 \cdot \frac{5}{2} \Rightarrow 2 \cdot \frac{25}{4} = \frac{25}{2} \Rightarrow \frac{25}{2} = \frac{25}{2} & \text{works out} \end{array}$$

Answer: $x = 0, x = \frac{5}{2}$

c) $9x^2y - 6xy = 0$

Rewrite: not necessary

Factor: The common factor is $3xy$, so this factors as $3xy(3x - 2) = 0$.

Set each factor equal to zero:

$3 = 0$ is never true, so this part does not give a solution. The factors we have left give us:

$$x = 0 \qquad \text{or} \qquad y = 0 \qquad \text{or} \qquad 3x - 2 = 0$$

Solve:

$$\begin{array}{ccccccc} \underline{x = 0} & & \text{or} & & \underline{y = 0} & & \text{or} & & 3x = 2 \\ & & & & x = \frac{2}{3} & & & & \end{array}$$

Check: Substitute each solution back into the original equation.

$$\begin{array}{ll} x = 0 \Rightarrow 9(0)y - 6(0)y = 0 - 0 = 0 & \text{works out} \\ y = 0 \Rightarrow 9x^2(0) - 6x(0) = 0 - 0 = 0 & \text{works out} \\ x = \frac{2}{3} \Rightarrow 9 \cdot \left(\frac{2}{3}\right)^2 y - 6 \cdot \frac{2}{3}y = 9 \cdot \frac{4}{9}y - 4y = 4y - 4y = 0 & \text{works out} \end{array}$$

Answer: $x = 0, y = 0, x = \frac{2}{3}$

Review Questions

Factor out the greatest common factor in the following polynomials.

1. $2x^2 - 5x$
2. $3x^3 - 21x$
3. $5x^6 + 15x^4$
4. $4x^3 + 10x^2 - 2x$
5. $-10x^6 + 12x^5 - 4x^4$
6. $12xy + 24xy^2 + 36xy^3$
7. $5a^3 - 7a$
8. $3y + 6z$
9. $10a^3 - 4ab$
10. $45y^{12} + 30y^{10}$
11. $16xy^2z + 4x^3y$
12. $2a - 4a^2 + 6$
13. $5xy^2 - 10xy + 5y^2$

Solve the following polynomial equations.

14. $x(x + 12) = 0$
15. $(2x + 1)(2x - 1) = 0$

16. $(x - 5)(2x + 7)(3x - 4) = 0$
17. $2x(x + 9)(7x - 20) = 0$
18. $x(3 + y) = 0$
19. $x(x - 2y) = 0$
20. $18y - 3y^2 = 0$
21. $9x^2 = 27x$
22. $4a^2 + a = 0$
23. $b^2 - \frac{5}{3}b = 0$
24. $4x^2 = 36$
25. $x^3 - 5x^2 = 0$

6.5 Factoring Quadratic Expressions

Learning Objectives

- Write quadratic equations in standard form.
- Factor quadratic expressions for different coefficient values.

Write Quadratic Expressions in Standard Form

Quadratic polynomials are polynomials of the 2nd degree. The standard form of a quadratic polynomial is written as

$$ax^2 + bx + c$$

where a, b , and c stand for constant numbers. Factoring these polynomials depends on the values of these constants. In this section we'll learn how to factor quadratic polynomials for different values of a, b , and c . (When none of the coefficients are zero, these expressions are also called quadratic **trinomials**, since they are polynomials with three terms.)

You've already learned how to factor quadratic polynomials where $c = 0$. For example, for the quadratic $ax^2 + bx$, the common factor is x and this expression is factored as $x(ax + b)$. Now we'll see how to factor quadratics where c is nonzero.

Factor when $a = 1$, b is Positive, and c is Positive

First, let's consider the case where $a = 1, b$ is positive and c is positive. The quadratic trinomials will take the form

$$x^2 + bx + c$$

You know from multiplying binomials that when you multiply two factors $(x + m)(x + n)$, you get a quadratic polynomial. Let's look at this process in more detail. First we use distribution:

$$(x + m)(x + n) = x^2 + nx + mx + mn$$

Then we simplify by combining the like terms in the middle. We get:

$$(x + m)(x + n) = x^2 + (n + m)x + mn$$

So to factor a quadratic, we just need to do this process in reverse.

$$\begin{array}{ll} \text{We see that} & x^2 + (n + m)x + mn \\ \text{is the same form as} & x^2 + bx + c \end{array}$$

This means that we need to find two numbers m and n where

$$n + m = b \quad \text{and} \quad mn = c$$

The factors of $x^2 + bx + c$ are always two binomials

$$(x + m)(x + n)$$

such that $n + m = b$ and $mn = c$.

Example 1

Factor $x^2 + 5x + 6$.

Solution

We are looking for an answer that is a product of two binomials in parentheses:

$$(x \quad)(x \quad)$$

We want two numbers m and n that multiply to 6 and add up to 5. A good strategy is to list the possible ways we can multiply two numbers to get 6 and then see which of these numbers add up to 5:

$$\begin{array}{lll} 6 = 1 \cdot 6 & \text{and} & 1 + 6 = 7 \\ 6 = 2 \cdot 3 & \text{and} & 2 + 3 = 5 \quad \textit{This is the correct choice.} \end{array}$$

So the answer is $(x + 2)(x + 3)$.

We can check to see if this is correct by multiplying $(x + 2)(x + 3)$:

$$\begin{array}{r} x + 2 \\ x + 3 \\ \hline 3x + 6 \\ x^2 + 2x \\ \hline x^2 + 5x + 6 \end{array}$$

The answer checks out.

Example 2

Factor $x^2 + 7x + 12$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

The number 12 can be written as the product of the following numbers:

$$\begin{array}{lll} 12 = 1 \cdot 12 & \text{and} & 1 + 12 = 13 \\ 12 = 2 \cdot 6 & \text{and} & 2 + 6 = 8 \\ 12 = 3 \cdot 4 & \text{and} & 3 + 4 = 7 \quad \textit{This is the correct choice.} \end{array}$$

The answer is $(x + 3)(x + 4)$.

Example 3

Factor $x^2 + 8x + 12$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

The number 12 can be written as the product of the following numbers:

| | | | |
|-------------------|-----|---------------|------------------------------------|
| $12 = 1 \cdot 12$ | and | $1 + 12 = 13$ | |
| $12 = 2 \cdot 6$ | and | $2 + 6 = 8$ | <i>This is the correct choice.</i> |
| $12 = 3 \cdot 4$ | and | $3 + 4 = 7$ | |

The answer is $(x + 2)(x + 6)$.

Example 4

Factor $x^2 + 12x + 36$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

The number 36 can be written as the product of the following numbers:

| | | | |
|-------------------|-----|---------------|------------------------------------|
| $36 = 1 \cdot 36$ | and | $1 + 36 = 37$ | |
| $36 = 2 \cdot 18$ | and | $2 + 18 = 20$ | |
| $36 = 3 \cdot 12$ | and | $3 + 12 = 15$ | |
| $36 = 4 \cdot 9$ | and | $4 + 9 = 13$ | |
| $36 = 6 \cdot 6$ | and | $6 + 6 = 12$ | <i>This is the correct choice.</i> |

The answer is $(x + 6)(x + 6)$.

Factor when a = 1, b is Negative and c is Positive

Now let's see how this method works if the middle coefficient is negative.

Example 5

Factor $x^2 - 6x + 8$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

When negative coefficients are involved, we have to remember that negative factors may be involved also.

The number 8 can be written as the product of the following numbers:

$$8 = 1 \cdot 8 \quad \text{and} \quad 1 + 8 = 9$$

but also

$$8 = (-1) \cdot (-8) \quad \text{and} \quad -1 + (-8) = -9$$

and

$$8 = 2 \cdot 4 \quad \text{and} \quad 2 + 4 = 6$$

but also

$$8 = (-2) \cdot (-4) \quad \text{and} \quad -2 + (-4) = -6 \quad \text{This is the correct choice.}$$

The answer is $(x - 2)(x - 4)$. We can check to see if this is correct by multiplying $(x - 2)(x - 4)$:

$$\begin{array}{r} x - 2 \\ \times x - 4 \\ \hline - 4x + 8 \\ x^2 - 2x \\ \hline x^2 - 6x + 8 \end{array}$$

The answer checks out.

Example 6

Factor $x^2 - 17x + 16$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

The number 16 can be written as the product of the following numbers:

| | | | |
|-------------------------|-----|--------------------|------------------------------------|
| $16 = 1 \cdot 16$ | and | $1 + 16 = 17$ | |
| $16 = (-1) \cdot (-16)$ | and | $-1 + (-16) = -17$ | <i>This is the correct choice.</i> |
| $16 = 2 \cdot 8$ | and | $2 + 8 = 10$ | |
| $16 = (-2) \cdot (-8)$ | and | $-2 + (-8) = -10$ | |
| $16 = 4 \cdot 4$ | and | $4 + 4 = 8$ | |
| $16 = (-4) \cdot (-4)$ | and | $-4 + (-4) = -8$ | |

The answer is $(x - 1)(x - 16)$.

In general, whenever b is negative and a and c are positive, the two binomial factors will have minus signs instead of plus signs.

Factor when $a = 1$ and c is Negative

Now let's see how this method works if the constant term is negative.

Example 7

Factor $x^2 + 2x - 15$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

Once again, we must take the negative sign into account. The number -15 can be written as the product of the following numbers:

| | | | |
|-----------------------|-----|-------------------|------------------------------------|
| $-15 = -1 \cdot 15$ | and | $-1 + 15 = 14$ | |
| $-15 = 1 \cdot (-15)$ | and | $1 + (-15) = -14$ | |
| $-15 = -3 \cdot 5$ | and | $-3 + 5 = 2$ | <i>This is the correct choice.</i> |
| $-15 = 3 \cdot (-5)$ | and | $3 + (-5) = -2$ | |

The answer is $(x - 3)(x + 5)$.

We can check to see if this is correct by multiplying:

$$\begin{array}{r}
 x - 3 \\
 \underline{x + 5} \\
 5x - 15 \\
 \underline{x^2 - 3x} \\
 x^2 + 2x - 15
 \end{array}$$

The answer checks out.

Example 8

Factor $x^2 - 10x - 24$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

The number -24 can be written as the product of the following numbers:

| | | | |
|-----------------------|-----|-------------------|------------------------------------|
| $-24 = -1 \cdot 24$ | and | $-1 + 24 = 23$ | |
| $-24 = 1 \cdot (-24)$ | and | $1 + (-24) = -23$ | |
| $-24 = -2 \cdot 12$ | and | $-2 + 12 = 10$ | |
| $-24 = 2 \cdot (-12)$ | and | $2 + (-12) = -10$ | <i>This is the correct choice.</i> |
| $-24 = -3 \cdot 8$ | and | $-3 + 8 = 5$ | |
| $-24 = 3 \cdot (-8)$ | and | $3 + (-8) = -5$ | |
| $-24 = -4 \cdot 6$ | and | $-4 + 6 = 2$ | |
| $-24 = 4 \cdot (-6)$ | and | $4 + (-6) = -2$ | |

The answer is $(x - 12)(x + 2)$.

Example 9

Factor $x^2 + 34x - 35$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

The number -35 can be written as the product of the following numbers:

| | | | |
|-----------------------|-----|-------------------|------------------------------------|
| $-35 = -1 \cdot 35$ | and | $-1 + 35 = 34$ | <i>This is the correct choice.</i> |
| $-35 = 1 \cdot (-35)$ | and | $1 + (-35) = -34$ | |
| $-35 = -5 \cdot 7$ | and | $-5 + 7 = 2$ | |
| $-35 = 5 \cdot (-7)$ | and | $5 + (-7) = -2$ | |

The answer is $(x - 1)(x + 35)$.

Factor when $a = -1$

When $a = -1$, the best strategy is to factor the common factor of -1 from all the terms in the quadratic polynomial and then apply the methods you learned so far in this section

Example 10

Factor $-x^2 + x + 6$.

Solution

First factor the common factor of -1 from each term in the trinomial. Factoring -1 just changes the signs of each term in the expression:

$$-x^2 + x + 6 = -(x^2 - x - 6)$$

We're looking for a product of two binomials in parentheses: $-(x \quad)(x \quad)$

Now our job is to factor $x^2 - x - 6$.

The number -6 can be written as the product of the following numbers:

| | | | |
|---------------------|-----|-----------------|------------------------------------|
| $-6 = -1 \cdot 6$ | and | $-1 + 6 = 5$ | |
| $-6 = 1 \cdot (-6)$ | and | $1 + (-6) = -5$ | |
| $-6 = -2 \cdot 3$ | and | $-2 + 3 = 1$ | |
| $-6 = 2 \cdot (-3)$ | and | $2 + (-3) = -1$ | <i>This is the correct choice.</i> |

The answer is $-(x - 3)(x + 2)$.

Lesson Summary

- A quadratic of the form $x^2 + bx + c$ factors as a product of two binomials in parentheses: $(x + m)(x + n)$
- If b and c are positive, then both m and n are positive.

Example: $x^2 + 8x + 12$ factors as $(x + 6)(x + 2)$.

- If b is negative and c is positive, then both m and n are negative.

Example: $x^2 - 6x + 8$ factors as $(x - 2)(x - 4)$.

- If c is negative, then either m is positive and n is negative or vice-versa.

Example: $x^2 + 2x - 15$ factors as $(x + 5)(x - 3)$.

Example: $x^2 + 34x - 35$ factors as $(x + 35)(x - 1)$.

- If $a = -1$, factor out -1 from each term in the trinomial and then factor as usual. The answer will have the form: $-(x + m)(x + n)$

Example: $-x^2 + x + 6$ factors as $-(x - 3)(x + 2)$.

Review Questions

Factor the following quadratic polynomials.

1. $x^2 + 10x + 9$
2. $x^2 + 15x + 50$
3. $x^2 + 10x + 21$
4. $x^2 + 16x + 48$
5. $x^2 - 11x + 24$
6. $x^2 - 13x + 42$
7. $x^2 - 14x + 33$
8. $x^2 - 9x + 20$
9. $x^2 + 5x - 14$
10. $x^2 + 6x - 27$
11. $x^2 + 7x - 78$
12. $x^2 + 4x - 32$
13. $x^2 - 12x - 45$
14. $x^2 - 5x - 50$
15. $x^2 - 3x - 40$
16. $x^2 - x - 56$
17. $-x^2 - 2x - 1$
18. $-x^2 - 5x + 24$
19. $-x^2 + 18x - 72$
20. $-x^2 + 25x - 150$
21. $x^2 + 21x + 108$
22. $-x^2 + 11x - 30$
23. $x^2 + 12x - 64$
24. $x^2 - 17x - 60$
25. $x^2 + 5x - 36$

6.6 Factoring Special Products

Learning Objectives

- Factor the difference of two squares.
- Factor perfect square trinomials.
- Solve quadratic polynomial equation by factoring.

Introduction

When you learned how to multiply binomials we talked about two special products.

The sum and difference formula: $(a + b)(a - b) = a^2 - b^2$

The square of a binomial formulas: $(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

In this section we'll learn how to recognize and factor these special products.

Factor the Difference of Two Squares

We use the sum and difference formula to factor a difference of two squares. A difference of two squares is any quadratic polynomial in the form $a^2 - b^2$, where a and b can be variables, constants, or just about anything else. The factors of $a^2 - b^2$ are always $(a + b)(a - b)$; the key is figuring out what the a and b terms are.

Example 1

Factor the difference of squares:

a) $x^2 - 9$

b) $x^2 - 100$

c) $x^2 - 1$

Solution

a) Rewrite $x^2 - 9$ as $x^2 - 3^2$. Now it is obvious that it is a difference of squares.

$$\text{The difference of squares formula is: } a^2 - b^2 = (a + b)(a - b)$$

$$\text{Let's see how our problem matches with the formula: } x^2 - 3^2 = (x + 3)(x - 3)$$

$$\text{The answer is: } x^2 - 9 = (x + 3)(x - 3)$$

We can check to see if this is correct by multiplying $(x + 3)(x - 3)$:

$$\begin{array}{r} x + 3 \\ \underline{x - 3} \\ -3x - 9 \\ \underline{x^2 + 3x} \\ x^2 + 0x - 9 \end{array}$$

The answer checks out.

Note: We could factor this polynomial without recognizing it as a difference of squares. With the methods we learned in the last section we know that a quadratic polynomial factors into the product of two binomials:

$$(x \quad)(x \quad)$$

We need to find two numbers that multiply to -9 and add to 0 (since there is no x -term, that's the same as if the x -term had a coefficient of 0). We can write -9 as the following products:

$$\begin{array}{lll} -9 = -1 \cdot 9 & \text{and} & -1 + 9 = 8 \\ -9 = 1 \cdot (-9) & \text{and} & 1 + (-9) = -8 \\ -9 = 3 \cdot (-3) & \text{and} & 3 + (-3) = 0 \end{array} \quad \text{These are the correct numbers.}$$

We can factor $x^2 - 9$ as $(x + 3)(x - 3)$, which is the same answer as before. You can always factor using the methods you learned in the previous section, but recognizing special products helps you factor them faster.

b) Rewrite $x^2 - 100$ as $x^2 - 10^2$. This factors as $(x + 10)(x - 10)$.

c) Rewrite $x^2 - 1$ as $x^2 - 1^2$. This factors as $(x + 1)(x - 1)$.

Example 2

Factor the difference of squares:

- a) $16x^2 - 25$
- b) $4x^2 - 81$
- c) $49x^2 - 64$

Solution

- a) Rewrite $16x^2 - 25$ as $(4x)^2 - 5^2$. This factors as $(4x + 5)(4x - 5)$.
- b) Rewrite $4x^2 - 81$ as $(2x)^2 - 9^2$. This factors as $(2x + 9)(2x - 9)$.
- c) Rewrite $49x^2 - 64$ as $(7x)^2 - 8^2$. This factors as $(7x + 8)(7x - 8)$.

Example 3

Factor the difference of squares:

- a) $x^2 - y^2$
- b) $9x^2 - 4y^2$
- c) $x^2y^2 - 1$

Solution

- a) $x^2 - y^2$ factors as $(x + y)(x - y)$.
- b) Rewrite $9x^2 - 4y^2$ as $(3x)^2 - (2y)^2$. This factors as $(3x + 2y)(3x - 2y)$.
- c) Rewrite $x^2y^2 - 1$ as $(xy)^2 - 1^2$. This factors as $(xy + 1)(xy - 1)$.

Example 4

Factor the difference of squares:

- a) $x^4 - 25$
- b) $16x^4 - y^2$
- c) $x^2y^8 - 64z^2$

Solution

- a) Rewrite $x^4 - 25$ as $(x^2)^2 - 5^2$. This factors as $(x^2 + 5)(x^2 - 5)$.
- b) Rewrite $16x^4 - y^2$ as $(4x^2)^2 - y^2$. This factors as $(4x^2 + y)(4x^2 - y)$.
- c) Rewrite $x^2y^8 - 64z^2$ as $(xy^4)^2 - (8z)^2$. This factors as $(xy^4 + 8z)(xy^4 - 8z)$.

Factor Perfect Square Trinomials

We use the square of a binomial formula to factor perfect square trinomials. A perfect square trinomial has the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$.

In these special kinds of trinomials, the first and last terms are perfect squares and the middle term is twice the product of the square roots of the first and last terms. In a case like this, the polynomial factors into perfect squares:

$$\begin{aligned}a^2 + 2ab + b^2 &= (a + b)^2 \\a^2 - 2ab + b^2 &= (a - b)^2\end{aligned}$$

Once again, the key is figuring out what the a and b terms are.

Example 5

Factor the following perfect square trinomials:

a) $x^2 + 8x + 16$

b) $x^2 - 4x + 4$

c) $x^2 + 14x + 49$

Solution

a) The first step is to recognize that this expression is a perfect square trinomial.

First, we can see that the first term and the last term are perfect squares. We can rewrite $x^2 + 8x + 16$ as $x^2 + 8x + 4^2$.

Next, we check that the middle term is twice the product of the square roots of the first and the last terms. This is true also since we can rewrite $x^2 + 8x + 16$ as $x^2 + 2 \cdot 4 \cdot x + 4^2$.

This means we can factor $x^2 + 8x + 16$ as $(x + 4)^2$. We can check to see if this is correct by multiplying $(x + 4)^2 = (x + 4)(x + 4)$:

$$\begin{array}{r} x + 4 \\ \times x + 4 \\ \hline 4x + 16 \\ x^2 + 4x \\ \hline x^2 + 8x + 16 \end{array}$$

The answer checks out.

Note: We could factor this trinomial without recognizing it as a perfect square. We know that a trinomial factors as a product of two binomials:

$$(x \quad)(x \quad)$$

We need to find two numbers that multiply to 16 and add to 8. We can write 16 as the following products:

$$\begin{array}{lll} 16 = 1 \cdot 16 & \text{and} & 1 + 16 = 17 \\ 16 = 2 \cdot 8 & \text{and} & 2 + 8 = 10 \\ 16 = 4 \cdot 4 & \text{and} & 4 + 4 = 8 \end{array} \quad \text{These are the correct numbers}$$

So we can factor $x^2 + 8x + 16$ as $(x + 4)(x + 4)$, which is the same as $(x + 4)^2$.

Once again, you can factor perfect square trinomials the normal way, but recognizing them as perfect squares gives you a useful shortcut.

b) Rewrite $x^2 + 4x + 4$ as $x^2 + 2 \cdot (-2) \cdot x + (-2)^2$.

We notice that this is a perfect square trinomial, so we can factor it as $(x - 2)^2$.

c) Rewrite $x^2 + 14x + 49$ as $x^2 + 2 \cdot 7 \cdot x + 7^2$.

We notice that this is a perfect square trinomial, so we can factor it as $(x + 7)^2$.

Example 6

Factor the following perfect square trinomials:

a) $4x^2 + 20x + 25$

b) $9x^2 - 24x + 16$

c) $x^2 + 2xy + y^2$

Solution

a) Rewrite $4x^2 + 20x + 25$ as $(2x)^2 + 2 \cdot 5 \cdot (2x) + 5^2$.

We notice that this is a perfect square trinomial and we can factor it as $(2x + 5)^2$.

b) Rewrite $9x^2 - 24x + 16$ as $(3x)^2 + 2 \cdot (-4) \cdot (3x) + (-4)^2$.

We notice that this is a perfect square trinomial and we can factor it as $(3x - 4)^2$.

We can check to see if this is correct by multiplying $(3x - 4)^2 = (3x - 4)(3x - 4)$:

$$\begin{array}{r} 3x - 4 \\ \underline{3x - 4} \\ -12x + 16 \\ \underline{9x^2 - 12x} \\ 9x^2 - 24x + 16 \end{array}$$

The answer checks out.

c) $x^2 + 2xy + y^2$

We notice that this is a perfect square trinomial and we can factor it as $(x + y)^2$.

For more examples of factoring perfect square trinomials, watch the videos at <http://www.onlinemathlearning.com/perfect-square-trinomial.html>.

Solve Quadratic Polynomial Equations by Factoring

With the methods we've learned in the last two sections, we can factor many kinds of quadratic polynomials. This is very helpful when we want to solve them. Remember the process we learned earlier:

1. If necessary, **rewrite** the equation in standard form so that the right-hand side equals zero.
2. Factor the polynomial completely.
3. Use the zero-product rule to set each factor equal to zero.
4. Solve each equation from step 3.
5. Check your answers by substituting your solutions into the original equation

We can use this process to solve quadratic polynomials using the factoring methods we just learned.

Example 7

Solve the following polynomial equations.

a) $x^2 + 7x + 6 = 0$

b) $x^2 - 8x = -12$

c) $x^2 = 2x + 15$

Solution

a) **Rewrite:** We can skip this since the equation is in the correct form already.

Factor: We can write 6 as a product of the following numbers:

$$\begin{array}{lll}
 6 = 1 \cdot 6 & \text{and} & 1 + 6 = 7 \quad \textit{This is the correct choice.} \\
 6 = 2 \cdot 3 & \text{and} & 2 + 3 = 5
 \end{array}$$

$x^2 + 7x + 6 = 0$ factors as $(x + 1)(x + 6) = 0$.

Set each factor equal to zero:

$$x + 1 = 0 \qquad \text{or} \qquad x + 6 = 0$$

Solve:

$$\underline{\underline{x = -1}} \qquad \text{or} \qquad \underline{\underline{x = -6}}$$

Check: Substitute each solution back into the original equation.

$$\begin{array}{lll}
 x = -1 & (-1)^2 + 7(-1) + 6 = 1 - 7 + 6 = 0 & \text{checks out} \\
 x = -6 & (-6)^2 + 7(-6) + 6 = 36 - 42 + 6 = 0 & \text{checks out}
 \end{array}$$

b) **Rewrite:** $x^2 - 8x = -12$ is rewritten as $x^2 - 8x + 12 = 0$

Factor: We can write 12 as a product of the following numbers:

$$\begin{array}{lll}
 12 = 1 \cdot 12 & \text{and} & 1 + 12 = 13 \\
 12 = -1 \cdot (-12) & \text{and} & -1 + (-12) = -13 \\
 12 = 2 \cdot 6 & \text{and} & 2 + 6 = 8 \\
 12 = -2 \cdot (-6) & \text{and} & -2 + (-6) = -8 \quad \textit{This is the correct choice.} \\
 12 = 3 \cdot 4 & \text{and} & 3 + 4 = 7 \\
 12 = -3 \cdot (-4) & \text{and} & -3 + (-4) = -7
 \end{array}$$

$x^2 + 8x + 12 = 0$ factors as $(x - 2)(x - 6) = 0$.

Set each factor equal to zero:

$$x - 2 = 0 \qquad \text{or} \qquad x - 6 = 0$$

Solve:

$$\underline{\underline{x = 2}} \qquad \text{or} \qquad \underline{\underline{x = 6}}$$

Check: Substitute each solution back into the original equation.

$$\begin{array}{lll}
 x = 2 & (2)^2 - 8(2) = 4 - 16 = -12 & \text{checks out} \\
 x = 6 & (6)^2 - 8(6) = 36 - 48 = -12 & \text{checks out}
 \end{array}$$

c) **Rewrite:** $x^2 = 2x + 15$ is rewritten as $x^2 - 2x - 15 = 0$

Factor: We can write -15 as a product of the following numbers:

| | | | |
|-----------------------|-----|-------------------|------------------------------------|
| $-15 = 1 \cdot (-15)$ | and | $1 + (-15) = -14$ | |
| $-15 = -1 \cdot (15)$ | and | $-1 + (15) = 14$ | |
| $-15 = -3 \cdot 5$ | and | $-3 + 5 = 2$ | |
| $-15 = 3 \cdot (-5)$ | and | $3 + (-5) = -2$ | <i>This is the correct choice.</i> |

$$x^2 - 2x - 15 = 0 \text{ factors as } (x + 3)(x - 5) = 0$$

Set each factor equal to zero:

$$x + 3 = 0 \qquad \text{or} \qquad x - 5 = 0$$

Solve:

$$\underline{\underline{x = -3}} \qquad \text{or} \qquad \underline{\underline{x = 5}}$$

Check: Substitute each solution back into the original equation.

| | | |
|----------|---|------------|
| $x = -3$ | $(-3)^2 = 2(-3) + 15 \Rightarrow 9 = 9$ | checks out |
| $x = 5$ | $(5)^2 = 2(5) + 15 \Rightarrow 25 = 25$ | checks out |

Example 8

Solve the following polynomial equations:

a) $x^2 - 12x + 36 = 0$

b) $x^2 - 81 = 0$

c) $x^2 + 20x + 100 = 0$

Solution

a) $x^2 - 12x + 36 = 0$

Rewrite: The equation is in the correct form already.

Factor: Rewrite $x^2 - 12x + 36 = 0$ as $x^2 - 2 \cdot (-6)x + (-6)^2$.

We recognize this as a perfect square. This factors as $(x - 6)^2 = 0$ or $(x - 6)(x - 6) = 0$

Set each factor equal to zero:

$$x - 6 = 0 \qquad \text{or} \qquad x - 6 = 0$$

Solve:

$$\underline{\underline{x = 6}} \qquad \text{or} \qquad \underline{\underline{x = 6}}$$

Notice that for a perfect square the two solutions are the same. This is called a double root.

Check: Substitute each solution back into the original equation.

$$x = 6 \qquad 6^2 - 12(6) + 36 = 36 - 72 + 36 = 0 \qquad \text{checks out}$$

b) $x^2 - 81 = 0$

Rewrite: this is not necessary since the equation is in the correct form already

Factor: Rewrite $x^2 - 81$ as $x^2 - 9^2$.

We recognize this as a difference of squares. This factors as $(x - 9)(x + 9) = 0$.

Set each factor equal to zero:

$$x - 9 = 0$$

or

$$x + 9 = 0$$

Solve:

$$\underline{\underline{x = 9}}$$

or

$$\underline{\underline{x = -9}}$$

Check: Substitute each solution back into the original equation.

$$x = 9$$

$$9^2 - 81 = 81 - 81 = 0$$

checks out

$$x = -9$$

$$(-9)^2 - 81 = 81 - 81 = 0$$

checks out

c) $x^2 + 20x + 100 = 0$

Rewrite: this is not necessary since the equation is in the correct form already

Factor: Rewrite $x^2 + 20x + 100$ as $x^2 + 2 \cdot 10 \cdot x + 10^2$.

We recognize this as a perfect square. This factors as $(x + 10)^2 = 0$ or $(x + 10)(x + 10) = 0$

Set each factor equal to zero:

$$x + 10 = 0$$

or

$$x + 10 = 0$$

Solve:

$$\underline{\underline{x = -10}}$$

or

$$\underline{\underline{x = -10}}$$

This is a double root.

Check: Substitute each solution back into the original equation.

$$x = 10$$

$$(-10)^2 + 20(-10) + 100 = 100 - 200 + 100 = 0$$

checks out

Review Questions

Factor the following perfect square trinomials.

1. $x^2 + 8x + 16$
2. $x^2 - 18x + 81$
3. $-x^2 + 24x - 144$
4. $x^2 + 14x + 49$
5. $4x^2 - 4x + 1$
6. $25x^2 + 60x + 36$
7. $4x^2 - 12xy + 9y^2$
8. $x^4 + 22x^2 + 121$

Factor the following differences of squares.

9. $x^2 - 4$
10. $x^2 - 36$
11. $-x^2 + 100$
12. $x^2 - 400$
13. $9x^2 - 4$
14. $25x^2 - 49$
15. $-36x^2 + 25$
16. $4x^2 - y^2$
17. $16x^2 - 81y^2$

Solve the following quadratic equations using factoring.

18. $x^2 - 11x + 30 = 0$
19. $x^2 + 4x = 21$
20. $x^2 + 49 = 14x$
21. $x^2 - 64 = 0$
22. $x^2 - 24x + 144 = 0$
23. $4x^2 - 25 = 0$
24. $x^2 + 26x = -169$
25. $-x^2 - 16x - 60 = 0$

6.7 Factoring Polynomials Completely

Learning Objectives

- Factor out a common binomial.
- Factor by grouping.
- Factor a quadratic trinomial where $a \neq 1$.
- Solve real world problems using polynomial equations.

Introduction

We say that a polynomial is **factored completely** when we can't factor it any more. Here are some suggestions that you should follow to make sure that you factor completely:

- Factor all common monomials first.
- Identify special products such as difference of squares or the square of a binomial. Factor according to their formulas.
- If there are no special products, factor using the methods we learned in the previous sections.
- Look at each factor and see if any of these can be factored further.

Example 1

Factor the following polynomials completely.

- a) $6x^2 - 30x + 24$
- b) $2x^2 - 8$
- c) $x^3 + 6x^2 + 9x$

Solution

a) Factor out the common monomial. In this case 6 can be divided from each term:

$$6(x^2 - 5x + 6)$$

There are no special products. We factor $x^2 - 5x + 6$ as a product of two binomials: $(x)(x)$

The two numbers that multiply to 6 and add to -5 are -2 and -3, so:

$$6(x^2 - 5x + 6) = 6(x - 2)(x - 3)$$

If we look at each factor we see that we can factor no more.

The answer is $6(x - 2)(x - 3)$.

b) Factor out common monomials: $2x^2 - 8 = 2(x^2 - 4)$

We recognize $x^2 - 4$ as a difference of squares. We factor it as $(x + 2)(x - 2)$.

If we look at each factor we see that we can factor no more.

The answer is $2(x + 2)(x - 2)$.

c) Factor out common monomials: $x^3 + 6x^2 + 9x = x(x^2 + 6x + 9)$

We recognize $x^2 + 6x + 9$ as a perfect square and factor it as $(x + 3)^2$.

If we look at each factor we see that we can factor no more.

The answer is $x(x + 3)^2$.

Example 2

Factor the following polynomials completely:

a) $-2x^4 + 162$

b) $x^5 - 8x^3 + 16x$

Solution

a) Factor out the common monomial. In this case, factor out -2 rather than 2. (It's always easier to factor out the negative number so that the highest degree term is positive.)

$$-2x^4 + 162 = -2(x^4 - 81)$$

We recognize expression in parenthesis as a difference of squares. We factor and get:

$$-2(x^2 - 9)(x^2 + 9)$$

If we look at each factor we see that the first parenthesis is a difference of squares. We factor and get:

$$-2(x + 3)(x - 3)(x^2 + 9)$$

If we look at each factor now we see that we can factor no more.

The answer is $-2(x + 3)(x - 3)(x^2 + 9)$.

b) Factor out the common monomial: $x^5 - 8x^3 + 14x = x(x^4 - 8x^2 + 16)$

We recognize $x^4 - 8x^2 + 16$ as a perfect square and we factor it as $x(x^2 - 4)^2$.

We look at each term and recognize that the term in parentheses is a difference of squares.

We factor it and get $((x + 2)(x - 2))^2$, which we can rewrite as $(x + 2)^2(x - 2)^2$.

If we look at each factor now we see that we can factor no more.

The final answer is $x(x + 2)^2(x - 2)^2$.

Factor out a Common Binomial

The first step in the factoring process is often factoring out the common monomials from a polynomial. But sometimes polynomials have common terms that are binomials. For example, consider the following expression:

$$x(3x + 2) - 5(3x + 2)$$

Since the term $(3x + 2)$ appears in both terms of the polynomial, we can factor it out. We write that term in front of a set of parentheses containing the terms that are left over:

$$(3x + 2)(x - 5)$$

This expression is now completely factored.

Let's look at some more examples.

Example 3

Factor out the common binomials.

a) $3x(x - 1) + 4(x - 1)$

b) $x(4x + 5) + (4x + 5)$

Solution

a) $3x(x - 1) + 4(x - 1)$ has a common binomial of $(x - 1)$.

When we factor out the common binomial we get $(x - 1)(3x + 4)$.

b) $x(4x + 5) + (4x + 5)$ has a common binomial of $(4x + 5)$.

When we factor out the common binomial we get $(4x + 5)(x + 1)$.

Factor by Grouping

Sometimes, we can factor a polynomial containing four or more terms by factoring common monomials from groups of terms. This method is called **factor by grouping**.

The next example illustrates how this process works.

Example 4

Factor $2x + 2y + ax + ay$.

Solution

There is no factor common to all the terms. However, the first two terms have a common factor of 2 and the last two terms have a common factor of a . Factor 2 from the first two terms and factor a from the last two terms:

$$2x + 2y + ax + ay = 2(x + y) + a(x + y)$$

Now we notice that the binomial $(x + y)$ is common to both terms. We factor the common binomial and get:

$$(x + y)(2 + a)$$

Example 5

Factor $3x^2 + 6x + 4x + 8$.

Solution

We factor 3 from the first two terms and factor 4 from the last two terms:

$$3x(x + 2) + 4(x + 2)$$

Now factor $(x + 2)$ from both terms: $(x + 2)(3x + 4)$.

Now the polynomial is factored completely.

Factor Quadratic Trinomials Where $a \neq 1$

Factoring by grouping is a very useful method for factoring quadratic trinomials of the form $ax^2 + bx + c$, where $a \neq 1$.

A quadratic like this doesn't factor as $(x \pm m)(x \pm n)$, so it's not as simple as looking for two numbers that multiply to c and add up to b . Instead, we also have to take into account the coefficient in the first term.

To factor a quadratic polynomial where $a \neq 1$, we follow these steps:

1. We find the product ac .
2. We look for two numbers that multiply to ac and add up to b .
3. We rewrite the middle term using the two numbers we just found.
4. We factor the expression by grouping.

Let's apply this method to the following examples.

Example 6

Factor the following quadratic trinomials by grouping.

a) $3x^2 + 8x + 4$

b) $6x^2 - 11x + 4$

c) $5x^2 - 6x + 1$

Solution

Let's follow the steps outlined above:

a) $3x^2 + 8x + 4$

Step 1: $ac = 3 \cdot 4 = 12$

Step 2: The number 12 can be written as a product of two numbers in any of these ways:

| | | | |
|-------------------|-----|---------------|------------------------------------|
| $12 = 1 \cdot 12$ | and | $1 + 12 = 13$ | |
| $12 = 2 \cdot 6$ | and | $2 + 6 = 8$ | <i>This is the correct choice.</i> |
| $12 = 3 \cdot 4$ | and | $3 + 4 = 7$ | |

Step 3: Re-write the middle term: $8x = 2x + 6x$, so the problem becomes:

$$3x^2 + 8x + 4 = 3x^2 + 2x + 6x + 4$$

Step 4: Factor an x from the first two terms and a 2 from the last two terms:

$$x(3x + 2) + 2(3x + 2)$$

Now factor the common binomial $(3x + 2)$:

$$(3x + 2)(x + 2) \quad \text{This is the answer.}$$

To check if this is correct we multiply $(3x + 2)(x + 2)$:

$$\begin{array}{r} 3x + 2 \\ x + 2 \\ \hline 6x + 4 \\ 3x^2 + 2x \\ \hline 3x^2 + 8x + 4 \end{array}$$

The solution checks out.

b) $6x^2 - 11x + 4$

Step 1: $ac = 6 \cdot 4 = 24$

Step 2: The number 24 can be written as a product of two numbers in any of these ways:

| | | | |
|-----------------------|-----|--------------------|------------------------------------|
| $24 = 1 \cdot 24$ | and | $1 + 24 = 25$ | |
| $24 = -1 \cdot (-24)$ | and | $-1 + (-24) = -25$ | |
| $24 = 2 \cdot 12$ | and | $2 + 12 = 14$ | |
| $24 = -2 \cdot (-12)$ | and | $-2 + (-12) = -14$ | |
| $24 = 3 \cdot 8$ | and | $3 + 8 = 11$ | |
| $24 = -3 \cdot (-8)$ | and | $-3 + (-8) = -11$ | <i>This is the correct choice.</i> |
| $24 = 4 \cdot 6$ | and | $4 + 6 = 10$ | |
| $24 = -4 \cdot (-6)$ | and | $-4 + (-6) = -10$ | |

Step 3: Re-write the middle term: $-11x = -3x - 8x$, so the problem becomes:

$$6x^2 - 11x + 4 = 6x^2 - 3x - 8x + 4$$

Step 4: Factor by grouping: factor a $3x$ from the first two terms and a -4 from the last two terms:

$$3x(2x - 1) - 4(2x - 1)$$

Now factor the common binomial $(2x - 1)$:

$$(2x - 1)(3x - 4) \quad \text{This is the answer.}$$

c) $5x^2 - 6x + 1$

Step 1: $ac = 5 \cdot 1 = 5$

Step 2: The number 5 can be written as a product of two numbers in any of these ways:

$$\begin{array}{lll} 5 = 1 \cdot 5 & \text{and} & 1 + 5 = 6 \\ 5 = -1 \cdot (-5) & \text{and} & -1 + (-5) = -6 \end{array} \quad \text{This is the correct choice.}$$

Step 3: Re-write the middle term: $-6x = -x - 5x$, so the problem becomes:

$$5x^2 - 6x + 1 = 5x^2 - x - 5x + 1$$

Step 4: Factor by grouping: factor an x from the first two terms and $a - 1$ from the last two terms:

$$x(5x - 1) - 1(5x - 1)$$

Now factor the common binomial $(5x - 1)$:

$$(5x - 1)(x - 1) \quad \text{This is the answer.}$$

Solve Real-World Problems Using Polynomial Equations

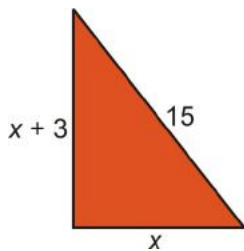
Now that we know most of the factoring strategies for quadratic polynomials, we can apply these methods to solving real world problems.

Example 7

One leg of a right triangle is 3 feet longer than the other leg. The hypotenuse is 15 feet. Find the dimensions of the triangle.

Solution

Let x = the length of the short leg of the triangle; then the other leg will measure $x + 3$.



Use the Pythagorean Theorem: $a^2 + b^2 = c^2$, where a and b are the lengths of the legs and c is the length of the hypotenuse. When we substitute the values from the diagram, we get $x^2 + (x + 3)^2 = 15^2$.

In order to solve this equation, we need to get the polynomial in standard form. We must first distribute, collect like terms and **rewrite** in the form “polynomial = 0.”

$$\begin{aligned} x^2 + x^2 + 6x + 9 &= 225 \\ 2x^2 + 6x + 9 &= 225 \\ 2x^2 + 6x - 216 &= 0 \end{aligned}$$

Factor out the common monomial: $2(x^2 + 3x - 108) = 0$

To factor the trinomial inside the parentheses, we need two numbers that multiply to -108 and add to 3. It would take a long time to go through all the options, so let's start by trying some of the bigger factors:

$$\begin{array}{lll}
 -108 = -12 \cdot 9 & \text{and} & -12 + 9 = -3 \\
 -108 = 12 \cdot (-9) & \text{and} & 12 + (-9) = 3 \quad \textit{This is the correct choice.}
 \end{array}$$

We factor the expression as $2(x - 9)(x + 12) = 0$.

Set each term equal to zero and solve:

$$\begin{array}{ll}
 x - 9 = 0 & x + 12 = 0 \\
 \text{or} & \\
 \underline{\underline{x = 9}} & \underline{\underline{x = -12}}
 \end{array}$$

It makes no sense to have a negative answer for the length of a side of the triangle, so the answer must be $x = 9$. That means **the short leg is 9 feet and the long leg is 12 feet.**

Check: $9^2 + 12^2 = 81 + 144 = 225 = 15^2$, so the answer checks.

Example 8

The product of two positive numbers is 60. Find the two numbers if one number is 4 more than the other.

Solution

Let $x =$ one of the numbers; then $x + 4$ is the other number.

The product of these two numbers is 60, so we can write the equation $x(x + 4) = 60$.

In order to solve we must write the polynomial in standard form. Distribute, collect like terms and **rewrite**:

$$\begin{aligned}
 x^2 + 4x &= 60 \\
 x^2 + 4x - 60 &= 0
 \end{aligned}$$

Factor by finding two numbers that multiply to -60 and add to 4. List some numbers that multiply to -60:

$$\begin{array}{lll}
 -60 = -4 \cdot 15 & \text{and} & -4 + 15 = 11 \\
 -60 = 4 \cdot (-15) & \text{and} & 4 + (-15) = -11 \\
 -60 = -5 \cdot 12 & \text{and} & -5 + 12 = 7 \\
 -60 = 5 \cdot (-12) & \text{and} & 5 + (-12) = -7 \\
 -60 = -6 \cdot 10 & \text{and} & -6 + 10 = 4 \quad \textit{This is the correct choice.} \\
 -60 = 6 \cdot (-10) & \text{and} & 6 + (-10) = -4
 \end{array}$$

The expression factors as $(x + 10)(x - 6) = 0$.

Set each term equal to zero and solve:

$$\begin{array}{ll}
 x + 10 = 0 & x - 6 = 0 \\
 \text{or} & \\
 \underline{\underline{x = -10}} & \underline{\underline{x = 6}}
 \end{array}$$

Since we are looking for positive numbers, the answer must be $x = 6$. **One number is 6, and the other number is 10.**

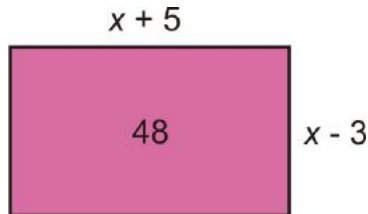
Check: $6 \cdot 10 = 60$, so the answer checks.

Example 9

A rectangle has sides of length $x + 5$ and $x - 3$. What is x if the area of the rectangle is 48?

Solution

Make a sketch of this situation:



Using the formula Area = length \times width, we have $(x + 5)(x - 3) = 48$.

In order to solve, we must write the polynomial in standard form. Distribute, collect like terms and **rewrite:**

$$x^2 + 2x - 15 = 48$$

$$x^2 + 2x - 63 = 0$$

Factor by finding two numbers that multiply to -63 and add to 2. List some numbers that multiply to -63:

| | | | |
|----------------------|-----|-----------------|------------------------------------|
| $-63 = -7 \cdot 9$ | and | $-7 + 9 = 2$ | <i>This is the correct choice.</i> |
| $-63 = 7 \cdot (-9)$ | and | $7 + (-9) = -2$ | |

The expression factors as $(x + 9)(x - 7) = 0$.

Set each term equal to zero and solve:

$$x + 9 = 0$$

$$x - 7 = 0$$

or

$$\underline{\underline{x = -9}}$$

$$\underline{\underline{x = 7}}$$

Since we are looking for positive numbers the answer must be $x = 7$. So **the width is** $x - 3 = 4$ **and the length is** $x + 5 = 12$.

Check: $4 \cdot 12 = 48$, so the answer checks.

Resources

The WTAMU Virtual Math Lab has a detailed page on factoring polynomials here: http://www.wtamu.edu/academic/alfa/algebra/col_alg_tut7_factor.htm. This page contains many videos showing example problems being solved.

Review Questions

Factor completely.

1. $2x^2 + 16x + 30$
2. $5x^2 - 70x + 245$
3. $-x^3 + 17x^2 - 70x$
4. $2x^4 - 512$
5. $25x^4 - 20x^3 + 4x^2$
6. $12x^3 + 12x^2 + 3x$

Factor by grouping.

7. $6x^2 - 9x + 10x - 15$
8. $5x^2 - 35x + x - 7$
9. $9x^2 - 9x - x + 1$
10. $4x^2 + 32x - 5x - 40$
11. $2a^2 - 6ab + 3ab - 9b^2$
12. $5x^2 + 15x - 2xy - 6y$

Factor the following quadratic trinomials by grouping.

13. $4x^2 + 25x - 21$
14. $6x^2 + 7x + 1$
15. $4x^2 + 8x - 5$
16. $3x^2 + 16x + 21$
17. $6x^2 - 2x - 4$
18. $8x^2 - 14x - 15$

Solve the following application problems:

19. One leg of a right triangle is 7 feet longer than the other leg. The hypotenuse is 13. Find the dimensions of the right triangle.
20. A rectangle has sides of $x + 2$ and $x - 1$. What value of x gives an area of 108?
21. The product of two positive numbers is 120. Find the two numbers if one number is 7 more than the other.
22. A rectangle has a 50-foot diagonal. What are the dimensions of the rectangle if it is 34 feet longer than it is wide?
23. Two positive numbers have a sum of 8, and their product is equal to the larger number plus 10. What are the numbers?
24. Two positive numbers have a sum of 8, and their product is equal to the smaller number plus 10. What are the numbers?
25. Framing Warehouse offers a picture framing service. The cost for framing a picture is made up of two parts: glass costs \$1 per square foot and the frame costs \$2 per foot. If the frame has to be a square, what size picture can you get framed for \$20?

Chapter 7

Quadratic Equations & Functions

7.1 Graphs of Quadratic Functions

Learning Objectives

- Graph quadratic functions.
- Compare graphs of quadratic functions.
- Graph quadratic functions in intercept form.
- Analyze graphs of real-world quadratic functions.

Introduction

The graphs of quadratic functions are curved lines called **parabolas**. You don't have to look hard to find parabolic shapes around you. Here are a few examples:

- The path that a ball or a rocket takes through the air.
- Water flowing out of a drinking fountain.
- The shape of a satellite dish.
- The shape of the mirror in car headlights or a flashlight.
- The cables in a suspension bridge.

Graph Quadratic Functions

Let's see what a parabola looks like by graphing the simplest quadratic function, $y = x^2$.

We'll graph this function by making a table of values. Since the graph will be curved, we need to plot a fair number of points to make it accurate.

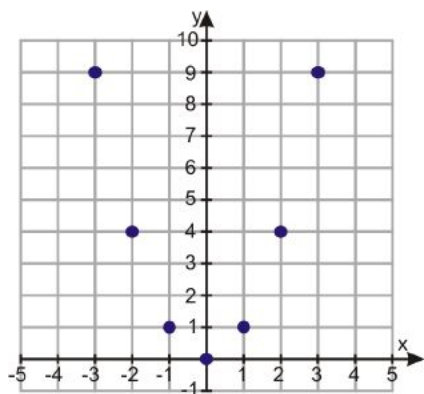
Table 7.1:

| x | $y = x^2$ |
|-----|--------------|
| -3 | $(-3)^2 = 9$ |

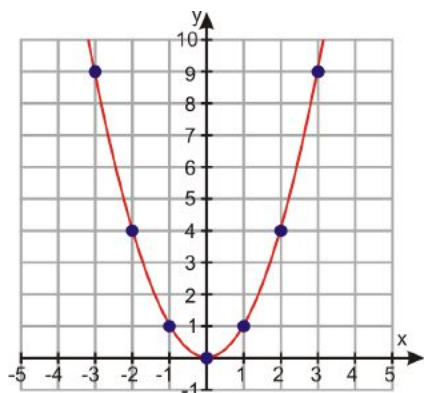
Table 7.1: (continued)

| x | $y = x^2$ |
|-----|--------------|
| -2 | $(-2)^2 = 4$ |
| -1 | $(-1)^2 = 1$ |
| 0 | $(0)^2 = 0$ |
| 1 | $(1)^2 = 1$ |
| 2 | $(2)^2 = 4$ |
| 3 | $(3)^2 = 9$ |

Here are the points plotted on a coordinate graph:



To draw the parabola, draw a smooth curve through all the points. (Do not connect the points with straight lines).



Let's graph a few more examples.

Example 1

Graph the following parabolas.

a) $y = 2x^2 + 4x + 1$

b) $y = -x^2 + 3$

c) $y = x^2 - 8x + 3$

Solution

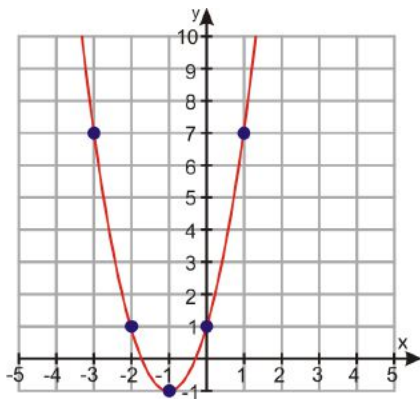
a) $y = 2x^2 + 4x + 1$

Make a table of values:

Table 7.2:

| x | $y = 2x^2 + 4x + 1$ |
|-----|----------------------------|
| -3 | $2(-3)^2 + 4(-3) + 1 = 7$ |
| -2 | $2(-2)^2 + 4(-2) + 1 = 1$ |
| -1 | $2(-1)^2 + 4(-1) + 1 = -1$ |
| 0 | $2(0)^2 + 4(0) + 1 = 1$ |
| 1 | $2(1)^2 + 4(1) + 1 = 7$ |
| 2 | $2(2)^2 + 4(2) + 1 = 17$ |
| 3 | $2(3)^2 + 4(3) + 1 = 31$ |

Notice that the last two points have very large y -values. Since we don't want to make our y -scale too big, we'll just skip graphing those two points. But we'll plot the remaining points and join them with a smooth curve.



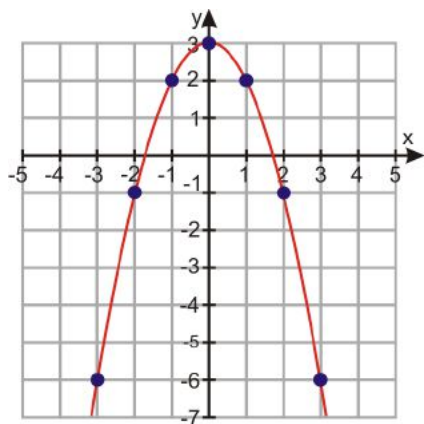
b) $y = -x^2 + 3$

Make a table of values:

Table 7.3:

| x | $y = -x^2 + 3$ |
|-----|--------------------|
| -3 | $-(-3)^2 + 3 = -6$ |
| -2 | $-(-2)^2 + 3 = -1$ |
| -1 | $-(-1)^2 + 3 = 2$ |
| 0 | $-(0)^2 + 3 = 3$ |
| 1 | $-(1)^2 + 3 = 2$ |
| 2 | $-(2)^2 + 3 = -1$ |
| 3 | $-(3)^2 + 3 = -6$ |

Plot the points and join them with a smooth curve.



Notice that this time we get an “upside down” parabola. That’s because our equation has a negative sign in front of the x^2 term. The sign of the coefficient of the x^2 term determines whether the parabola turns up or down: the parabola turns up if it’s positive and down if it’s negative.

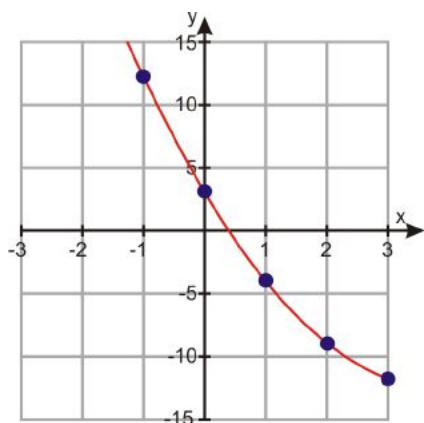
c) $y = x^2 - 8x + 3$

Make a table of values:

Table 7.4:

| x | $y = x^2 - 8x + 3$ |
|-----|---------------------------|
| -3 | $(-3)^2 - 8(-3) + 3 = 36$ |
| -2 | $(-2)^2 - 8(-2) + 3 = 23$ |
| -1 | $(-1)^2 - 8(-1) + 3 = 12$ |
| 0 | $(0)^2 - 8(0) + 3 = 3$ |
| 1 | $(1)^2 - 8(1) + 3 = -4$ |
| 2 | $(2)^2 - 8(2) + 3 = -9$ |
| 3 | $(3)^2 - 8(3) + 3 = -12$ |

Let’s not graph the first two points in the table since the values are so big. Plot the remaining points and join them with a smooth curve.



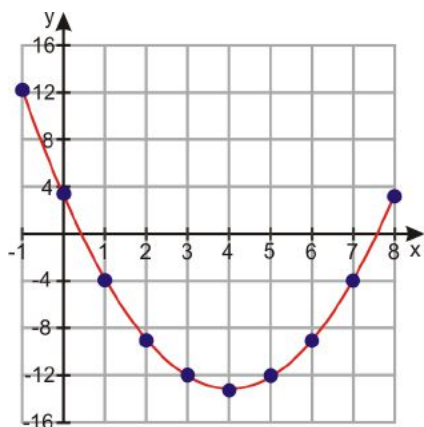
Wait—this doesn’t look like a parabola. What’s going on here?

Maybe if we graph more points, the curve will look more familiar. For negative values of x it looks like the values of y are just getting bigger and bigger, so let’s pick more positive values of x beyond $x = 3$.

Table 7.5:

| x | $y = x^2 - 8x + 3$ |
|-----|---------------------------|
| -1 | $(-1)^2 - 8(-1) + 3 = 12$ |
| 0 | $(0)^2 - 8(0) + 3 = 3$ |
| 1 | $(1)^2 - 8(1) + 3 = -4$ |
| 2 | $(2)^2 - 8(2) + 3 = -9$ |
| 3 | $(3)^2 - 8(3) + 3 = -12$ |
| 4 | $(4)^2 - 8(4) + 3 = -13$ |
| 5 | $(5)^2 - 8(5) + 3 = -12$ |
| 6 | $(6)^2 - 8(6) + 3 = -9$ |
| 7 | $(7)^2 - 8(7) + 3 = -4$ |
| 8 | $(8)^2 - 8(8) + 3 = 3$ |

Plot the points again and join them with a smooth curve.



Now we can see the familiar parabolic shape. And now we can see the drawback to graphing quadratics by making a table of values—if we don’t pick the right values, we won’t get to see the important parts of the graph.

In the next couple of lessons, we’ll find out how to graph quadratic equations more efficiently—but first we need to learn more about the properties of parabolas.

Compare Graphs of Quadratic Functions

The **general form** (or **standard form**) of a quadratic function is:

$$y = ax^2 + bx + c$$

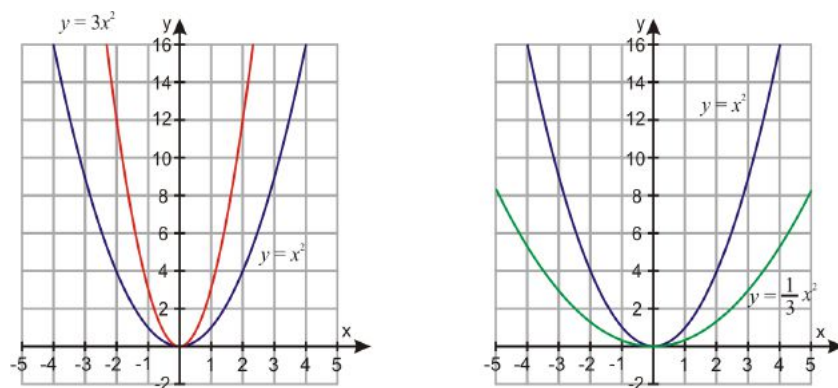
Here a , b and c are the **coefficients**. Remember, a coefficient is just a number (a constant term) that can go before a variable or appear alone.

Although the graph of a quadratic equation in standard form is always a parabola, the shape of the parabola depends on the values of the coefficients a , b and c . Let’s explore some of the ways the coefficients can affect the graph.

Dilation

Changing the value of a makes the graph “fatter” or “skinnier”. Let’s look at how graphs compare for

different positive values of a . The plot on the left shows the graphs of $y = x^2$ and $y = 3x^2$. The plot on the right shows the graphs of $y = x^2$ and $y = \frac{1}{3}x^2$.



Notice that the larger the value of a is, the skinnier the graph is – for example, in the first plot, the graph of $y = 3x^2$ is skinnier than the graph of $y = x^2$. Also, the smaller a is, the fatter the graph is – for example, in the second plot, the graph of $y = \frac{1}{3}x^2$ is fatter than the graph of $y = x^2$. This might seem counterintuitive, but if you think about it, it should make sense. Let's look at a table of values of these graphs and see if we can explain why this happens.

Table 7.6:

| x | $y = x^2$ | $y = 3x^2$ | $y = \frac{1}{3}x^2$ |
|-----|--------------|----------------|----------------------------------|
| -3 | $(-3)^2 = 9$ | $3(-3)^2 = 27$ | $\frac{(-3)^2}{3} = 3$ |
| -2 | $(-2)^2 = 4$ | $3(-2)^2 = 12$ | $\frac{(-2)^2}{3} = \frac{4}{3}$ |
| -1 | $(-1)^2 = 1$ | $3(-1)^2 = 3$ | $\frac{(-1)^2}{3} = \frac{1}{3}$ |
| 0 | $(0)^2 = 0$ | $3(0)^2 = 0$ | $\frac{(0)^2}{3} = 0$ |
| 1 | $(1)^2 = 1$ | $3(1)^2 = 3$ | $\frac{(1)^2}{3} = \frac{1}{3}$ |
| 2 | $(2)^2 = 4$ | $3(2)^2 = 12$ | $\frac{(2)^2}{3} = \frac{4}{3}$ |
| 3 | $(3)^2 = 9$ | $3(3)^2 = 27$ | $\frac{(3)^2}{3} = 3$ |

From the table, you can see that the values of $y = 3x^2$ are bigger than the values of $y = x^2$. This is because each value of y gets multiplied by 3. As a result the parabola will be skinnier because it grows three times faster than $y = x^2$. On the other hand, you can see that the values of $y = \frac{1}{3}x^2$ are smaller than the values of $y = x^2$, because each value of y gets divided by 3. As a result the parabola will be fatter because it grows at one third the rate of $y = x^2$.

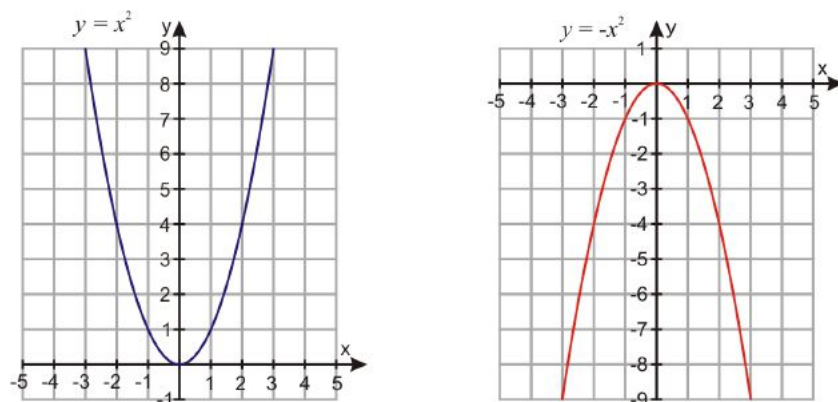
Orientation

As the value of a gets smaller and smaller, then, the parabola gets wider and flatter. What happens when a gets all the way down to zero? What happens when it's negative?

Well, when $a = 0$, the x^2 term drops out of the equation entirely, so the equation becomes linear and the graph is just a straight line. For example, we just saw what happens to $y = ax^2$ when we change the value of a ; if we tried to graph $y = 0x^2$, we would just be graphing $y = 0$, which would be a horizontal line.

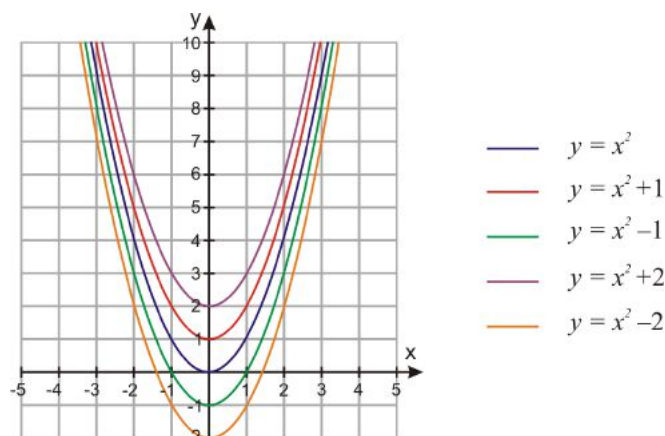
So as a gets smaller and smaller, the graph of $y = ax^2$ gets flattened all the way out into a horizontal line. Then, when a becomes negative, the graph of $y = ax^2$ starts to curve again, only it curves downward instead of upward. This fits with what you've already learned: the graph opens upward if a is positive and downward if a is negative.

For example, here are the graphs of $y = x^2$ and $y = -x^2$. You can see that the parabola has the same shape in both graphs, but the graph of $y = x^2$ is right-side-up and the graph of $y = -x^2$ is upside-down.



Vertical Shift

Changing the constant c just shifts the parabola up or down. The following plot shows the graphs of $y = x^2$, $y = x^2 + 1$, $y = x^2 - 1$, $y = x^2 + 2$, and $y = x^2 - 2$.



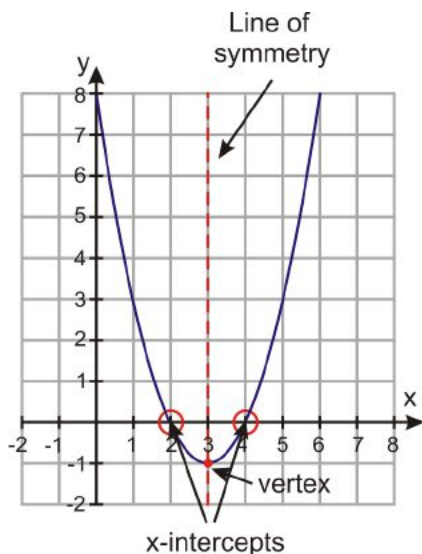
You can see that when c is positive, the graph shifts up, and when c is negative the graph shifts down; in either case, it shifts by $|c|$ units. In one of the later sections we'll learn about **horizontal shift** (i.e. moving to the right or to the left). Before we can do that, though, we need to learn how to rewrite quadratic equations in different forms.

Meanwhile, if you want to explore further what happens when you change the coefficients of a quadratic equation, the page at <http://www.analyzemath.com/quadraticg/quadraticg.htm> has an applet you can use. Click on the “Click here to start” button in section A, and then use the sliders to change the values of a , b , and c .

Graph Quadratic Functions in Intercept Form

Now it's time to learn how to graph a parabola without having to use a table with a large number of points.

Let's look at the graph of $y = x^2 - 6x + 8$.



There are several things we can notice:

- The parabola crosses the x -axis at two points: $x = 2$ and $x = 4$. These points are called the **x -intercepts** of the parabola.
- The lowest point of the parabola occurs at $(3, -1)$.
 - This point is called the **vertex** of the parabola.
 - The vertex is the lowest point in any parabola that turns upward, or the highest point in any parabola that turns downward.
 - The vertex is *exactly halfway between the two x -intercepts*. This will always be the case, and you can find the vertex using that property.
- The parabola is **symmetric**. If you draw a vertical line through the vertex, you see that the two halves of the parabola are mirror images of each other. This vertical line is called the **line of symmetry**.

We said that the general form of a quadratic function is $y = ax^2 + bx + c$. When we can factor a quadratic expression, we can rewrite the function in **intercept form**:

$$y = a(x - m)(x - n)$$

This form is very useful because it makes it easy for us to find the x -intercepts and the vertex of the parabola. The x -intercepts are the values of x where the graph crosses the x -axis; in other words, they are the values of x when $y = 0$. To find the x -intercepts from the quadratic function, we set $y = 0$ and solve:

$$0 = a(x - m)(x - n)$$

Since the equation is already factored, we use the zero-product property to set each factor equal to zero and solve the individual linear equations:

$$x - m = 0$$

$$x - n = 0$$

or

$$x = m$$

$$x = n$$

So the x -intercepts are at points $(m, 0)$ and $(n, 0)$.

Once we find the x -intercepts, it's simple to find the vertex. The x -value of the vertex is halfway between the two x -intercepts, so we can find it by taking the average of the two values: $\frac{m+n}{2}$. Then we can find the y -value by plugging the value of x back into the equation of the function.

Example 2

Find the x -intercepts and the vertex of the following quadratic functions:

a) $y = x^2 - 8x + 15$

b) $y = 3x^2 + 6x - 24$

Solution

a) $y = x^2 - 8x + 15$

Write the quadratic function in intercept form by factoring the right hand side of the equation. Remember, to factor we need two numbers whose product is 15 and whose sum is -8 . These numbers are -5 and -3 .

The function in intercept form is $y = (x - 5)(x - 3)$

We find the x -intercepts by setting $y = 0$.

We have:

$$\begin{array}{ccc} 0 = (x - 5)(x - 3) & & \\ -5 = 0 & & x - 3 = 0 \\ & \text{or} & \\ = 5 & & x = 3 \end{array}$$

So the x -intercepts are **(5, 0) and (3, 0)**.

The vertex is halfway between the two x -intercepts. We find the x -value by taking the average of the two x -intercepts: $x = \frac{5+3}{2} = 4$

We find the y -value by plugging the x -value we just found into the original equation:

$$y = x^2 - 8x + 15 \Rightarrow y = 4^2 - 8(4) + 15 = 16 - 32 + 15 = -1$$

So the vertex is **(4, -1)**.

b) $y = 3x^2 + 6x - 24$

Re-write the function in intercept form.

Factor the common term of 3 first: $y = 3(x^2 + 2x - 8)$

Then factor completely: $y = 3(x + 4)(x - 2)$

Set $y = 0$ and solve:

$$\begin{array}{ccc} 0 = 3(x + 4)(x - 2) \Rightarrow & x + 4 = 0 & x - 2 = 0 \\ & x = -4 & x = 2 \end{array}$$

The x -intercepts are **(-4, 0) and (2, 0)**.

For the vertex,

$$x = \frac{-4+2}{2} = -1 \text{ and } y = 3(-1)^2 + 6(-1) - 24 = 3 - 6 - 24 = -27$$

The vertex is: (-1, -27)

Knowing the vertex and x -intercepts is a useful first step toward being able to graph quadratic functions more easily. Knowing the vertex tells us where the middle of the parabola is. When making a table of values, we can make sure to pick the vertex as a point in the table. Then we choose just a few smaller and larger values of x . In this way, we get an accurate graph of the quadratic function without having to have too many points in our table.

Example 3

Find the x -intercepts and vertex. Use these points to create a table of values and graph each function.

a) $y = x^2 - 4$

b) $y = -x^2 + 14x - 48$

Solution

a) $y = x^2 - 4$

Let's find the x -intercepts and the vertex:

Factor the right-hand side of the function to put the equation in intercept form:

$$y = (x - 2)(x + 2)$$

Set $y = 0$ and solve:

$$\begin{array}{ccc} 0 = (x - 2)(x + 2) & & \\ -2 = 0 & & x + 2 = 0 \\ & \text{or} & \\ = 2 & & x = -2 \end{array}$$

The x -intercepts are (2, 0) and (-2, 0).

Find the vertex:

$$x = \frac{2 - 2}{2} = 0 \quad y = (0)^2 - 4 = -4$$

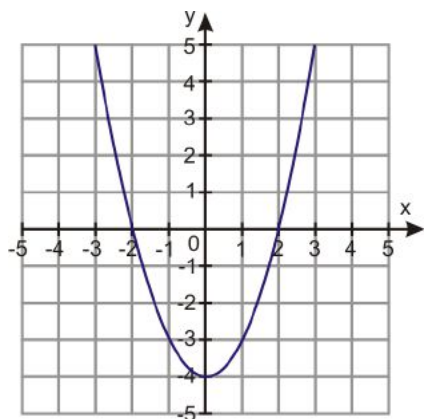
The vertex is (0, -4).

Make a table of values using the vertex as the middle point. Pick a few values of x smaller and larger than $x = 0$. Include the x -intercepts in the table.

Table 7.7:

| x | $y = x^2 - 4$ | |
|-----|-----------------------|----------------|
| -3 | $y = (-3)^2 - 4 = 5$ | |
| -2 | $y = (-2)^2 - 4 = 0$ | x -intercept |
| -1 | $y = (-1)^2 - 4 = -3$ | |
| 0 | $y = (0)^2 - 4 = -4$ | vertex |
| 1 | $y = (1)^2 - 4 = -3$ | |
| 2 | $y = (2)^2 - 4 = 0$ | x -intercept |
| 3 | $y = (3)^2 - 4 = 5$ | |

Then plot the graph:



b) $y = -x^2 + 14x - 48$

Let's find the x -intercepts and the vertex:

Factor the right-hand-side of the function to put the equation in intercept form:

$$y = -(x^2 - 14x + 48) = -(x - 6)(x - 8)$$

Set $y = 0$ and solve:

$$\begin{aligned} 0 &= -(x - 6)(x - 8) \\ -6 &= 0 & x - 8 &= 0 \\ & & \text{or} & \\ &= 6 & x &= 8 \end{aligned}$$

The x -intercepts are $(6, 0)$ and $(8, 0)$.

Find the vertex:

$$x = \frac{6 + 8}{2} = 7 \quad y = -(7)^2 + 14(7) - 48 = 1$$

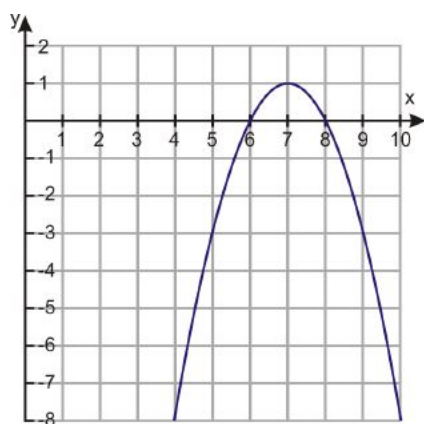
The vertex is $(7, 1)$.

Make a table of values using the vertex as the middle point. Pick a few values of x smaller and larger than $x = 7$. Include the x -intercepts in the table.

Table 7.8:

| x | $y = -x^2 + 14x - 48$ |
|-----|----------------------------------|
| 4 | $y = -(4)^2 + 14(4) - 48 = -8$ |
| 5 | $y = -(5)^2 + 14(5) - 48 = -3$ |
| 6 | $y = -(6)^2 + 14(6) - 48 = 0$ |
| 7 | $y = -(7)^2 + 14(7) - 48 = 1$ |
| 8 | $y = -(8)^2 + 14(8) - 48 = 0$ |
| 9 | $y = -(9)^2 + 14(9) - 48 = -3$ |
| 10 | $y = -(10)^2 + 14(10) - 48 = -8$ |

Then plot the graph:



Analyze Graphs of Real-World Quadratic Functions.

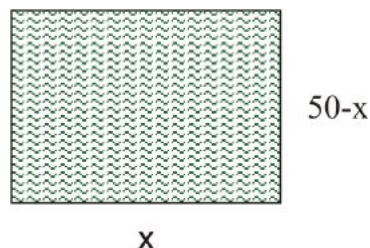
As we mentioned at the beginning of this section, parabolic curves are common in real-world applications. Here we will look at a few graphs that represent some examples of real-life application of quadratic functions.

Example 4

Andrew has 100 feet of fence to enclose a rectangular tomato patch. What should the dimensions of the rectangle be in order for the rectangle to have the greatest possible area?

Solution

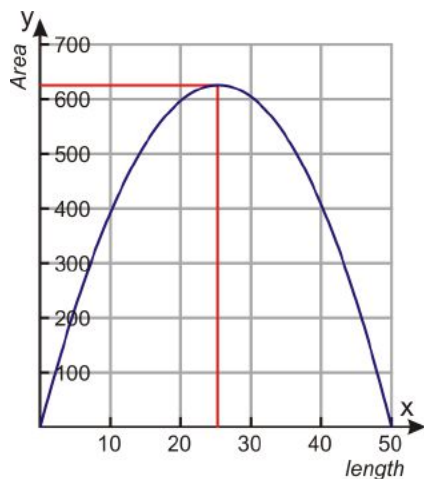
Drawing a picture will help us find an equation to describe this situation:



If the length of the rectangle is x , then the width is $50 - x$. (The length and the width add up to 50, not 100, because two lengths and two widths together add up to 100.)

If we let y be the area of the triangle, then we know that the area is length \times width, so $y = x(50 - x) = 50x - x^2$.

Here's the graph of that function, so we can see how the area of the rectangle depends on the length of the rectangle:



We can see from the graph that the highest value of the area occurs when the length of the rectangle is 25. The area of the rectangle for this side length equals 625. (Notice that the width is also 25, which makes the shape a square with side length 25.)

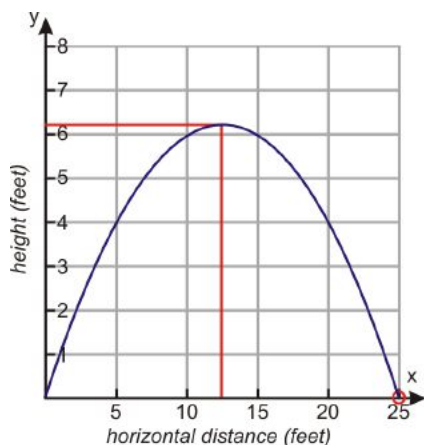
This is an example of an *optimization problem*. These problems show up often in the real world, and if you ever study calculus, you'll learn how to solve them without graphs.

Example 5

Anne is playing golf. On the 4th tee, she hits a slow shot down the level fairway. The ball follows a parabolic path described by the equation $y = x - 0.04x^2$, where y is the ball's height in the air and x is the horizontal distance it has traveled from the tee. The distances are measured in feet. How far from the tee does the ball hit the ground? At what distance from the tee does the ball attain its maximum height? What is the maximum height?

Solution

Let's graph the equation of the path of the ball:



$x(1 - 0.04x) = 0$ has solutions $x = 0$ and $x = 25$.

From the graph, we see that the ball hits the ground **25 feet from the tee**. (The other x -intercept, $x = 0$, tells us that the ball was also on the ground when it was on the tee!)

We can also see that the ball reaches its maximum height of **about 6.25 feet** when it is **12.5 feet from the tee**.

Review Questions

Rewrite the following functions in intercept form. Find the x -intercepts and the vertex.

1. $y = x^2 - 2x - 8$
2. $y = -x^2 + 10x - 21$
3. $y = 2x^2 + 6x + 4$
4. $y = 3(x + 5)(x - 2)$

Does the graph of the parabola turn up or down?

5. $y = -2x^2 - 2x - 3$
6. $y = 3x^2$
7. $y = 16 - 4x^2$
8. $y = 3x^2 - 2x - 4x^2 + 3$

The vertex of which parabola is higher?

9. $y = x^2 + 4$ or $y = x^2 + 1$
10. $y = -2x^2$ or $y = -2x^2 - 2$
11. $y = 3x^2 - 3$ or $y = 3x^2 - 6$
12. $y = 5 - 2x^2$ or $y = 8 - 2x^2$

Which parabola is wider?

13. $y = x^2$ or $y = 4x^2$
14. $y = 2x^2 + 4$ or $y = \frac{1}{2}x^2 + 4$
15. $y = -2x^2 - 2$ or $y = -x^2 - 2$
16. $y = x^2 + 3x^2$ or $y = x^2 + 3$

Graph the following functions by making a table of values. Use the vertex and x -intercepts to help you pick values for the table.

17. $y = 4x^2 - 4$
18. $y = -x^2 + x + 12$
19. $y = 2x^2 + 10x + 8$
20. $y = \frac{1}{2}x^2 - 2x$
21. $y = x - 2x^2$
22. $y = 4x^2 - 8x + 4$
23. Nadia is throwing a ball to Peter. Peter does not catch the ball and it hits the ground. The graph shows the path of the ball as it flies through the air. The equation that describes the path of the ball is $y = 4 + 2x - 0.16x^2$. Here y is the height of the ball and x is the horizontal distance from Nadia. Both distances are measured in feet.
 - (a) How far from Nadia does the ball hit the ground?
 - (b) At what distance x from Nadia, does the ball attain its maximum height?
 - (c) What is the maximum height?
24. Jasreel wants to enclose a vegetable patch with 120 feet of fencing. He wants to put the vegetable against an existing wall, so he only needs fence for three of the sides. The equation for the area is given by $A = 120x - x^2$. From the graph, find what dimensions of the rectangle would give him the greatest area.

7.2 Quadratic Equations by Graphing

Learning Objectives

- Identify the number of solutions of a quadratic equation.
- Solve quadratic equations by graphing.
- Analyze quadratic functions using a graphing calculator.
- Solve real-world problems by graphing quadratic functions.

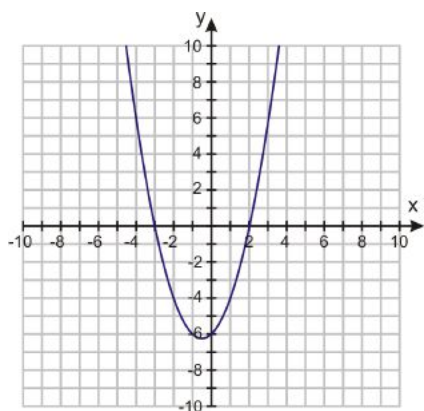
Introduction

Solving a quadratic equation means finding the x -values that will make the quadratic function equal zero; in other words, it means finding the points where the graph of the function crosses the x -axis. The solutions to a quadratic equation are also called the **roots** or **zeros** of the function, and in this section we'll learn how to find them by graphing the function.

Identify the Number of Solutions of a Quadratic Equation

Three different situations can occur when graphing a quadratic function:

Case 1: The parabola crosses the x -axis at two points. An example of this is $y = x^2 + x - 6$:



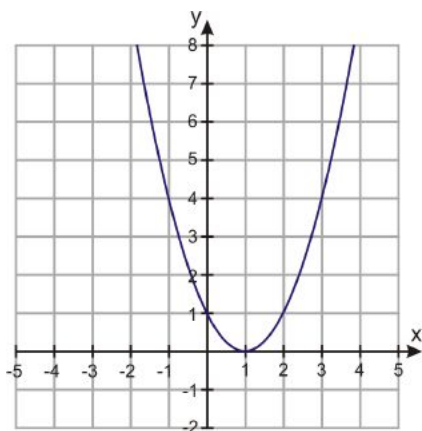
Looking at the graph, we see that the parabola crosses the x -axis at $x = -3$ and $x = 2$.

We can also find the solutions to the equation $x^2 + x - 6 = 0$ by setting $y = 0$. We solve the equation by factoring:

$$(x + 3)(x - 2) = 0, \text{ so } x = -3 \text{ or } x = 2.$$

When the graph of a quadratic function crosses the x -axis at two points, we get **two distinct solutions** to the quadratic equation.

Case 2: The parabola touches the x -axis at one point. An example of this is $y = x^2 - 2x + 1$:

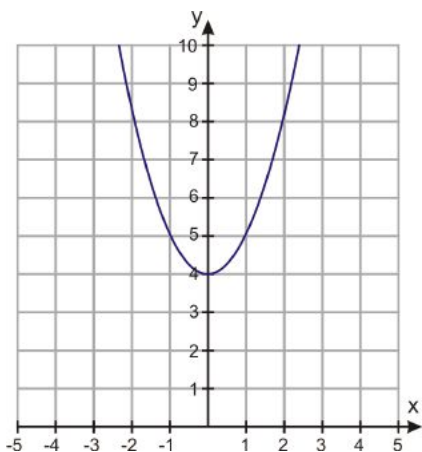


We can see that the graph touches the x -axis at $x = 1$.

We can also solve this equation by factoring. If we set $y = 0$ and factor, we obtain $(x - 1)^2 = 0$, so $x = 1$. Since the quadratic function is a perfect square, we get only one solution for the equation—it's just the same solution repeated twice over.

When the graph of a quadratic function touches the x -axis at one point, the quadratic equation has one solution and the solution is called a **double root**.

Case 3: The parabola does not cross or touch the x -axis. An example of this is $y = x^2 + 4$:



If we set $y = 0$ we get $x^2 + 4 = 0$. This quadratic polynomial does not factor.

When the graph of a quadratic function does not cross or touch the x -axis, the quadratic equation has **no real solutions**.

Solve Quadratic Equations by Graphing

So far we've found the solutions to quadratic equations using factoring. However, in real life very few functions factor easily. As you just saw, graphing a function gives a lot of information about the solutions. We can find exact or approximate solutions to a quadratic equation by graphing the function associated with it.

Example 1

Find the solutions to the following quadratic equations by graphing.

a) $-x^2 + 3 = 0$

b) $2x^2 + 5x - 7 = 0$

c) $-x^2 + x - 3 = 0$

d) $y = -x^2 + 4x - 4$

Solution

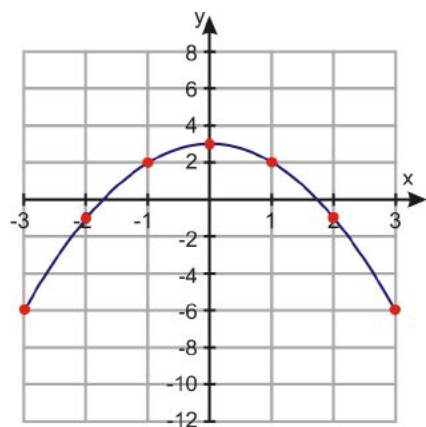
Since we can't factor any of these equations, we won't be able to graph them using intercept form (if we could, we wouldn't need to use the graphs to find the intercepts!) We'll just have to make a table of arbitrary values to graph each one.

a)

Table 7.9:

| x | $y = -x^2 + 3$ |
|-----|------------------------|
| -3 | $y = -(-3)^2 + 3 = -6$ |
| -2 | $y = -(-2)^2 + 3 = -1$ |
| -1 | $y = -(-1)^2 + 3 = 2$ |
| 0 | $y = -(0)^2 + 3 = 3$ |
| 1 | $y = -(1)^2 + 3 = 2$ |
| 2 | $y = -(2)^2 + 3 = -1$ |
| 3 | $y = -(3)^2 + 3 = -6$ |

We plot the points and get the following graph:



From the graph we can read that the x -intercepts are approximately $x = 1.7$ **and** $x = -1.7$. These are the solutions to the equation.

b)

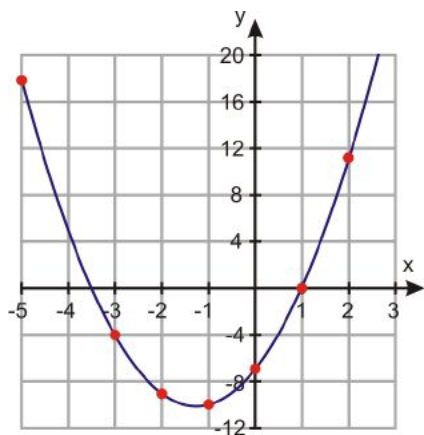
Table 7.10:

| x | $y = 2x^2 + 5x - 7$ |
|-----|---------------------------------|
| -5 | $y = 2(-5)^2 + 5(-5) - 7 = 18$ |
| -4 | $y = 2(-4)^2 + 5(-4) - 7 = 5$ |
| -3 | $y = 2(-3)^2 + 5(-3) - 7 = -4$ |
| -2 | $y = 2(-2)^2 + 5(-2) - 7 = -9$ |
| -1 | $y = 2(-1)^2 + 5(-1) - 7 = -10$ |
| 0 | $y = 2(0)^2 + 5(0) - 7 = -7$ |

Table 7.10: (continued)

| x | $y = 2x^2 + 5x - 7$ |
|-----|------------------------------|
| 1 | $y = 2(1)^2 + 5(1) - 7 = 0$ |
| 2 | $y = 2(2)^2 + 5(2) - 7 = 11$ |
| 3 | $y = 2(3)^2 + 5(3) - 7 = 26$ |

We plot the points and get the following graph:



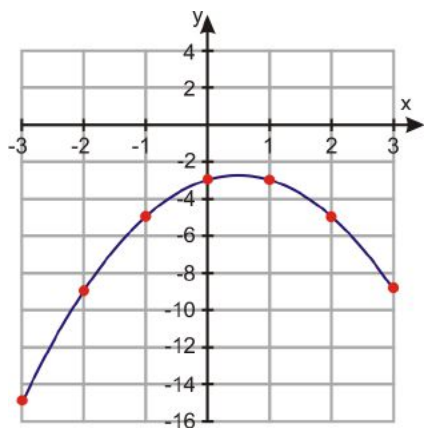
From the graph we can read that the x -intercepts are $x = 1$ **and** $x = -3.5$. These are the solutions to the equation.

c)

Table 7.11:

| x | $y = -x^2 + x - 3$ |
|-----|--------------------------------|
| -3 | $y = -(-3)^2 + (-3) - 3 = -15$ |
| -2 | $y = -(-2)^2 + (-2) - 3 = -9$ |
| -1 | $y = -(-1)^2 + (-1) - 3 = -5$ |
| 0 | $y = -(0)^2 + (0) - 3 = -3$ |
| 1 | $y = -(1)^2 + (1) - 3 = -3$ |
| 2 | $y = -(2)^2 + (2) - 3 = -5$ |
| 3 | $y = -(3)^2 + (3) - 3 = -9$ |

We plot the points and get the following graph:



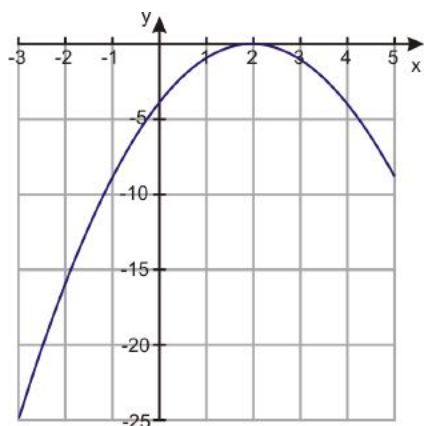
The graph curves up toward the x -axis and then back down without ever reaching it. This means that the graph never intercepts the x -axis, and so the corresponding equation has **no real solutions**.

d)

Table 7.12:

| x | $y = -x^2 + 4x - 4$ |
|-----|---------------------------------|
| -3 | $y = -(-3)^2 + 4(-3) - 4 = -25$ |
| -2 | $y = -(-2)^2 + 4(-2) - 4 = -16$ |
| -1 | $y = -(-1)^2 + 4(-1) - 4 = -9$ |
| 0 | $y = -(0)^2 + 4(0) - 4 = -4$ |
| 1 | $y = -(1)^2 + 4(1) - 4 = -1$ |
| 2 | $y = -(2)^2 + 4(2) - 4 = 0$ |
| 3 | $y = -(3)^2 + 4(3) - 4 = -1$ |
| 4 | $y = -(4)^2 + 4(4) - 4 = -4$ |
| 5 | $y = -(5)^2 + 4(5) - 4 = -9$ |

Here is the graph of this function:



The graph just touches the x -axis at $x = 2$, so the function has a **double root** there. $x = 2$ is the only solution to the equation.

Analyze Quadratic Functions Using a Graphing Calculator

A graphing calculator is very useful for graphing quadratic functions. Once the function is graphed, we can use the calculator to find important information such as the roots or the vertex of the function.

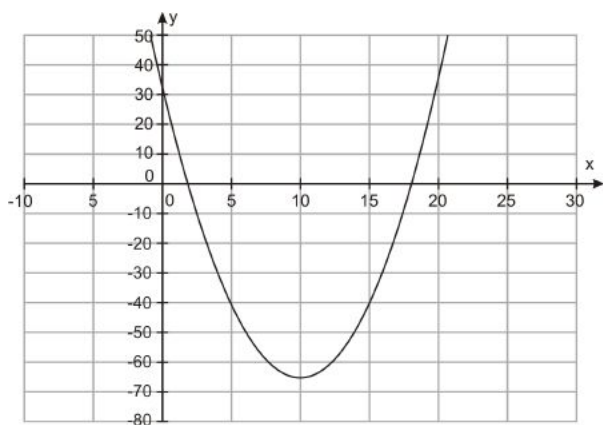
Example 2

Use a graphing calculator to analyze the graph of $y = x^2 - 20x + 35$.

Solution

1. Graph the function.

Press the **[Y=]** button and enter " $x^2 - 20x + 35$ " next to $[Y_1 =]$. Press the **[GRAPH]** button. This is the plot you should see:



If this is not what you see, press the **[WINDOW]** button to change the window size. For the graph shown here, the x -values should range from -10 to 30 and the y -values from -80 to 50.

2. Find the **roots**.

There are at least three ways to find the roots:

Use **[TRACE]** to scroll over the x -intercepts. The approximate value of the roots will be shown on the screen. You can improve your estimate by zooming in.

OR

Use **[TABLE]** and scroll through the values until you find values of y equal to zero. You can change the accuracy of the solution by setting the step size with the **[TBLSET]** function.

OR

Use **[2nd] [TRACE]** (i.e. 'calc' button) and use option 'zero'.

Move the cursor to the left of one of the roots and press **[ENTER]**.

Move the cursor to the right of the same root and press **[ENTER]**.

Move the cursor close to the root and press **[ENTER]**.

The screen will show the value of the root. Repeat the procedure for the other root.

Whichever technique you use, you should get about $x = 1.9$ and $x = 18$ for the two roots.

3. Find the **vertex**.

There are three ways to find the vertex:

Use **[TRACE]** to scroll over the highest or lowest point on the graph. The approximate value of the roots

will be shown on the screen.

OR

Use [TABLE] and scroll through the values until you find values the lowest or highest value of y . You can change the accuracy of the solution by setting the step size with the [TBLSET] function.

OR

Use [2nd] [TRACE] and use the option 'maximum' if the vertex is a maximum or 'minimum' if the vertex is a minimum.

Move the cursor to the left of the vertex and press [ENTER].

Move the cursor to the right of the vertex and press [ENTER].

Move the cursor close to the vertex and press [ENTER].

The screen will show the x - and y -values of the vertex.

Whichever method you use, you should find that the vertex is at (10, -65).

Solve Real-World Problems by Graphing Quadratic Functions

Here's a real-world problem we can solve using the graphing methods we've learned.

Example 3

Andrew is an avid archer. He launches an arrow that takes a parabolic path. The equation of the height of the ball with respect to time is $y = -4.9t^2 + 48t$, where y is the height of the arrow in meters and t is the time in seconds since Andrew shot the arrow. Find how long it takes the arrow to come back to the ground.

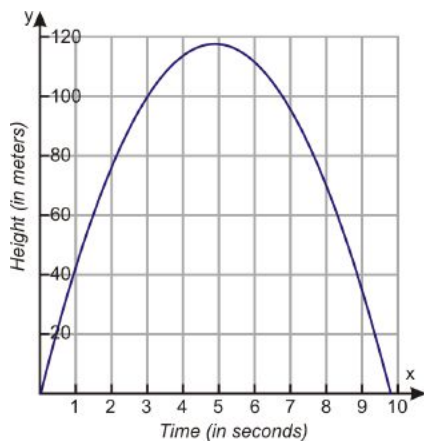
Solution

Let's graph the equation by making a table of values.

Table 7.13:

| t | $y = -4.9t^2 + 48t$ |
|-----|---------------------------------|
| 0 | $y = -4.9(0)^2 + 48(0) = 0$ |
| 1 | $y = -4.9(1)^2 + 48(1) = 43.1$ |
| 2 | $y = -4.9(2)^2 + 48(2) = 76.4$ |
| 3 | $y = -4.9(3)^2 + 48(3) = 99.9$ |
| 4 | $y = -4.9(4)^2 + 48(4) = 113.6$ |
| 5 | $y = -4.9(5)^2 + 48(5) = 117.6$ |
| 6 | $y = -4.9(6)^2 + 48(6) = 111.6$ |
| 7 | $y = -4.9(7)^2 + 48(7) = 95.9$ |
| 8 | $y = -4.9(8)^2 + 48(8) = 70.4$ |
| 9 | $y = -4.9(9)^2 + 48(9) = 35.1$ |
| 10 | $y = -4.9(10)^2 + 48(10) = -10$ |

Here's the graph of the function:



The roots of the function are approximately $x = 0$ sec and $x = 9.8$ sec. The first root tells us that the height of the arrow was 0 meters when Andrew first shot it. The second root says that it takes approximately **9.8 seconds** for the arrow to return to the ground.

Further Practice

Now that you've learned how to solve quadratic equations by graphing them, you can sharpen your skills even more by learning how to find an equation from the graph alone. Go to the page linked in the previous section, <http://www.analyzemath.com/quadraticg/quadraticg.htm>, and scroll down to section E. Read the example there to learn how to find the equation of a quadratic function by reading off a few key values from the graph; then click the “Click here to start” button to try a problem yourself. The “New graph” button will give you a new problem when you finish the first one.

Review Questions

Find the solutions of the following equations by graphing.

1. $x^2 + 3x + 6 = 0$
2. $-2x^2 + x + 4 = 0$
3. $x^2 - 9 = 0$
4. $x^2 + 6x + 9 = 0$
5. $10x - 3x^2 = 0$
6. $\frac{1}{2}x^2 - 2x + 3 = 0$

Find the roots of the following quadratic functions by graphing.

7. $y = -3x^2 + 4x - 1$
8. $y = 9 - 4x^2$
9. $y = x^2 + 7x + 2$
10. $y = -x^2 - 10x - 25$
11. $y = 2x^2 - 3x$
12. $y = x^2 - 2x + 5$

Using your graphing calculator, find the roots and the vertex of each polynomial.

13. $y = x^2 + 12x + 5$

14. $y = x^2 + 3x + 6$
15. $y = -x^2 - 3x + 9$
16. $y = -x^2 + 4x - 12$
17. $y = 2x^2 - 4x + 8$
18. $y = -5x^2 - 3x + 2$
19. Graph the equations $y = 2x^2 - 4x + 8$ and $y = x^2 - 2x + 4$ on the same screen. Find their roots and vertices.
 - (a) What is the same about the graphs? What is different?
 - (b) How are the two equations related to each other? (Hint: factor them.)
 - (c) What might be another equation with the same roots? Graph it and see.
20. Graph the equations $y = x^2 - 2x + 2$ and $y = x^2 - 2x + 4$ on the same screen. Find their roots and vertices.
 - (a) What is the same about the graphs? What is different?
 - (b) How are the two equations related to each other?
21. Phillip throws a ball and it takes a parabolic path. The equation of the height of the ball with respect to time is $y = -16t^2 + 60t$, where y is the height in feet and t is the time in seconds. Find how long it takes the ball to come back to the ground.
22. Use your graphing calculator to solve Ex. 3. You should get the same answers as we did graphing by hand, but a lot quicker!

7.3 Quadratic Equations by Square Roots

Learning Objectives

- Solve quadratic equations involving perfect squares.
- Approximate solutions of quadratic equations.
- Solve real-world problems using quadratic functions and square roots.

Introduction

So far you know how to solve quadratic equations by factoring. However, this method works only if a quadratic polynomial can be factored. In the real world, most quadratics can't be factored, so now we'll start to learn other methods we can use to solve them. In this lesson, we'll examine equations in which we can take the square root of both sides of the equation in order to arrive at the result.

Solve Quadratic Equations Involving Perfect Squares

Let's first examine quadratic equations of the type

$$x^2 - c = 0$$

We can solve this equation by isolating the x^2 term: $x^2 = c$

Once the x^2 term is isolated we can take the square root of both sides of the equation. Remember that when we take the square root we get two answers: the positive square root and the negative square root:

$$x = \sqrt{c} \quad \text{and} \quad x = -\sqrt{c}$$

Often this is written as $x = \pm \sqrt{c}$.

Example 1

Solve the following quadratic equations:

a) $x^2 - 4 = 0$

b) $x^2 - 25 = 0$

Solution

a) $x^2 - 4 = 0$

Isolate the x^2 : $x^2 = 4$

Take the square root of both sides: $x = \sqrt{4}$ and $x = -\sqrt{4}$

The solutions are $x = 2$ **and** $x = -2$.

b) $x^2 - 25 = 0$

Isolate the x^2 : $x^2 = 25$

Take the square root of both sides: $x = \sqrt{25}$ and $x = -\sqrt{25}$

The solutions are $x = 5$ **and** $x = -5$.

We can also find the solution using the square root when the x^2 term is multiplied by a constant—in other words, when the equation takes the form

$$ax^2 - c = 0$$

We just have to isolate the x^2 :

$$\begin{aligned} ax^2 &= b \\ x^2 &= \frac{b}{a} \end{aligned}$$

Then we can take the square root of both sides of the equation:

$$x = \sqrt{\frac{b}{a}} \quad \text{and} \quad x = -\sqrt{\frac{b}{a}}$$

Often this is written as: $x = \pm \sqrt{\frac{b}{a}}$.

Example 2

Solve the following quadratic equations.

a) $9x^2 - 16 = 0$

b) $81x^2 - 1 = 0$

Solution

a) $9x^2 - 16 = 0$

Isolate the x^2 :

$$\begin{aligned} 9x^2 &= 16 \\ x^2 &= \frac{16}{9} \end{aligned}$$

Take the square root of both sides: $x = \sqrt{\frac{16}{9}}$ and $x = -\sqrt{\frac{16}{9}}$

Answer: $x = \frac{4}{3}$ and $x = -\frac{4}{3}$

b) $81x^2 - 1 = 0$

Isolate the x^2 :

$$\begin{aligned}81x^2 &= 1 \\ x^2 &= \frac{1}{81}\end{aligned}$$

Take the square root of both sides: $x = \sqrt{\frac{1}{81}}$ and $x = -\sqrt{\frac{1}{81}}$

Answer: $x = \frac{1}{9}$ and $x = -\frac{1}{9}$

As you've seen previously, some quadratic equations have no real solutions.

Example 3

Solve the following quadratic equations.

a) $x^2 + 1 = 0$

b) $4x^2 + 9 = 0$

Solution

a) $x^2 + 1 = 0$

Isolate the x^2 : $x^2 = -1$

Take the square root of both sides: $x = \sqrt{-1}$ and $x = -\sqrt{-1}$

Square roots of negative numbers do not give real number results, so there are **no real solutions** to this equation.

b) $4x^2 + 9 = 0$

Isolate the x^2 :

$$\begin{aligned}4x^2 &= -9 \\ x^2 &= -\frac{9}{4}\end{aligned}$$

Take the square root of both sides: $x = \sqrt{-\frac{9}{4}}$ and $x = -\sqrt{-\frac{9}{4}}$

There are **no real solutions**.

We can also use the square root function in some quadratic equations where one side of the equation is a perfect square. This is true if an equation is of this form:

$$(x - 2)^2 = 9$$

Both sides of the equation are perfect squares. We take the square root of both sides and end up with two equations: $x - 2 = 3$ and $x - 2 = -3$.

Solving both equations gives us $x = 5$ and $x = -1$.

Example 4

Solve the following quadratic equations.

a) $(x - 1)^2 = 4$

b) $(x + 3)^2 = 1$

Solution

a) $(x - 1)^2 = 4$

Take the square root of both sides :

$$x - 1 = 2 \text{ and } x - 1 = -2$$

Solve each equation :

$$x = 3 \text{ and } x = -1$$

Answer: $x = 3$ and $x = -1$

b) $(x + 3)^2 = 1$

Take the square root of both sides :

$$x + 3 = 1 \text{ and } x + 3 = -1$$

Solve each equation :

$$x = -2 \text{ and } x = -4$$

Answer: $x = -2$ and $x = -4$

It might be necessary to factor the right-hand side of the equation as a perfect square before applying the method outlined above.

Example 5

Solve the following quadratic equations.

a) $x^2 + 8x + 16 = 25$

b) $4x^2 - 40x + 25 = 9$

Solution

a) $x^2 + 8x + 16 = 25$

Factor the right-hand-side :

$$x^2 + 8x + 16 = (x + 4)^2 \quad \text{so} \quad (x + 4)^2 = 25$$

Take the square root of both sides :

$$x + 4 = 5 \text{ and } x + 4 = -5$$

Solve each equation :

$$x = 1 \text{ and } x = -9$$

Answer: $x = 1$ and $x = -9$

b) $4x^2 - 20x + 25 = 9$

Factor the right-hand-side :

$$4x^2 - 20x + 25 = (2x - 5)^2 \quad \text{so} \quad (2x - 5)^2 = 9$$

Take the square root of both sides :

$$2x - 5 = 3 \text{ and } 2x - 5 = -3$$

Solve each equation :

$$2x = 8 \text{ and } 2x = 2$$

Answer: $x = 4$ and $x = 1$

Approximate Solutions of Quadratic Equations

We can use the methods we've learned so far in this section to find approximate solutions to quadratic equations, when taking the square root doesn't give an exact answer.

Example 6

Solve the following quadratic equations.

a) $x^2 - 3 = 0$

b) $2x^2 - 9 = 0$

Solution

a) Isolate the x^2 :

$$x^2 = 3$$

Take the square root of both sides : $x = \sqrt{3}$ and $x = -\sqrt{3}$

Answer: $x \approx 1.73$ and $x \approx -1.73$

b) Isolate the x^2 :

$$2x^2 = 9 \text{ so } x^2 = \frac{9}{2}$$

Take the square root of both sides : $x = \sqrt{\frac{9}{2}}$ and $x = -\sqrt{\frac{9}{2}}$

Answer: $x \approx 2.12$ and $x \approx -2.12$

Example 7

Solve the following quadratic equations.

a) $(2x + 5)^2 = 10$

b) $x^2 - 2x + 1 = 5$

Solution

a) Take the square root of both sides :

$$2x + 5 = \sqrt{10} \text{ and } 2x + 5 = -\sqrt{10}$$

Solve both equations :

$$x = \frac{-5 + \sqrt{10}}{2} \text{ and } x = \frac{-5 - \sqrt{10}}{2}$$

Answer: $x \approx -0.92$ and $x \approx -4.08$

b) Factor the right-hand-side :

$$(x - 1)^2 = 5$$

Take the square root of both sides :

$$x - 1 = \sqrt{5} \text{ and } x - 1 = -\sqrt{5}$$

Solve each equation :

$$x = 1 + \sqrt{5} \text{ and } x = 1 - \sqrt{5}$$

Answer: $x \approx 3.24$ and $x \approx -1.24$

Solve Real-World Problems Using Quadratic Functions and Square Roots

Quadratic equations are needed to solve many real-world problems. In this section, we'll examine problems about objects falling under the influence of gravity. When objects are **dropped** from a height, they have no initial velocity; the force that makes them move towards the ground is due to gravity. The acceleration of gravity on earth is given by the equation

$$g = -9.8 \text{ m/s}^2 \quad \text{or} \quad g = -32 \text{ ft/s}^2$$

The negative sign indicates a downward direction. We can assume that gravity is constant for the problems we'll be examining, because we will be staying close to the surface of the earth. The acceleration of gravity decreases as an object moves very far from the earth. It is also different on other celestial bodies such as the moon.

The equation that shows the height of an object in free fall is

$$y = \frac{1}{2}gt^2 + y_0$$

The term y_0 represents the initial height of the object, t is time, and g is the constant representing the force of gravity. You then plug in one of the two values for g above, depending on whether you want the answer in feet or meters. Thus the equation works out to $y = -4.9t^2 + y_0$ if you want the height in meters, and $y = -16t^2 + y_0$ if you want it in feet.

Example 8

How long does it take a ball to fall from a roof to the ground 25 feet below?

Solution

| | |
|--|---|
| Since we are given the height in feet, use equation : | $y = -16t^2 + y_0$ |
| The initial height is $y_0 = 25$ feet, so : | $y = -16t^2 + 25$ |
| The height when the ball hits the ground is $y = 0$, so : | $0 = -16t^2 + 25$ |
| Solve for t : | $16t^2 = 25$ |
| | $t^2 = \frac{25}{16}$ |
| | $t = \frac{5}{4}$ or $t = -\frac{5}{4}$ |

Since only positive time makes sense in this case, **it takes the ball 1.25 seconds to fall to the ground.**

Example 9

A rock is dropped from the top of a cliff and strikes the ground 7.2 seconds later. How high is the cliff in meters?

Solution

| | |
|--|---------------------------------|
| Since we want the height in meters, use equation : | $y = -4.9t^2 + y_0$ |
| The time of flight is $t = 7.2$ seconds : | $y = -4.9(7.2)^2 + y_0$ |
| The height when the ball hits the ground is $y = 0$, so : | $0 = -4.9(7.2)^2 + y_0$ |
| Simplify : | $0 = -254 + y_0$ so $y_0 = 254$ |

The cliff is 254 meters high.

Example 10

Victor throws an apple out of a window on the 10th floor which is 120 feet above ground. One second later Juan throws an orange out of a 6th floor window which is 72 feet above the ground. Which fruit reaches the ground first, and how much faster does it get there?

Solution

Let's find the time of flight for each piece of fruit.

Apple:

| | |
|--|---|
| Since we have the height in feet, use this equation : | $y = -16t^2 + y_0$ |
| The initial height is $y_0 = 120$ feet : | $y = -16t^2 + 120$ |
| The height when the ball hits the ground is $y = 0$, so : | $0 = -16t^2 + 120$ |
| Solve for t : | $16t^2 = 120$ |
| | $t^2 = \frac{120}{16} = 7.5$ |
| | $t = \underline{2.74}$ or $t = -2.74$ seconds |

Orange:

The initial height is $y_0 = 72$ feet :

Solve for t :

$$0 = -16t^2 + 72$$

$$16t^2 = 72$$

$$t^2 = \frac{72}{16} = 4.5$$

$$t = \underline{2.12} \text{ or } t = -2.12 \text{ seconds}$$

The orange was thrown one second later, so add 1 second to the time of the orange: $t = 3.12$ seconds

The apple hits the ground first. It gets there 0.38 seconds faster than the orange.

Review Questions

Solve the following quadratic equations.

1. $x^2 - 1 = 0$
2. $x^2 - 100 = 0$
3. $x^2 + 16 = 0$
4. $9x^2 - 1 = 0$
5. $4x^2 - 49 = 0$
6. $64x^2 - 9 = 0$
7. $x^2 - 81 = 0$
8. $25x^2 - 36 = 0$
9. $x^2 + 9 = 0$
10. $x^2 - 16 = 0$
11. $x^2 - 36 = 0$
12. $16x^2 - 49 = 0$
13. $(x - 2)^2 = 1$
14. $(x + 5)^2 = 16$
15. $(2x - 1)^2 - 4 = 0$
16. $(3x + 4)^2 = 9$
17. $(x - 3)^2 + 25 = 0$
18. $x^2 - 6 = 0$
19. $x^2 - 20 = 0$
20. $3x^2 + 14 = 0$
21. $(x - 6)^2 = 5$
22. $(4x + 1)^2 - 8 = 0$
23. $x^2 - 10x + 25 = 9$
24. $x^2 + 18x + 81 = 1$
25. $4x^2 - 12x + 9 = 16$
26. $(x + 10)^2 = 2$
27. $x^2 + 14x + 49 = 3$
28. $2(x + 3)^2 = 8$
29. Susan drops her camera in the river from a bridge that is 400 feet high. How long is it before she hears the splash?
30. It takes a rock 5.3 seconds to splash in the water when it is dropped from the top of a cliff. How high is the cliff in meters?
31. Nisha drops a rock from the roof of a building 50 feet high. Ashaan drops a quarter from the top

story window, 40 feet high, exactly half a second after Nisha drops the rock. Which hits the ground first?

7.4 Solving Quadratic Equations by Completing the Square

Learning Objectives

- Complete the square of a quadratic expression.
- Solve quadratic equations by completing the square.
- Solve quadratic equations in standard form.
- Graph quadratic equations in vertex form.
- Solve real-world problems using functions by completing the square.

Introduction

You saw in the last section that if you have a quadratic equation of the form $(x - 2)^2 = 5$, you can easily solve it by taking the square root of each side:

$$x - 2 = \sqrt{5} \quad \text{and} \quad x - 2 = -\sqrt{5}$$

Simplify to get:

$$x = 2 + \sqrt{5} \approx 4.24 \quad \text{and} \quad x = 2 - \sqrt{5} \approx -0.24$$

So what do you do with an equation that isn't written in this nice form? In this section, you'll learn how to rewrite any quadratic equation in this form by **completing the square**.

Complete the Square of a Quadratic Expression

Completing the square lets you rewrite a quadratic expression so that it contains a perfect square trinomial that you can factor as the square of a binomial.

Remember that the square of a binomial takes one of the following forms:

$$\begin{aligned}(x + a)^2 &= x^2 + 2ax + a^2 \\ (x - a)^2 &= x^2 - 2ax + a^2\end{aligned}$$

So in order to have a perfect square trinomial, we need two terms that are perfect squares and one term that is twice the product of the square roots of the other terms.

Example 1

Complete the square for the quadratic expression $x^2 + 4x$.

Solution

To complete the square we need a constant term that turns the expression into a perfect square trinomial. Since the middle term in a perfect square trinomial is always 2 times the product of the square roots of the other two terms, we re-write our expression as:

$$x^2 + 2(2)(x)$$

We see that the constant we are seeking must be 2^2 :

$$x^2 + 2(2)(x) + 2^2$$

Answer: By adding 4 to both sides, this can be factored as: $(x + 2)^2$

Notice, though, that we just changed the value of the whole expression by adding 4 to it. If it had been an equation, we would have needed to add 4 to the other side as well to make up for this.

Also, this was a relatively easy example because a , the coefficient of the x^2 term, was 1. When that coefficient doesn't equal 1, we have to factor it out from the whole expression before completing the square.

Example 2

Complete the square for the quadratic expression $4x^2 + 32x$.

Solution

Factor the coefficient of the x^2 term :

$$4(x^2 + 8x)$$

Now complete the square of the expression in parentheses.

Re-write the expression :

$$4(x^2 + 2(4)(x))$$

We complete the square by adding the constant 4^2 :

$$4(x^2 + 2(4)(x) + 4^2)$$

Factor the perfect square trinomial inside the parenthesis :

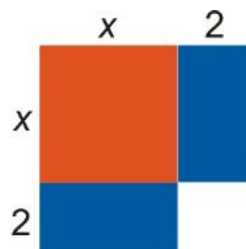
$$4(x + 4)^2 \quad \text{Answer}$$

The expression “**completing the square**” comes from a geometric interpretation of this situation. Let's revisit the quadratic expression in Example 1: $x^2 + 4x$.

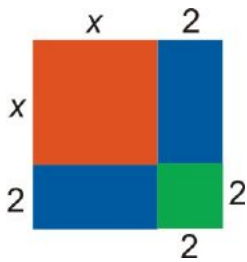
We can think of this expression as the sum of three areas. The first term represents the area of a square of side x . The second expression represents the areas of two rectangles with a length of 2 and a width of x :

$$\begin{array}{c} x \\ \text{[Orange Square]} \\ x \end{array} = x^2 \quad \begin{array}{c} 2 \\ \text{[Blue Rectangle]} \\ x \end{array} \quad \begin{array}{c} x \\ \text{[Blue Rectangle]} \\ 2 \end{array} = 2x$$

We can combine these shapes as follows:



We obtain a square that is not quite complete. To complete the square, we need to add a smaller square of side length 2.



We end up with a square of side length $(x + 2)$; its area is therefore $(x + 2)^2$.

Solve Quadratic Equations by Completing the Square

Let's demonstrate the method of **completing the square** with an example.

Example 3

Solve the following quadratic equation: $3x^2 - 10x = -1$

Solution

Divide all terms by the coefficient of the x^2 term:

$$x^2 - \frac{10}{3}x = -\frac{1}{3}$$

Rewrite: $x^2 - 2\left(\frac{5}{3}\right)(x) = -\frac{1}{3}$ In order to have a perfect square trinomial on the right-hand-side we need to add the constant $\left(\frac{5}{3}\right)^2$. Add this constant to **both** sides of the equation:

$$x^2 - 2\left(\frac{5}{3}\right)(x) + \left(\frac{5}{3}\right)^2 = -\frac{1}{3} + \left(\frac{5}{3}\right)^2$$

Factor the perfect square trinomial and simplify:

$$\begin{aligned} \left(x - \frac{5}{3}\right)^2 &= -\frac{1}{3} + \frac{25}{9} \\ \left(x - \frac{5}{3}\right)^2 &= \frac{22}{9} \end{aligned}$$

Take the square root of both sides:

$$\begin{aligned} x - \frac{5}{3} &= \sqrt{\frac{22}{9}} & \text{and} & & x - \frac{5}{3} &= -\sqrt{\frac{22}{9}} \\ x &= \frac{5}{3} + \sqrt{\frac{22}{9}} \approx 3.23 & \text{and} & & x &= \frac{5}{3} - \sqrt{\frac{22}{9}} \approx 0.1 \end{aligned}$$

Answer: $x = 3.23$ and $x = 0.1$

If an equation is in standard form ($ax^2 + bx + c = 0$), we can still solve it by the method of completing the square. All we have to do is start by moving the constant term to the right-hand-side of the equation.

Example 4

Solve the following quadratic equation: $x^2 + 15x + 12 = 0$

Solution

Move the constant to the other side of the equation:

$$x^2 + 15x = -12$$

Rewrite: $x^2 + 2\left(\frac{15}{2}\right)(x) = -12$

Add the constant $\left(\frac{15}{2}\right)^2$ to both sides of the equation:

$$x^2 + 2\left(\frac{15}{2}\right)(x) + \left(\frac{15}{2}\right)^2 = -12 + \left(\frac{15}{2}\right)^2$$

Factor the perfect square trinomial and simplify:

$$\begin{aligned}\left(x + \frac{15}{2}\right)^2 &= -12 + \frac{225}{4} \\ \left(x + \frac{15}{2}\right)^2 &= \frac{177}{4}\end{aligned}$$

Take the square root of both sides:

$$\begin{aligned}x + \frac{15}{2} &= \sqrt{\frac{177}{4}} & \text{and} & & x + \frac{15}{2} &= -\sqrt{\frac{177}{4}} \\ x &= -\frac{15}{2} + \sqrt{\frac{177}{4}} \approx -0.85 & \text{and} & & x &= -\frac{15}{2} - \sqrt{\frac{177}{4}} \approx -14.15\end{aligned}$$

Answer: $x = -0.85$ and $x = -14.15$

Graph Quadratic Functions in Vertex Form

Probably one of the best applications of the method of completing the square is using it to rewrite a quadratic function in vertex form. The vertex form of a quadratic function is

$$y - k = a(x - h)^2$$

This form is very useful for graphing because it gives the vertex of the parabola explicitly. The vertex is at the point (h, k) .

It is also simple to find the x -intercepts from the vertex form: just set $y = 0$ and take the square root of both sides of the resulting equation.

To find the y -intercept, set $x = 0$ and simplify.

Example 5

Find the vertex, the x -intercepts and the y -intercept of the following parabolas:

a) $y - 2 = (x - 1)^2$

b) $y + 8 = 2(x - 3)^2$

Solution

a) $y - 2 = (x - 1)^2$

Vertex: $(1, 2)$

To find the x -intercepts,

$$\begin{array}{llll} \text{Set } y = 0 : & -2 = (x - 1)^2 & & \\ \text{Take the square root of both sides :} & \sqrt{-2} = x - 1 & \text{and} & -\sqrt{-2} = x - 1 \end{array}$$

The solutions are not real so there are **no x -intercepts**.

To find the y -intercept,

$$\begin{array}{ll} \text{Set } x = 0 : & y - 2 = (-1)^2 \\ \text{Simplify :} & y - 2 = 1 \Rightarrow \underline{y = 3} \end{array}$$

$$\text{b) } y + 8 = 2(x - 3)^2$$

$$\begin{array}{ll} \text{Rewrite :} & y - (-8) = 2(x - 3)^2 \\ \text{Vertex :} & \underline{(3, -8)} \end{array}$$

To find the x -intercepts,

$$\begin{array}{llll} \text{Set } y = 0 : & 8 = 2(x - 3)^2 & & \\ \text{Divide both sides by 2 :} & 4 = (x - 3)^2 & & \\ \text{Take the square root of both sides :} & 4 = x - 3 & \text{and} & -4 = x - 3 \\ \text{Simplify :} & \underline{x = 7} & \text{and} & \underline{x = -1} \end{array}$$

To find the y -intercept,

$$\begin{array}{ll} \text{Set } x = 0 : & y + 8 = 2(-3)^2 \\ \text{Simplify :} & y + 8 = 18 \Rightarrow \underline{y = 10} \end{array}$$

To graph a parabola, we only need to know the following information:

- the vertex
- the x -intercepts
- the y -intercept
- whether the parabola turns up or down (remember that it turns up if $a > 0$ and down if $a < 0$)

Example 6

Graph the parabola given by the function $y + 1 = (x + 3)^2$.

Solution

$$\begin{array}{llll} \text{Rewrite :} & y - (-1) = (x - (-3))^2 & & \\ \text{Vertex :} & \underline{(-3, -1)} & \text{vertex :} & (-3, -1) \end{array}$$

To find the x -intercepts,

$$\begin{array}{llll} \text{Set } y = 0 : & 1 = (x + 3)^2 & & \\ \text{Take the square root of both sides :} & 1 = x + 3 & \text{and} & -1 = x + 3 \\ \text{Simplify :} & \underline{x = -2} & \text{and} & \underline{x = -4} \\ & x\text{-intercepts : } (-2, 0) \text{ and } (-4, 0) & & \end{array}$$

To find the y -intercept,

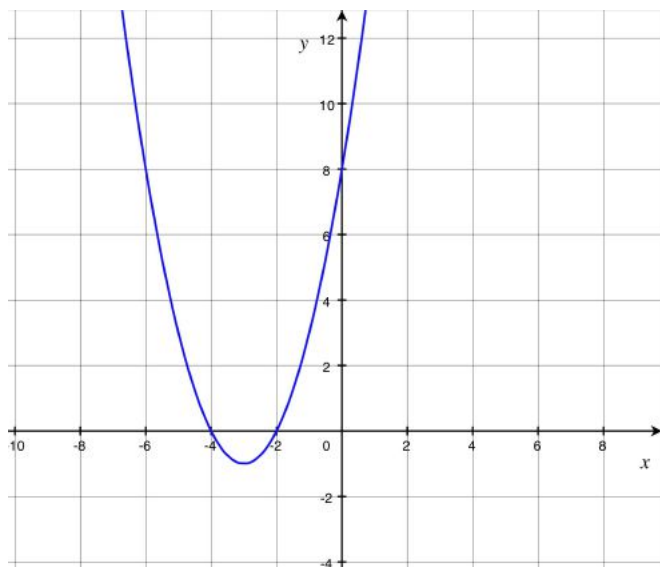
$$\text{Set } x = 0 : \quad y + 1 = (3)^2$$

$$\text{Simplify:} \quad \underline{\underline{y = 8}}$$

$$y - \text{intercept} : (0, 8)$$

And since $a > 0$, the parabola **turns up**.

Graph all the points and connect them with a smooth curve:



Example 7

Graph the parabola given by the function $y = -\frac{1}{2}(x - 2)^2$.

Solution:

$$\text{Rewrite} \quad y - (0) = -\frac{1}{2}(x - 2)^2$$

$$\text{Vertex:} \quad \underline{\underline{(2, 0)}}$$

$$\text{vertex:}(2, 0)$$

To find the x -intercepts,

$$\text{Set } y = 0 : \quad 0 = -\frac{1}{2}(x - 2)^2$$

$$\text{Multiply both sides by } -2 : \quad 0 = (x - 2)^2$$

$$\text{Take the square root of both sides :} \quad 0 = x - 2$$

$$\text{Simplify :} \quad \underline{\underline{x = 2}}$$

$$x - \text{intercept:}(2, 0)$$

Note: there is only one x -intercept, indicating that the vertex is located at this point, $(2, 0)$.

To find the y -intercept,

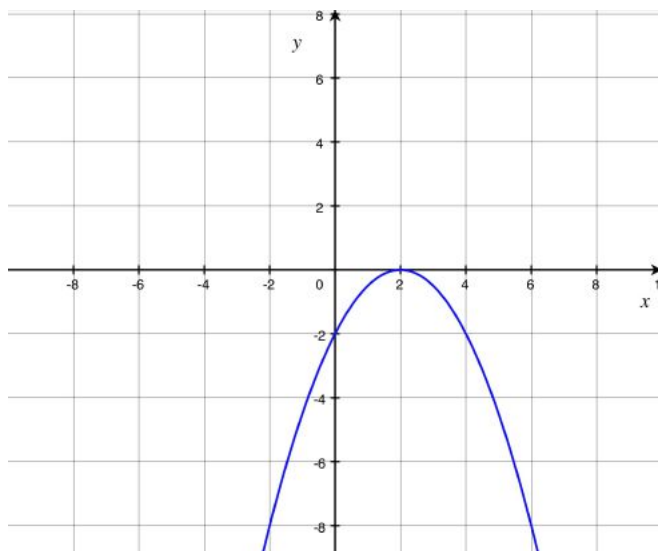
$$\text{Set } x = 0 : \quad y = -\frac{1}{2}(-2)^2$$

$$\text{Simplify:} \quad y = -\frac{1}{2}(4) \Rightarrow \underline{\underline{y = -2}}$$

$$y - \text{intercept:}(0, -2)$$

Since $a < 0$, the parabola **turns down**.

Graph all the points and connect them with a smooth curve:



Solve Real-World Problems Using Quadratic Functions by Completing the Square

In the last section you learned that an object that is dropped falls under the influence of gravity. The equation for its height with respect to time is given by $y = \frac{1}{2}gt^2 + y_0$, where y_0 represents the initial height of the object and g is the coefficient of gravity on earth, which equals -9.8 m/s^2 or -32 ft/s^2 .

On the other hand, if an object is thrown straight up or straight down in the air, it has an initial vertical velocity. This term is usually represented by the notation v_{0y} . Its value is positive if the object is thrown up in the air and is negative if the object is thrown down. The equation for the height of the object in this case is

$$y = \frac{1}{2}gt^2 + v_{0y}t + y_0$$

Plugging in the appropriate value for g turns this equation into

$$y = -4.9t^2 + v_{0y}t + y_0 \text{ if you wish to have the height in meters}$$

$$y = -16t^2 + v_{0y}t + y_0 \text{ if you wish to have the height in feet}$$

Example 8

An arrow is shot straight up from a height of 2 meters with a velocity of 50 m/s.

- How high will the arrow be 4 seconds after being shot? After 8 seconds?*
- At what time will the arrow hit the ground again?*
- What is the maximum height that the arrow will reach and at what time will that happen?*

Solution

Since we are given the velocity in m/s, use: $y = -4.9t^2 + v_{0y}t + y_0$

We know $v_{0y} = 50 \text{ m/s}$ and $y_0 = 2 \text{ meters}$ so: $y = -4.9t^2 + 50t + 2$

- To find how high the arrow will be 4 seconds after being shot we plug in $t = 4$:

$$\begin{aligned}
 y &= -4.9(4)^2 + 50(4) + 2 \\
 &= -4.9(16) + 200 + 2 = \underline{\underline{123.6 \text{ feet}}}
 \end{aligned}$$

we plug in $t = 8$:

$$\begin{aligned}
 y &= -4.9(8)^2 + 50(8) + 2 \\
 &= -4.9(64) + 400 + 2 = \underline{\underline{88.4 \text{ feet}}}
 \end{aligned}$$

b) The height of the ball arrow on the ground is $y = 0$, so: $0 = -4.9t^2 + 50t + 2$

Solve for t by completing the square:

$$\begin{aligned}
 -4.9t^2 + 50t &= -2 \\
 -4.9(t^2 - 10.2t) &= -2 \\
 t^2 - 10.2t &= 0.41 \\
 t^2 - 2(5.1)t + (5.1)^2 &= 0.41 + (5.1)^2 \\
 (t - 5.1)^2 &= 26.43 \\
 t - 5.1 &= 5.14 \text{ and } t - 5.1 = -5.14 \\
 t &= \underline{\underline{10.2 \text{ sec}}} \text{ and } t = -0.04 \text{ sec}
 \end{aligned}$$

The arrow will hit the ground about 10.2 seconds after it is shot.

c) If we graph the height of the arrow with respect to time we would get an upside down parabola ($a < 0$). The maximum height and the time when this occurs is really the vertex of this parabola: (t, h) .

| | |
|--|--|
| We re-write the equation in vertex form: | $y = -4.9t^2 + 50t + 2$ |
| | $y - 2 = -4.9t^2 + 50t$ |
| | $y - 2 = -4.9(t^2 - 10.2t)$ |
| Complete the square: | $y - 2 - 4.9(5.1)^2 = -4.9(t^2 - 10.2t + (5.1)^2)$ |
| | $y - 129.45 = -4.9(t - 5.1)^2$ |

The vertex is at $(5.1, 129.45)$. In other words, **when $t = 5.1$ seconds, the height is $y = 129$ meters.**

Another type of application problem that can be solved using quadratic equations is one where two objects are moving away from each other in perpendicular directions. Here is an example of this type of problem.

Example 9

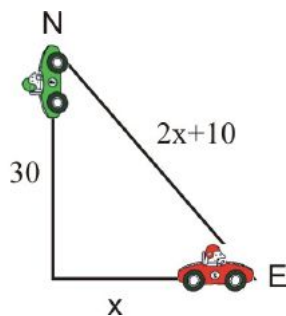
Two cars leave an intersection. One car travels north; the other travels east. When the car traveling north had gone 30 miles, the distance between the cars was 10 miles more than twice the distance traveled by the car heading east. Find the distance between the cars at that time.

Solution

Let x = the distance traveled by the car heading east

Then $2x + 10$ = the distance between the two cars

Let's make a sketch:



We can use the Pythagorean Theorem to find an equation for x :

$$x^2 + 30^2 = (2x + 10)^2$$

Expand parentheses and simplify:

$$\begin{aligned} x^2 + 900 &= 4x^2 + 40x + 100 \\ 800 &= 3x^2 + 40x \end{aligned}$$

Solve by completing the square:

$$\begin{aligned} \frac{800}{3} &= x^2 + \frac{40}{3}x \\ \frac{800}{3} + \left(\frac{20}{3}\right)^2 &= x^2 + 2\left(\frac{20}{3}\right)x + \left(\frac{20}{3}\right)^2 \\ \frac{2800}{9} &= \left(x + \frac{20}{3}\right)^2 \\ x + \frac{20}{3} &= 17.6 \text{ and } x + \frac{20}{3} = -17.6 \\ x &= 11 \text{ and } x = -24.3 \end{aligned}$$

Since only positive distances make sense here, the distance between the two cars is: $2(11) + 10 = 32$ miles

Review Questions

Complete the square for each expression.

1. $x^2 + 5x$
2. $x^2 - 2x$
3. $x^2 + 3x$
4. $x^2 - 4x$
5. $3x^2 + 18x$
6. $2x^2 - 22x$
7. $8x^2 - 10x$
8. $5x^2 + 12x$

Solve each quadratic equation by completing the square.

9. $x^2 - 4x = 5$

10. $x^2 - 5x = 10$
11. $x^2 + 10x + 15 = 0$
12. $x^2 + 15x + 20 = 0$
13. $2x^2 - 18x = 3$
14. $4x^2 + 5x = -1$
15. $10x^2 - 30x - 8 = 0$
16. $5x^2 + 15x - 40 = 0$

Rewrite each quadratic function in vertex form.

17. $y = x^2 - 6x$
18. $y + 1 = -2x^2 - x$
19. $y = 9x^2 + 3x - 10$
20. $y = -32x^2 + 60x + 10$

For each parabola, find the vertex; the x - and y -intercepts; and if it turns up or down. Then graph the parabola.

21. $y - 4 = x^2 + 8x$
22. $y = -4x^2 + 20x - 24$
23. $y = 3x^2 + 15x$
24. $y + 6 = -x^2 + x$
25. Sam throws an egg straight down from a height of 25 feet. The initial velocity of the egg is 16 ft/sec. How long does it take the egg to reach the ground?
26. Amanda and Dolvin leave their house at the same time. Amanda walks south and Dolvin bikes east. Half an hour later they are 5.5 miles away from each other and Dolvin has covered three miles more than the distance that Amanda covered. How far did Amanda walk and how far did Dolvin bike?

7.5 Solving Quadratic Equations by the Quadratic Formula

Learning Objectives

- Solve quadratic equations using the quadratic formula.
- Identify and choose methods for solving quadratic equations.
- Solve real-world problems using functions by completing the square.

Introduction

The **Quadratic Formula** is probably the most used method for solving quadratic equations. For a quadratic equation in standard form, $ax^2 + bx + c = 0$, the quadratic formula looks like this:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula is derived by solving a general quadratic equation using the method of completing the square that you learned in the previous section.

We start with a general quadratic equation: $ax^2 + bx + c = 0$

Subtract the constant term from both sides: $ax^2 + bx = -c$

Divide by the coefficient of the x^2 term: $x^2 + \frac{b}{a}x = -\frac{c}{a}$

Rewrite: $x^2 + 2\left(\frac{b}{2a}\right)x = -\frac{c}{a}$

Add the constant $\left(\frac{b}{2a}\right)^2$ to both sides: $x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$

Factor the perfect square trinomial: $\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$

Simplify: $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

Take the square root of both sides: $x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$ and $x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}$

Simplify: $x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a}$ and $x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{2a}$
 $x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$ and $x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$
 $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

This can be written more compactly as $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

You can see that the familiar formula comes directly from applying the method of completing the square. Applying the method of completing the square to solve quadratic equations can be tedious, so the quadratic formula is a more straightforward way of finding the solutions.

Solve Quadratic Equations Using the Quadratic Formula

To use the quadratic formula, just plug in the values of a , b , and c .

Example 1

Solve the following quadratic equations using the quadratic formula.

a) $2x^2 + 3x + 1 = 0$

b) $x^2 - 6x + 5 = 0$

c) $-4x^2 + x + 1 = 0$

Solution

Start with the quadratic formula and plug in the values of a , b and c .

a)

Quadratic formula:

Plug in the values $a = 2$, $b = 3$, $c = 1$

Simplify:

Separate the two options:

Solve:

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\x &= \frac{-3 \pm \sqrt{(3)^2 - 4(2)(1)}}{2(2)} \\x &= \frac{-3 \pm \sqrt{9 - 8}}{4} = \frac{-3 \pm \sqrt{1}}{4} \\x &= \frac{-3 + 1}{4} \quad \text{and} \quad x = \frac{-3 - 1}{4} \\x &= \frac{-2}{4} = -\frac{1}{2} \quad \text{and} \quad x = \frac{-4}{4} = -1\end{aligned}$$

Answer: $x = -\frac{1}{2}$ and $x = -1$

b)

Quadratic formula:

Plug in the values $a = 1$, $b = -6$, $c = 5$

Simplify:

Separate the two options:

Solve:

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)} \\x &= \frac{6 \pm \sqrt{36 - 20}}{2} = \frac{6 \pm \sqrt{16}}{2} \\x &= \frac{6 + 4}{2} \quad \text{and} \quad x = \frac{6 - 4}{2} \\x &= \frac{10}{2} = 5 \quad \text{and} \quad x = \frac{2}{2} = 1\end{aligned}$$

Answer: $x = 5$ and $x = 1$

c)

Quadratic formula:

Plug in the values $a = -4$, $b = 1$, $c = 1$

Simplify:

Separate the two options:

Solve:

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\x &= \frac{-1 \pm \sqrt{(1)^2 - 4(-4)(1)}}{2(-4)} \\x &= \frac{-1 \pm \sqrt{1 + 16}}{-8} = \frac{-1 \pm \sqrt{17}}{-8} \\x &= \frac{-1 + \sqrt{17}}{-8} \quad \text{and} \quad x = \frac{-1 - \sqrt{17}}{-8} \\x &= -.39 \quad \text{and} \quad x = .64\end{aligned}$$

Answer: $x = -.39$ and $x = .64$

Often when we plug the values of the coefficients into the quadratic formula, we end up with a negative number inside the square root. Since the square root of a negative number does not give real answers, we say that the equation has no real solutions. In more advanced math classes, you'll learn how to work with "complex" (or "imaginary") solutions to quadratic equations.

Example 2

Use the quadratic formula to solve the equation $x^2 + 2x + 7 = 0$.

Solution

$$\text{Quadratic formula: } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Plug in the values } a = 1, b = 2, c = 7 \quad x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(7)}}{2(1)}$$

$$\text{Simplify: } x = \frac{-2 \pm \sqrt{4 - 28}}{2} = \frac{-2 \pm \sqrt{-24}}{2}$$

Answer: There are no real solutions.

To apply the quadratic formula, we must make sure that the equation is written in standard form. For some problems, that means we have to start by rewriting the equation.

Example 3

Solve the following equations using the quadratic formula.

a) $x^2 - 6x = 10$

b) $-8x^2 = 5x + 6$

Solution

a)

$$\text{Re-write the equation in standard form: } x^2 - 6x - 10 = 0$$

$$\text{Quadratic formula: } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Plug in the values } a = 1, b = -6, c = -10 \quad x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-10)}}{2(1)}$$

$$\text{Simplify: } x = \frac{6 \pm \sqrt{36 + 40}}{2} = \frac{6 \pm \sqrt{76}}{2}$$

$$\text{Separate the two options: } x = \frac{6 + \sqrt{76}}{2} \text{ and } x = \frac{6 - \sqrt{76}}{2}$$

$$\text{Solve: } x = 7.36 \text{ and } x = -1.36$$

Answer: $x = 7.36$ and $x = -1.36$

b)

$$\text{Re-write the equation in standard form: } 8x^2 + 5x + 6 = 0$$

$$\text{Quadratic formula: } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Plug in the values } a = 8, b = 5, c = 6 \quad x = \frac{-5 \pm \sqrt{(5)^2 - 4(8)(6)}}{2(8)}$$

$$\text{Simplify: } x = \frac{-5 \pm \sqrt{25 - 192}}{16} = \frac{-5 \pm \sqrt{-167}}{16}$$

Answer: no real solutions

For more examples of solving quadratic equations using the quadratic formula, see the Khan Academy video at <http://www.youtube.com/watch?v=y19jYxzY8Y8>. This video isn't necessarily different from the examples above, but it does help reinforce the procedure of using the quadratic formula to solve equations.



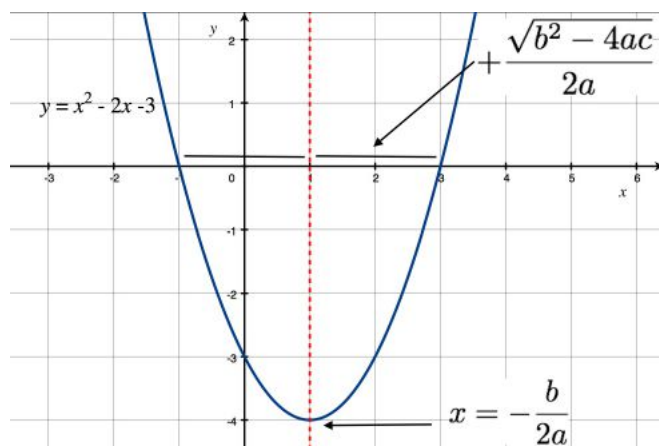
Figure 7.1: Quadratic Equation part 2 ([Watch Youtube Video](http://www.ck12.org/flexbook/embed/view/94))
<http://www.ck12.org/flexbook/embed/view/94>

Finding the Vertex of a Parabola with the Quadratic Formula

Sometimes a formula gives you even more information than you were looking for. For example, the quadratic formula also gives us an easy way to locate the vertex of a parabola.

Remember that the quadratic formula tells us the **roots** or **solutions** of the equation $ax^2 + bx + c = 0$. Those roots are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, and we can rewrite that as $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

Recall that the roots are **symmetric** about the vertex. In the form above, we can see that the roots of a quadratic equation are symmetric around the x -coordinate $\frac{-b}{2a}$, because they are $\frac{\sqrt{b^2 - 4ac}}{2a}$ units to the left and right (recall the \pm sign) from the vertical line $x = \frac{-b}{2a}$. For example, in the equation $x^2 - 2x - 3 = 0$, the roots -1 and 3 are both 2 units from the vertical line $x = 1$, as you can see in the graph below:



Identify and Choose Methods for Solving Quadratic Equations.

In mathematics, you'll need to solve quadratic equations that describe application problems or that are part of more complicated problems. You've learned four ways of solving a quadratic equation:

- Factoring
- Taking the square root
- Completing the square
- Quadratic formula

Usually you'll have to decide for yourself which method to use. However, here are some guidelines as to which methods are better in different situations.

Factoring is always best if the quadratic expression is easily factorable. It is always worthwhile to check if you can factor because this is the fastest method. Many expressions are not factorable so this method is not used very often in practice.

Taking the square root is best used when there is no x -term in the equation.

Completing the square can be used to solve any quadratic equation. This is usually not any better than using the quadratic formula (in terms of difficult computations), but it is very useful if you need to rewrite a quadratic function in vertex form. It's also used to rewrite the equations of circles, ellipses and hyperbolas in standard form (something you'll do in algebra II, trigonometry, physics, calculus, and beyond).

Quadratic formula is the method that is used most often for solving a quadratic equation. When solving directly by taking square root and factoring does not work, this is the method that most people prefer to use.

If you are using factoring or the quadratic formula, make sure that the equation is in standard form.

Example 4

Solve each quadratic equation.

a) $x^2 - 4x - 5 = 0$

b) $x^2 = 8$

c) $-4x^2 + x = 2$

d) $25x^2 - 9 = 0$

e) $3x^2 = 8x$

Solution

a) This expression is easily factorable so we can factor and apply the zero-product property:

| | |
|------------------------------|-----------------------------|
| Factor: | $(x - 5)(x + 1) = 0$ |
| Apply zero-product property: | $x - 5 = 0$ and $x + 1 = 0$ |
| Solve: | $x = 5$ and $x = -1$ |

Answer: $x = 5$ and $x = -1$

b) Since the expression is missing the x term we can take the square root:

Take the square root of both sides: $x = \sqrt{8}$ and $x = -\sqrt{8}$

Answer: $x = 2.83$ and $x = -2.83$

c) Re-write the equation in standard form: $-4x^2 + x - 2 = 0$

It is not apparent right away if the expression is factorable so we will use the quadratic formula:

| | |
|--|--|
| Quadratic formula: | $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| Plug in the values $a = -4$, $b = 1$, $c = -2$: | $x = \frac{-1 \pm \sqrt{1^2 - 4(-4)(-2)}}{2(-4)}$ |
| Simplify: | $x = \frac{-1 \pm \sqrt{1 - 32}}{-8} = \frac{-1 \pm \sqrt{-31}}{-8}$ |

Answer: no real solution

d) This problem can be solved easily either with factoring or taking the square root. Let's take the square root in this case:

Add 9 to both sides of the equation:

$$25x^2 = 9$$

Divide both sides by 25 :

$$x^2 = \frac{9}{25}$$

Take the square root of both sides:

$$x = \sqrt{\frac{9}{25}} \text{ and } x = -\sqrt{\frac{9}{25}}$$

Simplify:

$$x = \frac{3}{5} \text{ and } x = -\frac{3}{5}$$

Answer: $x = \frac{3}{5}$ and $x = -\frac{3}{5}$

e) Re-write the equation in standard form:

$$3x^2 - 8x = 0$$

Factor out common x term:

$$x(3x - 8) = 0$$

Set both terms to zero:

$$x = 0 \text{ and } 3x = 8$$

Solve:

$$x = 0 \text{ and } x = \frac{8}{3} = 2.67$$

Answer: $x = 0$ and $x = 2.67$

Solve Real-World Problems Using Quadratic Functions by any Method

Here are some application problems that arise from number relationships and geometry applications.

Example 5

The product of two positive consecutive integers is 156. Find the integers.

Solution

Define: Let x = the smaller integer

Then $x + 1$ = the next integer

Translate: The product of the two numbers is 156. We can write the equation:

$$x(x + 1) = 156$$

Solve:

$$x^2 + x = 156$$

$$x^2 + x - 156 = 0$$

Apply the quadratic formula with: $a = 1$, $b = 1$, $c = -156$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-156)}}{2(1)}$$

$$x = \frac{-1 \pm \sqrt{625}}{2} = \frac{-1 \pm 25}{2}$$

$$x = \frac{-1 + 25}{2} \text{ and } x = \frac{-1 - 25}{2}$$

$$x = \frac{24}{2} = 12 \text{ and } x = \frac{-26}{2} = -13$$

Since we are looking for positive integers, we want $x = 12$. So the numbers are **12 and 13**.

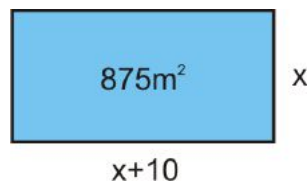
Check: $12 \times 13 = 156$. The answer checks out.

Example 6

The length of a rectangular pool is 10 meters more than its width. The area of the pool is 875 square meters. Find the dimensions of the pool.

Solution

Draw a sketch:



Define: Let x = the width of the pool

Then $x + 10$ = the length of the pool

Translate: The area of a rectangle is $A = \text{length} \times \text{width}$, so we have $x(x + 10) = 875$.

Solve:

$$\begin{aligned}x^2 + 10x &= 875 \\x^2 + 10x - 875 &= 0\end{aligned}$$

Apply the quadratic formula with $a = 1$, $b = 10$ and $c = -875$

$$\begin{aligned}x &= \frac{-10 \pm \sqrt{(10)^2 - 4(1)(-875)}}{2(1)} \\x &= \frac{-10 \pm \sqrt{100 + 3500}}{2} \\x &= \frac{-10 \pm \sqrt{3600}}{2} = \frac{-10 \pm 60}{2} \\x &= \frac{-10 + 60}{2} \text{ and } x = \frac{-10 - 60}{2} \\x &= \frac{50}{2} = 25 \text{ and } x = \frac{-70}{2} = -35\end{aligned}$$

Since the dimensions of the pool should be positive, we want $x = 25$ meters. So the pool is 25 meters \times 35 meters.

Check: $25 \times 35 = 875 \text{ m}^2$. The answer checks out.

Example 7

Suzie wants to build a garden that has three separate rectangular sections. She wants to fence around the whole garden and between each section as shown. The plot is twice as long as it is wide and the total area is 200ft^2 . How much fencing does Suzie need?

Solution

Define: Let x = the width of the plot

Then $2x$ = the length of the plot

Translate: area of a rectangle is $A = \text{length} \times \text{width}$, so

$$x(2x) = 200$$

Solve: $2x^2 = 200$

Solve by taking the square root:

$$x^2 = 100$$

$$x = \sqrt{100} \text{ and } x = -\sqrt{100}$$

$$x = 10 \text{ and } x = -10$$

We take $x = 10$ since only positive dimensions make sense.

The plot of land is 10 *feet* \times 20 *feet*.

To fence the garden the way Suzie wants, we need 2 lengths and 4 widths $= 2(20) + 4(10) = 80$ **feet of fence**.

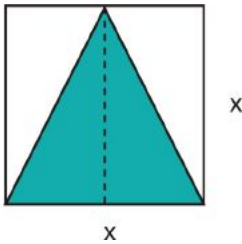
Check: $10 \times 20 = 200 \text{ ft}^2$ and $2(20) + 4(10) = 80 \text{ feet}$. The answer checks out.

Example 8

An isosceles triangle is enclosed in a square so that its base coincides with one of the sides of the square and the tip of the triangle touches the opposite side of the square. If the area of the triangle is 20 in^2 what is the length of one side of the square?

Solution

Draw a sketch:



Define: Let $x = \text{base of the triangle}$

Then $x = \text{height of the triangle}$

Translate: Area of a triangle is $\frac{1}{2} \times \text{base} \times \text{height}$, so $\frac{1}{2} \cdot x \cdot x = 20$

Solve: $\frac{1}{2}x^2 = 20$

Solve by taking the square root:

$$x^2 = 40$$

$$x = \sqrt{40} \text{ and } x = -\sqrt{40}$$

$$x = 6.32 \text{ and } x = -6.32$$

The side of the square is **6.32 inches**. That means the area of the square is $(6.32)^2 = 40 \text{ in}^2$, twice as big as the area of the triangle.

Check: It makes sense that the area of the square will be twice that of the triangle. If you look at the figure you can see that you could fit two triangles inside the square.

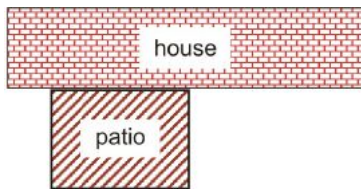
Review Questions

Solve the following quadratic equations using the quadratic formula.

1. $x^2 + 4x - 21 = 0$
2. $x^2 - 6x = 12$
3. $3x^2 - \frac{1}{2}x = \frac{3}{8}$
4. $2x^2 + x - 3 = 0$
5. $-x^2 - 7x + 12 = 0$
6. $-3x^2 + 5x = 2$
7. $4x^2 = x$
8. $x^2 + 2x + 6 = 0$

Solve the following quadratic equations using the method of your choice.

9. $x^2 - x = 6$
10. $x^2 - 12 = 0$
11. $-2x^2 + 5x - 3 = 0$
12. $x^2 + 7x - 18 = 0$
13. $3x^2 + 6x = -10$
14. $-4x^2 + 4000x = 0$
15. $-3x^2 + 12x + 1 = 0$
16. $x^2 + 6x + 9 = 0$
17. $81x^2 + 1 = 0$
18. $-4x^2 + 4x = 9$
19. $36x^2 - 21 = 0$
20. $x^2 - 2x - 3 = 0$
21. The product of two consecutive integers is 72. Find the two numbers.
22. The product of two consecutive odd integers is 1 less than 3 times their sum. Find the integers.
23. The length of a rectangle exceeds its width by 3 inches. The area of the rectangle is 70 square inches, find its dimensions.
24. Angel wants to cut off a square piece from the corner of a rectangular piece of plywood. The larger piece of wood is 4 *feet* \times 8 *feet* and the cut off part is $\frac{1}{3}$ of the total area of the plywood sheet. What is the length of the side of the square?
25. Mike wants to fence three sides of a rectangular patio that is adjacent the back of his house. The area of the patio is 192 ft^2 and the length is 4 feet longer than the width.



Find how much fencing Mike will need.

7.6 The Discriminant

Learning Objectives

- Find the discriminant of a quadratic equation.

- Interpret the discriminant of a quadratic equation.
- Solve real-world problems using quadratic functions and interpreting the discriminant.

Introduction

In the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the expression inside the square root is called the **discriminant**. The discriminant can be used to analyze the types of solutions to a quadratic equation without actually solving the equation. Here's how:

- If $b^2 - 4ac > 0$, the equation has two separate real solutions.
- If $b^2 - 4ac < 0$, the equation has only non-real solutions.
- If $b^2 - 4ac = 0$, the equation has one real solution, a **double root**.

Find the Discriminant of a Quadratic Equation

To find the discriminant of a quadratic equation we calculate $D = b^2 - 4ac$.

Example 1

Find the discriminant of each quadratic equation. Then tell how many solutions there will be to the quadratic equation without solving.

a) $x^2 - 5x + 3 = 0$

b) $4x^2 - 4x + 1 = 0$

c) $-2x^2 + x = 4$

Solution

a) Plug $a = 1$, $b = -5$ and $c = 3$ into the discriminant formula: $D = (-5)^2 - 4(1)(3) = 13$ $D > 0$, so there are **two real solutions**.

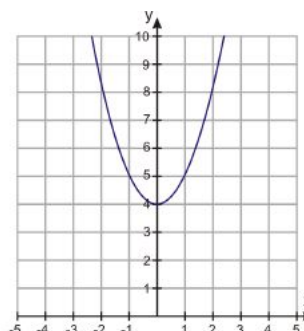
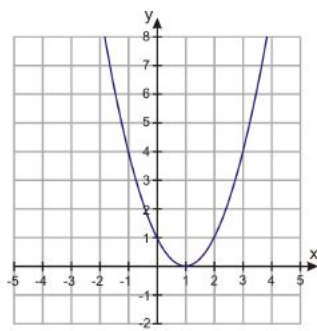
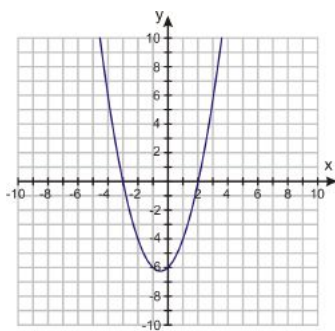
b) Plug $a = 4$, $b = -4$ and $c = 1$ into the discriminant formula: $D = (-4)^2 - 4(4)(1) = 0$ $D = 0$, so there is **one real solution**.

c) Rewrite the equation in standard form: $-2x^2 + x - 4 = 0$

Plug $a = -2$, $b = 1$ and $c = -4$ into the discriminant formula: $D = (1)^2 - 4(-2)(-4) = -31$ $D < 0$, so there are **no real solutions**.

Interpret the Discriminant of a Quadratic Equation

The sign of the discriminant tells us the nature of the solutions (or roots) of a quadratic equation. We can obtain two distinct real solutions if $D > 0$, two non-real solutions if $D < 0$ or one solution (called a double root) if $D = 0$. Recall that the number of solutions of a quadratic equation tells us how many times its graph crosses the x -axis. If $D > 0$, the graph crosses the x -axis in two places; if $D = 0$ it crosses in one place; if $D < 0$ it doesn't cross at all:



Example 2

Determine the nature of the solutions of each quadratic equation.

- a) $4x^2 - 1 = 0$
- b) $10x^2 - 3x = -4$
- c) $x^2 - 10x + 25 = 0$

Solution

Use the value of the discriminant to determine the nature of the solutions to the quadratic equation.

- a) Plug $a = 4$, $b = 0$ and $c = -1$ into the discriminant formula: $D = (0)^2 - 4(4)(-1) = 16$

The discriminant is positive, so the equation has **two distinct real solutions**.

The solutions to the equation are: $\frac{0 \pm \sqrt{16}}{8} = \pm \frac{4}{8} = \pm \frac{1}{2}$

- b) Re-write the equation in standard form: $10x^2 - 3x + 4 = 0$

Plug $a = 10$, $b = -3$ and $c = 4$ into the discriminant formula: $D = (-3)^2 - 4(10)(4) = -151$

The discriminant is negative, so the equation has **two non-real solutions**.

- c) Plug $a = 1$, $b = -10$ and $c = 25$ into the discriminant formula: $D = (-10)^2 - 4(1)(25) = 0$

The discriminant is 0, so the equation has a **double root**.

The solution to the equation is: $\frac{10 \pm \sqrt{0}}{2} = \frac{10}{2} = 5$

If the discriminant is a perfect square, then the solutions to the equation are not only real, but also rational. If the discriminant is positive but not a perfect square, then the solutions to the equation are real but irrational.

Example 3

Determine the nature of the solutions to each quadratic equation.

- a) $2x^2 + x - 3 = 0$
- b) $5x^2 - x - 1 = 0$

Solution

Use the discriminant to determine the nature of the solutions.

- a) Plug $a = 2$, $b = 1$ and $c = -3$ into the discriminant formula: $D = (1)^2 - 4(2)(-3) = 25$

The discriminant is a positive perfect square, so the solutions are **two real rational numbers**.

The solutions to the equation are: $\frac{-1 \pm \sqrt{25}}{4} = \frac{-1 \pm 5}{4}$, so $x = 1$ and $x = -\frac{3}{2}$.

- b) Plug $a = 5$, $b = -1$ and $c = -1$ into the discriminant formula: $D = (-1)^2 - 4(5)(-1) = 21$

The discriminant is positive but not a perfect square, so the solutions are **two real irrational numbers**.

The solutions to the equation are: $\frac{1 \pm \sqrt{21}}{10}$, so $x \approx 0.56$ and $x \approx -0.36$.

Solve Real-World Problems Using Quadratic Functions and Interpreting the Discriminant

You've seen that calculating the discriminant shows what types of solutions a quadratic equation possesses. Knowing the types of solutions is very useful in applied problems. Consider the following situation.

Example 4

Marcus kicks a football in order to score a field goal. The height of the ball is given by the equation $y = -\frac{32}{6400}x^2 + x$. If the goalpost is 10 feet high, can Marcus kick the ball high enough to go over the goalpost? What is the farthest distance that Marcus can kick the ball from and still make it over the goalpost?

Solution

Define: Let y = height of the ball in feet.

Let x = distance from the ball to the goalpost.

Translate: We want to know if it is possible for the height of the ball to equal 10 feet at some real distance from the goalpost.

Solve:

| | |
|--------------------------------------|------------------------------------|
| Write the equation in standard form: | $-\frac{32}{6400}x^2 + x - 10 = 0$ |
| Simplify: | $-0.005x^2 + x - 10 = 0$ |
| Find the discriminant: | $D = (1)^2 - 4(-0.005)(-10) = 0.8$ |

Since the discriminant is positive, we know that it is possible for the ball to go over the goal post, if Marcus kicks it from an acceptable distance x from the goalpost.

To find the value of x that will work, we need to use the quadratic formula:

$$x = \frac{-1 \pm \sqrt{0.8}}{-0.01} = 189.4 \text{ feet or } 10.56 \text{ feet}$$

What does this answer mean? It means that if Marcus is exactly 189.4 feet or exactly 10.56 feet from the goalposts, the ball will just barely go over them. Are these the only distances that will work? No; those are just the distances at which the ball will be exactly 10 feet high, but *between* those two distances, the ball will go even higher than that. (It travels in a downward-opening parabola from the place where it is kicked to the spot where it hits the ground.) This means that Marcus will make the goal if he is anywhere **between 10.56 and 189.4 feet from the goalposts**.

Example 5

Emma and Bradon own a factory that produces bike helmets. Their accountant says that their profit per year is given by the function $P = -0.003x^2 + 12x + 27760$, where x is the number of helmets produced. Their goal is to make a profit of \$40,000 this year. Is this possible?

Solution

We want to know if it is possible for the profit to equal \$40,000.

$$40000 = -0.003x^2 + 12x + 27760$$

Write the equation in standard form: $-0.003x^2 + 12x - 12240 = 0$

Find the discriminant: $D = (12)^2 - 4(-0.003)(-12240) = -2.88$

Since the discriminant is negative, we know that **it is not possible** for Emma and Bradon to make a profit of \$40,000 this year no matter how many helmets they make.

Review Questions

Find the discriminant of each quadratic equation.

1. $2x^2 - 4x + 5 = 0$
2. $x^2 - 5x = 8$
3. $4x^2 - 12x + 9 = 0$
4. $x^2 + 3x + 2 = 0$
5. $x^2 - 16x = 32$
6. $-5x^2 + 5x - 6 = 0$
7. $x^2 + 4x = 2$
8. $-3x^2 + 2x + 5 = 0$

Determine the nature of the solutions of each quadratic equation.

9. $-x^2 + 3x - 6 = 0$
10. $5x^2 = 6x$
11. $41x^2 - 31x - 52 = 0$
12. $x^2 - 8x + 16 = 0$
13. $-x^2 + 3x - 10 = 0$
14. $x^2 - 64 = 0$
15. $3x^2 = 7$
16. $x^2 + 30 + 225 = 0$

Without solving the equation, determine whether the solutions will be rational or irrational.

17. $x^2 = -4x + 20$
18. $x^2 + 2x - 3 = 0$
19. $3x^2 - 11x = 10$
20. $\frac{1}{2}x^2 + 2x + \frac{2}{3} = 0$
21. $x^2 - 10x + 25 = 0$
22. $x^2 = 5x$
23. $2x^2 - 5x = 12$
24. Marty is outside his apartment building. He needs to give his roommate Yolanda her cell phone but he does not have time to run upstairs to the third floor to give it to her. He throws it straight up with a vertical velocity of 55 feet/second. Will the phone reach her if she is 36 feet up? (Hint: the equation for the height is $y = -32t^2 + 55t + 4$.)
25. Bryson owns a business that manufactures and sells tires. The revenue from selling the tires in the month of July is given by the function $R = x(200 - 0.4x)$ where x is the number of tires sold. Can Bryson's business generate revenue of \$20,000 in the month of July?

7.7 Linear, Exponential and Quadratic Models

Learning Objectives

- Identify functions using differences and ratios.
- Write equations for functions.
- Perform exponential and quadratic regressions with a graphing calculator.
- Solve real-world problems by comparing function models.

Introduction

In this course we've learned about three types of functions, linear, quadratic and exponential.

- Linear functions take the form $y = mx + b$.
- Quadratic functions take the form $y = ax^2 + bx + c$.
- Exponential functions take the form $y = a \cdot b^x$.

In real-world applications, the function that describes some physical situation is not given; it has to be found before the problem can be solved. For example, scientific data such as observations of planetary motion are often collected as a set of measurements given in a table. Part of the scientist's job is to figure out which function best fits the data. In this section, you'll learn some methods that are used to identify which function describes the relationship between the variables in a problem.

Identify Functions Using Differences or Ratios

One method for identifying functions is to look at the difference or the ratio of different values of the dependent variable. For example, **if the difference between values of the dependent variable is the same each time we change the independent variable by the same amount, then the function is *linear*.**

Example 1

Determine if the function represented by the following table of values is linear.

Table 7.14:

| x | y |
|-----|-----|
| -2 | -4 |
| -1 | -1 |
| 0 | 2 |
| 1 | 5 |
| 2 | 8 |

If we take the difference between consecutive y -values, we see that each time the x -value increases by one, the y -value always increases by 3.

Since the difference is always the same, **the function is linear.**

When we look at the difference of the y -values, we have to make sure that we examine entries for which

the x -values increase by the same amount.
 For example, examine the values in this table:

Table 7.15:

| x | y |
|-----|-----|
| 0 | 5 |
| 1 | 10 |
| 3 | 20 |
| 4 | 25 |
| 6 | 35 |

At first glance this function might not look linear, because the difference in the y -values is not always the same. But if we look closer, we can see that when the y -value increases by 10 instead of 5, it's because the x -value increased by 2 instead of 1. Whenever the x -value increases by the *same* amount, the y -value does too, so the function is linear.

Another way to think of this is in mathematical notation. We can say that a function is linear if $\frac{y_2-y_1}{x_2-x_1}$ is always the same for any two pairs of x - and y -values. Notice that the expression we used here is simply the definition of the slope of a line.

Differences can also be used to identify quadratic functions. **For a quadratic function, when we increase the x -values by the same amount, the difference between y -values will not be the same. However, the difference of the *differences* of the y -values will be the same.**

Here are some examples of quadratic relationships represented by tables of values:

| x | $y = x^2$ | difference of y -values | difference of differences |
|-----|-----------|------------------------------|---|
| 0 | 0 | $1 - 0 = 1$ | $\left. \begin{array}{l} 3 - 1 = 2 \\ 5 - 3 = 2 \\ 7 - 5 = 2 \\ 9 - 7 = 2 \\ 11 - 9 = 2 \end{array} \right\}$ |
| 1 | 1 | $4 - 1 = 3$ | |
| 2 | 4 | $9 - 4 = 5$ | |
| 3 | 9 | $16 - 9 = 7$ | |
| 4 | 16 | $25 - 16 = 9$ | |
| 5 | 25 | $36 - 25 = 11$ | |
| 6 | 36 | | |

In this quadratic function, $y = x^2$, when we increase the x -value by one, the value of y increases by different values. However, it increases at a constant rate, so the difference of the difference is always 2.

| x | $y = 2x^2 - 3x + 1$ | difference of y -values | | difference of differences | |
|-----|---------------------|------------------------------|----------------|------------------------------|---------------|
| 0 | 0 | { | $0 - 1 = -1$ | { | $3 + 1 = 4$ |
| 1 | 1 | | $3 - 0 = 3$ | | |
| 2 | 3 | { | $10 - 3 = 7$ | { | $7 - 3 = 4$ |
| 3 | 10 | | $21 - 10 = 11$ | | |
| 4 | 21 | { | $36 - 21 = 15$ | { | $15 - 11 = 4$ |
| 5 | 36 | | $55 - 36 = 19$ | | |
| 6 | 55 | | | | |

In this quadratic function, $y = 2x^2 - 3x + 1$, when we increase the x -value by one, the value of y increases by different values. However, the increase is constant: the difference of the difference is always 4.

To identify exponential functions, we use ratios instead of differences. **If the ratio between values of the dependent variable is the same each time we change the independent variable by the same amount, then the function is *exponential*.**

Example 2

Determine if the function represented by each table of values is exponential.

a)

| x | y | ratio of y -values | |
|-----|-----|-------------------------|-----------------------|
| 0 | 4 | { | $\frac{12}{4} = 3$ |
| 1 | 12 | | $\frac{36}{12} = 3$ |
| 2 | 36 | { | $\frac{108}{36} = 3$ |
| 3 | 108 | | $\frac{324}{108} = 3$ |
| 4 | 324 | | |

b)

| x | y | ratio of y -values | |
|-----|-----|-------------------------|---------------------------------|
| 0 | 240 | { | $\frac{120}{240} = \frac{1}{2}$ |
| 1 | 120 | | $\frac{60}{120} = \frac{1}{2}$ |
| 2 | 60 | { | $\frac{30}{60} = \frac{1}{2}$ |
| 3 | 30 | | $\frac{15}{30} = \frac{1}{2}$ |
| 4 | 15 | | |

- a) If we take the ratio of consecutive y -values, we see that each time the x -value increases by one, the y -value is multiplied by 3. Since the ratio is always the same, **the function is exponential.**
- b) If we take the ratio of consecutive y -values, we see that each time the x -value increases by one, the y -value is multiplied by $\frac{1}{2}$. Since the ratio is always the same, **the function is exponential.**

Write Equations for Functions

Once we identify which type of function fits the given values, we can write an equation for the function by starting with the general form for that type of function.

Example 3

Determine what type of function represents the values in the following table.

Table 7.16:

| x | y |
|-----|-----|
| 0 | 5 |
| 1 | 1 |
| 2 | -3 |
| 3 | -7 |
| 4 | -11 |

Solution

Let's first check the difference of consecutive values of y .

| x | y | difference of y -values | |
|-----|-----|------------------------------|----------------|
| 0 | 5 | } | $1 - 5 = -4$ |
| 1 | 1 | | $-3 - 1 = -4$ |
| 2 | -3 | } | $-7 + 3 = -4$ |
| 3 | -7 | | $-11 + 7 = -4$ |
| 4 | -11 | | |

If we take the difference between consecutive y -values, we see that each time the x -value increases by one, the y -value always decreases by 4. Since the difference is always the same, **the function is linear.**

To find the equation for the function, we start with the general form of a linear function: $y = mx + b$. Since m is the slope of the line, it's also the quantity by which y increases every time the value of x increases by one. The constant b is the value of the function when $x = 0$. Therefore, the function is $y = -4x + 5$.

Example 4

Determine what type of function represents the values in the following table.

Table 7.17:

| x | y |
|-----|-----|
| 0 | 0 |
| 1 | 5 |
| 2 | 20 |
| 3 | 45 |
| 4 | 80 |
| 5 | 125 |
| 6 | 180 |

Solution

Here, the difference between consecutive y -values isn't constant, so the function is not linear. Let's look at those differences more closely.

Table 7.18:

| x | y | |
|-----|-----|------------------|
| 0 | 0 | |
| 1 | 5 | $5 - 0 = 5$ |
| 2 | 20 | $20 - 5 = 15$ |
| 3 | 45 | $45 - 20 = 25$ |
| 4 | 80 | $80 - 45 = 35$ |
| 5 | 125 | $125 - 80 = 45$ |
| 6 | 180 | $180 - 125 = 55$ |

When the x -value increases by one, the difference between the y -values increases by 10 every time. Since the difference of the differences is constant, the function describing this set of values is **quadratic**.

To find the equation for the function that represents these values, we start with the general form of a quadratic function: $y = ax^2 + bx + c$.

We need to use the values in the table to find the values of the constants: a, b and c .

The value of c represents the value of the function when $x = 0$, so $c = 0$.

| | |
|---|--|
| Plug in the point (1, 5) : | $5 = a + b$ |
| Plug in the point (2, 20) : | $20 = 4a + 2b \Rightarrow 10 = 2a + b$ |
| To find a and b , we solve the system of equations: | $5 = a + b$ $10 = 2a + b$ |
| Solve the first equation for b : | $5 = a + b \Rightarrow b = 5 - a$ |
| Plug the first equation into the second: | $10 = 2a + 5 - a$ |
| Solve for a and b | $a = 5$ and $b = 0$ |

Therefore the equation of the quadratic function is $y = 5x^2$.

Example 5

Determine what type of function represents the values in the following table.

Table 7.19:

| x | y |
|-----|--------|
| 0 | 400 |
| 1 | 500 |
| 2 | 25 |
| 3 | 6.25 |
| 4 | 1.5625 |

Solution

The differences between consecutive y -values aren't the same, and the differences between those differences aren't the same either. So let's check the ratios instead.

| x | y | ratio of y - values |
|-----|--------|-------------------------------------|
| 0 | 400 | $\frac{100}{400} = \frac{1}{4}$ |
| 1 | 100 | |
| 2 | 25 | $\frac{25}{100} = \frac{1}{4}$ |
| 3 | 6.25 | |
| 4 | 1.5625 | $\frac{1.5625}{6.25} = \frac{1}{4}$ |
| | | |

Each time the x -value increases by one, the y -value is multiplied by $\frac{1}{4}$. Since the ratio is always the same, **the function is exponential.**

To find the equation for the function that represents these values, we start with the general form of an exponential function, $y = a \cdot b^x$.

Here b is the ratio between the values of y each time x is increased by one. The constant a is the value of the function when $x = 0$. Therefore, the function is $y = 400\left(\frac{1}{4}\right)^x$.

Perform Exponential and Quadratic Regressions with a Graphing Calculator

Earlier, you learned how to perform linear regression with a graphing calculator to find the equation of a straight line that fits a linear data set. In this section, you'll learn how to perform exponential and quadratic regression to find equations for curves that fit non-linear data sets.

Example 6

The following table shows how many miles per gallon a car gets at different speeds.

Table 7.20:

| Speed (mph) | Miles per gallon |
|-------------|------------------|
| 30 | 18 |

Table 7.20: (continued)

| Speed (mph) | Miles per gallon |
|-------------|------------------|
| 35 | 20 |
| 40 | 23 |
| 45 | 25 |
| 50 | 28 |
| 55 | 30 |
| 60 | 29 |
| 65 | 25 |
| 70 | 25 |

Using a graphing calculator:

- Draw the scatterplot of the data.
- Find the quadratic function of best fit.
- Draw the quadratic function of best fit on the scatterplot.
- Find the speed that maximizes the miles per gallon.
- Predict the miles per gallon of the car if you drive at a speed of 48 mph.

Solution

Step 1: Input the data.

Press **[STAT]** and choose the **[EDIT]** option.

Input the values of x in the first column (L_1) and the values of y in the second column (L_2). (**Note:** in order to clear a list, move the cursor to the top so that L_1 or L_2 is highlighted. Then press **[CLEAR]** and then **[ENTER]**.)

Step 2: Draw the scatterplot.

First press **[Y=]** and clear any function on the screen by pressing **[CLEAR]** when the old function is highlighted.

Press **[STATPLOT]** **[STAT]** and **[Y=]** and choose option 1.

Choose the ON option; after TYPE, choose the first graph type (scatterplot) and make sure that the Xlist and Ylist names match the names on top of the columns in the input table.

Press **[GRAPH]** and make sure that the window is set so you see all the points in the scatterplot. In this case, the settings should be $30 \leq x \leq 80$ and $0 \leq y \leq 40$. You can set the window size by pressing the **[WINDOW]** key at the top.

Step 3: Perform quadratic regression.

Press **[STAT]** and use the right arrow to choose **[CALC]**.

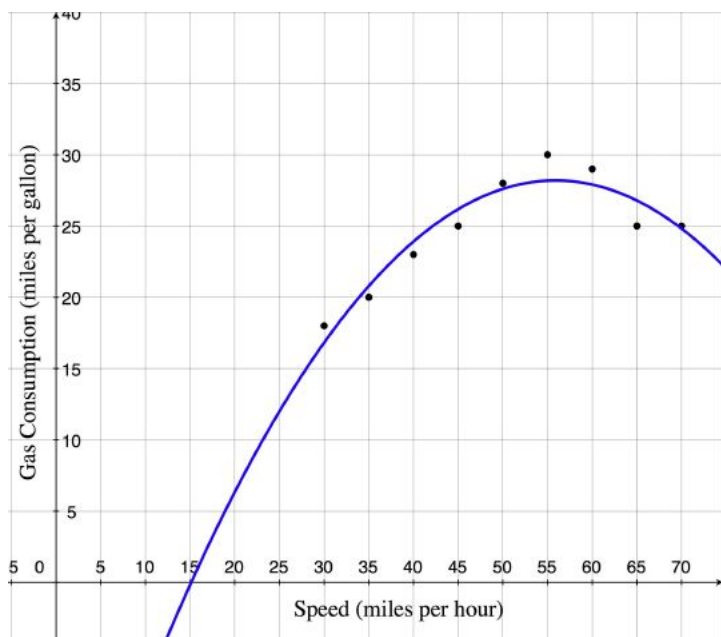
Choose Option 5 (QuadReg) and press **[ENTER]**. You will see “QuadReg” on the screen.

Type in L_1, L_2 after ‘QuadReg’ and press **[ENTER]**. The calculator shows the quadratic function: $y = -0.017x^2 + 1.9x - 25$

Step 4: Graph the function.

Press **[Y=]** and input the function you just found.

Press **[GRAPH]** and you will see the curve fit drawn over the data points.



To find the speed that maximizes the miles per gallon, use **[TRACE]** and move the cursor to the top of the parabola. You can also use **[CALC]** **[2nd]** **[TRACE]** and option 4:Maximum, for a more accurate answer. The speed that maximizes miles per gallon is **56 mph**.

Finally, plug $x = 48$ into the equation you found: $y = -0.017(48)^2 + 1.9(48) - 25 = 27.032$ miles per gallon.

Note: The image above shows our function plotted on the same graph as the data points from the table. One thing that is clear from this graph is that predictions made with this function won't make sense for all values of x . For example, if $x < 15$, this graph predicts that we will get negative mileage, which is impossible. Part of the skill of using regression on your calculator is being aware of the strengths and limitations of this method of fitting functions to data.

Example 7

The following table shows the amount of money an investor has in an account each year for 10 years.

Table 7.21:

| Year | Value of account |
|------|------------------|
| 1996 | \$5000 |
| 1997 | \$5400 |
| 1998 | \$5800 |
| 1999 | \$6300 |
| 2000 | \$6800 |
| 2001 | \$7300 |
| 2002 | \$7900 |
| 2003 | \$8600 |
| 2004 | \$9300 |
| 2005 | \$10000 |
| 2006 | \$11000 |

Using a graphing calculator:

a) Draw a scatterplot of the value of the account as the dependent variable, and the number of years since

1996 as the independent variable.

- b) Find the exponential function that fits the data.
- c) Draw the exponential function on the scatterplot.
- d) What will be the value of the account in 2020?

Solution

Step 1: Input the data.

Press **[STAT]** and choose the **[EDIT]** option.

Input the values of x in the first column (L_1) and the values of y in the second column (L_2).

Step 2: Draw the scatterplot.

First press **[Y=]** and clear any function on the screen.

Press **[GRAPH]** and choose Option 1.

Choose the ON option and make sure that the Xlist and Ylist names match the names on top of the columns in the input table.

Press **[GRAPH]** make sure that the window is set so you see all the points in the scatterplot. In this case the settings should be $0 \leq x \leq 10$ and $0 \leq y \leq 11000$.

Step 3: Perform exponential regression.

Press **[STAT]** and use the right arrow to choose **[CALC]**.

Choose Option 0 and press **[ENTER]**. You will see “ExpReg” on the screen.

Press **[ENTER]**. The calculator shows the exponential function: $y = 4975.7(1.08)^x$

Step 4: Graph the function.

Press **[Y=]** and input the function you just found. Press **[GRAPH]**.

Finally, plug $x = 2020 - 1996 = 24$ into the function: $y = 4975.7(1.08)^{24} = \underline{\$31551.81}$

In 2020, the account will have a value of **\$31551.81**.

Note: The function above is the curve that comes *closest* to all the data points. It won't return y -values that are exactly the same as in the data table, but they will be close. It is actually more accurate to use the curve fit values than the data points.

If you don't have a graphing calculator, there are resources available on the Internet for finding lines and curves of best fit. For example, the applet at <http://science.kennesaw.edu/~plaval/applets/LRegression.html> does linear regression on a set of data points; the one at <http://science.kennesaw.edu/~plaval/applets/QRegression.html> does quadratic regression; and the one at <http://science.kennesaw.edu/~plaval/applets/ERegression.html> does exponential regression. Also, programs like Microsoft Office or OpenOffice have the ability to create graphs and charts that include lines and curves of best fit.

Solve Real-World Problems by Comparing Function Models

Example 8

The following table shows the number of students enrolled in public elementary schools in the US (source: US Census Bureau). Make a scatterplot with the number of students as the dependent variable, and the number of years since 1990 as the independent variable. Find which curve fits this data the best and predict the school enrollment in the year 2007.

Table 7.22:

| Year | Number of students (millions) |
|------|-------------------------------|
| 1990 | 26.6 |
| 1991 | 26.6 |
| 1992 | 27.1 |
| 1993 | 27.7 |
| 1994 | 28.1 |
| 1995 | 28.4 |
| 1996 | 28.1 |
| 1997 | 29.1 |
| 1998 | 29.3 |
| 2003 | 32.5 |

Solution

We need to perform linear, quadratic and exponential regression on this data set to see which function represents the values in the table the best.

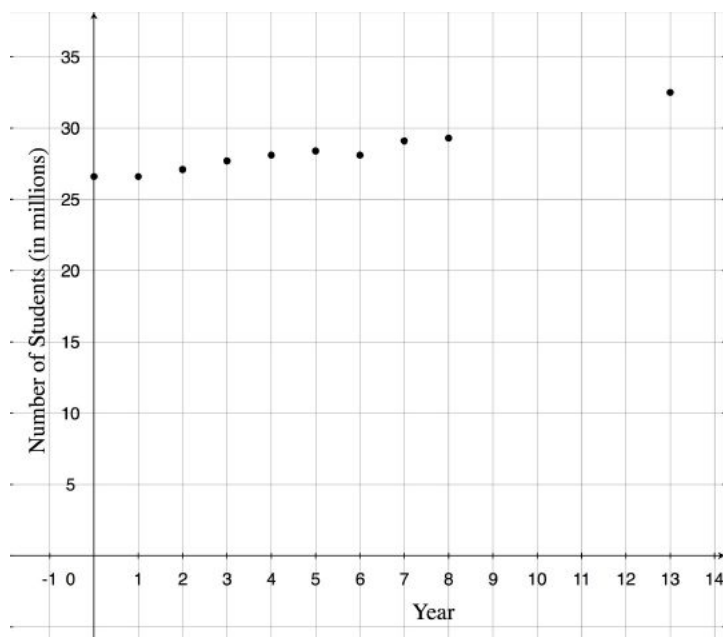
Step 1: Input the data.

Input the values of x in the first column (L_1) and the values of y in the second column (L_2).

Step 2: Draw the scatterplot.

Set the window size: $0 \leq x \leq 10$ and $20 \leq y \leq 40$.

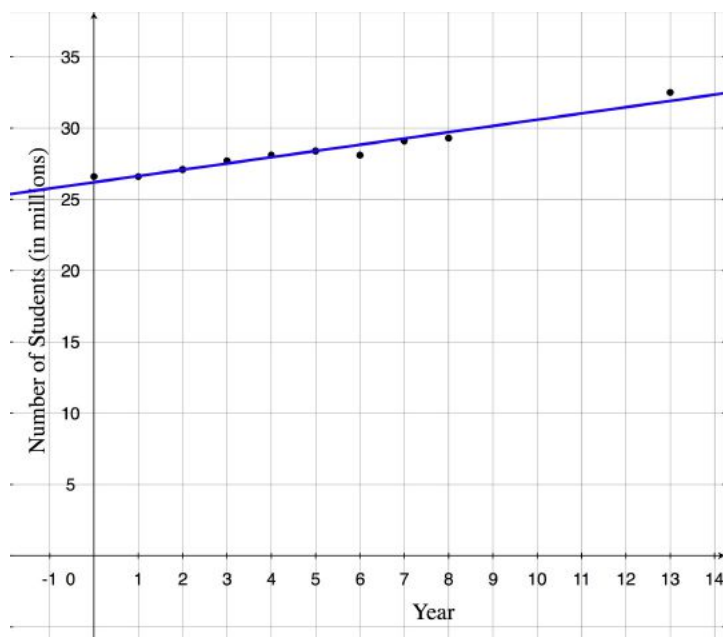
Here is the scatterplot:



Step 3: Perform Regression.

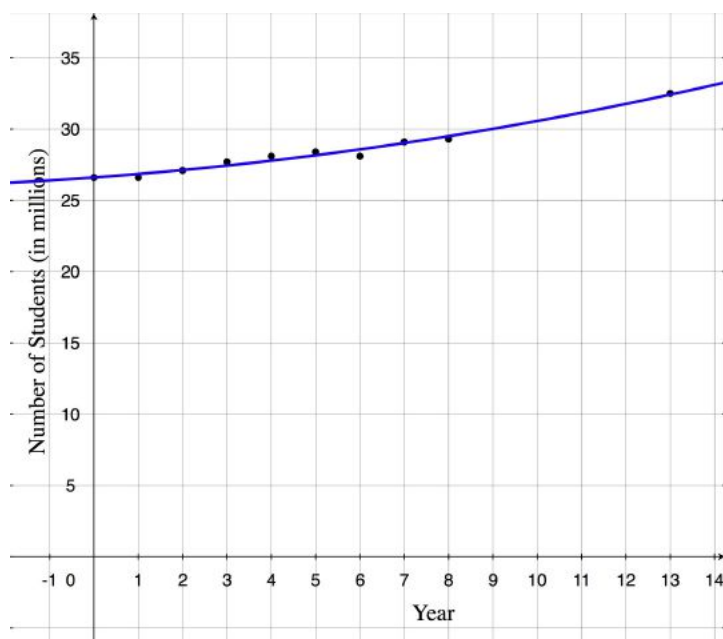
Linear Regression

The function of the line of best fit is $y = 0.51x + 26.1$. Here is the graph of the function on the scatterplot:



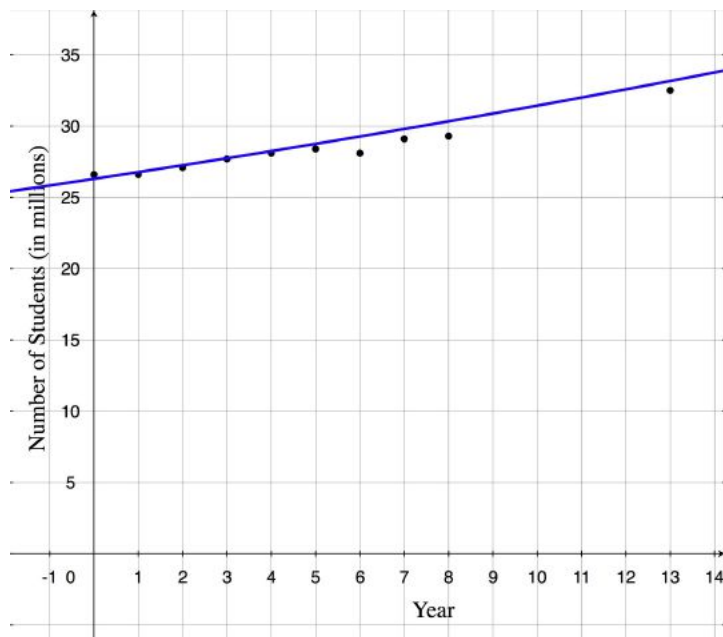
Quadratic Regression

The quadratic function of best fit is $y = 0.064x^2 - .067x + 26.84$. Here is the graph of the function on the scatterplot:



Exponential Regression

The exponential function of best fit is $y = 26.2(1.018)^x$. Here is the graph of the function on the scatterplot:



From the graphs, it looks like the quadratic function is the best fit for this data set. We'll use this function to predict school enrollment in 2007.

Plug in $x = 2007 - 1990 = 17$: $y = 0.064(17)^2 - .067(17) + 26.84 = 44.2$ *million students*.

Review Questions

Determine whether the data in the following tables can be represented by a linear function.

Table 7.23:

| x | y |
|-----|-----|
| -4 | 10 |
| -3 | 7 |
| -2 | 4 |
| -1 | 1 |
| 0 | -2 |
| 1 | -5 |

Table 7.24:

| x | y |
|-----|-----|
| -2 | 4 |
| -1 | 3 |
| 0 | 2 |
| 1 | 3 |
| 2 | 6 |
| 3 | 11 |

Table 7.25:

| x | y |
|-----|-----|
| 0 | 50 |
| 1 | 75 |
| 2 | 100 |
| 3 | 125 |
| 4 | 150 |
| 5 | 175 |

Determine whether the data in the following tables can be represented by a quadratic function.

Table 7.26:

| x | y |
|-----|------|
| -10 | 10 |
| -5 | 2.5 |
| 0 | 0 |
| 5 | 2.5 |
| 10 | 10 |
| 15 | 22.5 |

Table 7.27:

| x | y |
|-----|-----|
| 1 | 4 |
| 2 | 6 |
| 3 | 6 |
| 4 | 4 |
| 5 | 0 |
| 6 | -6 |

Table 7.28:

| x | y |
|-----|-----|
| -3 | -27 |
| -2 | -8 |
| -1 | -1 |
| 0 | 0 |
| 1 | 1 |
| 2 | 8 |
| 3 | 27 |

Determine whether the data in the following tables can be represented by an exponential function.

Table 7.29:

| x | y |
|-----|-------|
| 0 | 200 |
| 1 | 300 |
| 2 | 1800 |
| 3 | 8300 |
| 4 | 25800 |
| 5 | 62700 |

Table 7.30:

| x | y |
|-----|--------|
| 0 | 120 |
| 1 | 180 |
| 2 | 270 |
| 3 | 405 |
| 4 | 607.5 |
| 5 | 911.25 |

Table 7.31:

| x | y |
|-----|--------|
| 0 | 4000 |
| 1 | 2400 |
| 2 | 1440 |
| 3 | 864 |
| 4 | 518.4 |
| 5 | 311.04 |

Determine what type of function represents the values in the following tables and find the equation of each function.

Table 7.32:

| x | y |
|-----|----------|
| 0 | 400 |
| 1 | 500 |
| 2 | 625 |
| 3 | 781.25 |
| 4 | 976.5625 |

Table 7.33:

| x | y |
|-----|-----|
| -9 | -3 |
| -7 | -2 |
| -5 | -1 |
| -3 | 0 |
| -1 | 1 |
| 1 | 2 |

Table 7.34:

| x | y |
|-----|-----|
| -3 | 14 |
| -2 | 4 |
| -1 | -2 |
| 0 | -4 |
| 1 | -2 |
| 2 | 4 |
| 3 | 14 |

13. As a ball bounces up and down, the maximum height that the ball reaches continually decreases from one bounce to the next. For a given bounce, this table shows the height of the ball with respect to time:

Table 7.35:

| Time (seconds) | Height (inches) |
|----------------|-----------------|
| 2 | 2 |
| 2.2 | 16 |
| 2.4 | 24 |
| 2.6 | 33 |
| 2.8 | 38 |
| 3.0 | 42 |
| 3.2 | 36 |
| 3.4 | 30 |
| 3.6 | 28 |
| 3.8 | 14 |
| 4.0 | 6 |

Using a graphing calculator:

- Draw the scatterplot of the data
- Find the quadratic function of best fit
- Draw the quadratic function of best fit on the scatterplot
- Find the maximum height the ball reaches on the bounce

e) Predict how high the ball is at time $t = 2.5$ seconds

14. A chemist has a 250 gram sample of a radioactive material. She records the amount of radioactive material remaining in the sample every day for a week and obtains the data in the following table:

Table 7.36:

| Day | Weight (grams) |
|-----|----------------|
| 0 | 250 |
| 1 | 208 |
| 2 | 158 |
| 3 | 130 |
| 4 | 102 |
| 5 | 80 |
| 6 | 65 |
| 7 | 50 |

Using a graphing calculator:

- a) Draw a scatterplot of the data
- b) Find the exponential function of best fit
- c) Draw the exponential function of best fit on the scatterplot
- d) Predict the amount of material after 10 days.

15. The following table shows the rate of pregnancies (per 1000) for US women aged 15 to 19. (source: US Census Bureau).

- (a) Make a scatterplot with the rate of pregnancies as the dependent variable and the number of years since 1990 as the independent variable.
- (b) Find which type of curve fits this data best.
- (c) Predict the rate of teen pregnancies in the year 2010.

Table 7.37:

| Year | Rate of pregnancy (per 1000) |
|------|------------------------------|
| 1990 | 116.9 |
| 1991 | 115.3 |
| 1992 | 111.0 |
| 1993 | 108.0 |
| 1994 | 104.6 |
| 1995 | 99.6 |
| 1996 | 95.6 |
| 1997 | 91.4 |
| 1998 | 88.7 |
| 1999 | 85.7 |
| 2000 | 83.6 |
| 2001 | 79.5 |
| 2002 | 75.4 |

Chapter 8

Rational Equations and Functions; Statistics

The final chapter of this text introduces the concept of **rational functions**, that is, equations in which the variable appears in the denominator of a fraction. A common rational function in the **inverse variation model**, similar to the direct variation model you studied in chapter 4 lesson 6. We finish the chapter with solving rational equations and using graphical representations to display data.

8.1 Inverse Variation Models

In chapter 4, lesson 6, you learned how to write direct variation models. In direct variation, the variables changed in the same way and the graph contained the origin. But what happens when the variables change in different ways? Consider the following situation.

A group of friends rent a beach house and decide to split the cost of the rent and food. Four friends pay \$170 each. Five friends pay \$162 each. Six friends pay \$157. If nine people were to share the expense, how much would each pay?

Let's look at this in a table.

Table 8.1:

| n (number of friends) | t (share of expense) |
|-------------------------|------------------------|
| 4 | 170 |
| 5 | 162 |
| 6 | 157 |
| 9 | ??? |

As the number of friends gets larger, the cost per person gets smaller. This is an example of **inverse variation**.

An **inverse variation** function has the form $f(x) = \frac{k}{x}$, where k is called the constant of variation and must be a counting number and $x \neq 0$.

To show an inverse variation relationship, use either of the phrases

- *Is inversely proportional to*

- *Varies inversely as*

Example 1: Find the constant of variation of the beach house situation.

Solution: Use the inverse variation equation to find k , the constant of variation.

$$y = \frac{k}{x}$$

$$170 = \frac{k}{4}$$

Solve for k :

$$170 \times 4 = \frac{k}{4} \times 4$$

$$k = 680$$

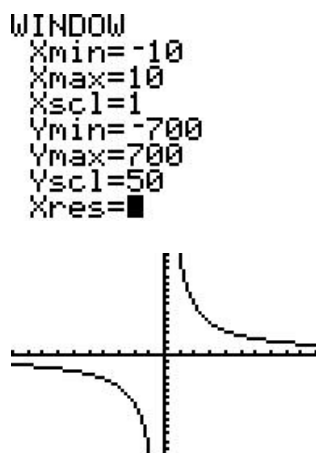
You can use this information to determine the amount of expense per person if nine people split the cost.

$$y = \frac{680}{x}$$

$$y = \frac{680}{9} = 75.56$$

If nine people split the expense, each would pay \$75.56.

Using a graphing calculator, look at a graph of this situation.



The graph of an inverse variation function $f(x) = \frac{k}{x}$ is a **hyperbola**. It has two branches in opposite quadrants.

If $k > 0$, the branches are in quadrants I and III.

If $k < 0$, the branches are in quadrants II and IV.

The graph appears to not cross the axes. In fact, this is true of any inverse variation equation of the form $y = \frac{k}{x^n}$. These lines are called **asymptotes**. Because of this, an inverse variation function has a special domain and range.

Domain : $\neq 0$

Range : $\neq 0$

You will investigate these excluded values in later lessons of this chapter.

Example 2: *The frequency, f , of sound varies inversely with wavelength, λ .* A sound signal that has a wavelength of 34 meters has a frequency of 10 hertz. What frequency does a sound signal of 120 meters have?

Solution: Use the inverse variation equation to find k , the constant of variation.

$$f = \frac{k}{\lambda}$$

$$10 = \frac{k}{34}$$

Solve for k :

$$10 \times 34 = \frac{k}{4} \times 34$$

$$k = 340$$

Use k to answer the question:

$$f = \frac{340}{120} = 2.83 \text{ hertz}$$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. [CK-12 Basic Algebra: Proportionality](#) (17:03)



Figure 8.1: Proportionality ([Watch Youtube Video](#))

http://www.youtube.com/v/AQFZuih2odo?f=videos&c=ytagapi-CK12Fondation-Flexrwikiimport-fg5akohk-0&d=AT8BNcsNZiISDLhsoSt-gqIO88HsQjpE1a8d1GxQnGDm&app=youtube_gdata

1. Define *inverse variation*.
2. Using 4.6 as a reference, explain three main differences between direct variation and inverse variation.

Read each statement and decide if the relationship is *direct*, *inverse*, or *neither*.

3. The weight of a book _____ as the number of pages it contains.
4. The temperature outside _____ as the time of day.
5. The amount of prize money you receive from winning the lottery _____ as the number of people who split the ticket cost.
6. The cost of a ferry ride _____ as the number of times you ride.
7. The area of a square _____ as the length of its side.

8. The height from the ground _____ as the number of seconds you have been on a roller coaster.
9. The time it takes to wash a car _____ as the number of people helping.
10. The number of tiles it takes to tile a floor _____ as the size of each tile.

Graph each inverse equation. State the domain and range.

11. $y = \frac{3}{x}$
12. $y = \frac{1}{x^2}$
13. $f(x) = -\frac{4}{x}$
14. $y = \frac{10}{x}$
15. $h(x) = -\frac{1}{x}$
16. $y = \frac{1}{4x}$
17. $g(x) = -\frac{2}{x^2}$
18. $y = \frac{4}{x^2}$
19. $y = \frac{5}{6x}$

Model each situation with an inverse variation equation, finding k . Then answer the question.

20. y varies inversely as x . If $y = 24$ when $x = 3$, find y when $x = -1.5$.
21. d varies inversely as the cube of t . If $d = -23.5$ when $t = 3$, find d when $x = \frac{1}{4}$.
22. If z is inversely proportional to w and $z = 81$ when $w = 9$, find w when $z = 24$.
23. If y is inversely proportional to x and $y = 2$ when $x = 8$, find y when $x = 12$.
24. If a is inversely proportional to the square root of b , and $a = 32$ when $b = 9$, find b when $a = 6$.
25. If w is inversely proportional to the square of u and $w = 4$ when $u = 2$, find w when $u = 8$.
26. The law of the fulcrum states the distance from the fulcrum varies inversely as the weight of the object. Joey and Josh are on a seesaw. If Joey weighs 40 pounds and sits six feet from the fulcrum, how far would Josh have to sit to balance the seesaw? (Josh weighs 65 pounds)?
27. The intensity of light is inversely proportional to the square of the distance between the light source and the object being illuminated. A light meter that is 10 meters from a light source registers 35 lux. What intensity would it register 25 meters from the light source?
28. Ohm's Law states that current flowing in a wire is inversely proportional to the resistance of the wire. If the current is 2.5 amperes when the resistance is 20 ohms, find the resistance when the current is 5 amperes.
29. The number of tiles it takes to tile a bathroom floor varies inversely as the square of the side of the tile. If it takes 112 six-inch tiles to cover a floor, how many eight-inches tiles are needed?

Mixed Review

30. Solve and graph the solutions on a number line: $16 \geq -3x + 5$.
31. Graph on a coordinate plane $x = \frac{7}{14}$
32. Simplify $\sqrt[3]{320}$
33. State the Commutative Property of Multiplication.
34. Draw the real number hierarchy and provide an example for each category.
35. Find 17.5% of 96.

8.2 Graphs of Rational Functions

In the previous lesson, you learned the basics of graphing an inverse variation function. The **hyperbola** forms two **branches** in opposite quadrants. The axes are **asymptotes** to the graph. This lesson will

compare graphs of inverse variation functions. You will also learn how to graph other rational equations.

Example: Graph the function $f(x) = \frac{k}{x}$ for the following values of k :

$$k = -2, -1, -\frac{1}{2}, 1, 2, 4$$

Each graph is shown separately then on one coordinate plane.

As mentioned in the previous lesson, if k is positive, then the branches of the hyperbola are located in quadrants I and III. If k is negative, the branches are located in quadrants II and IV. Also notice how the hyperbola changes as k gets larger.

Rational Functions

A **rational function** is a ratio of two polynomials (a polynomial divided by another polynomial). The formal definition is

$$f(x) = \frac{g(x)}{h(x)}, \text{ where } h(x) \neq 0$$

An **asymptote** is a value for which the equation or function is undefined. Asymptotes can be vertical, horizontal, or oblique. This text will focus on vertical asymptotes; other math courses will also show you how to find horizontal and oblique asymptotes. A function is undefined when the denominator of a fraction is zero. To find the asymptotes, find where the denominator of the rational function is zero. These are called **points of discontinuity** of the function.

The formal definition for asymptote is as follows:

An **asymptote** is a straight line to which, as the distance from the origin gets larger, a curve gets closer and closer but never intersects.

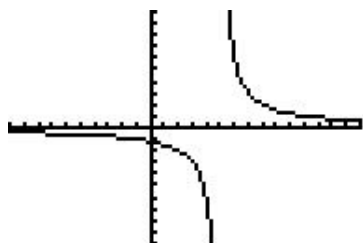
Example: Find the points of discontinuity and the asymptote for the function $y = \frac{6}{x-5}$.

Solution: Find the value of x for which the denominator of the rational function is zero.

$$0 = x - 5 \rightarrow x = 5$$

The point at which $x = 5$ is a point of discontinuity. Therefore, the asymptote has the equation $x = 5$.

Look at the graph of the function. There is a clear separation of the branches at the vertical line five units to the right of the origin.



The domain is “all real numbers except five” or symbolically written $x \neq 5$.

Example 1: Determine the asymptotes of $t(x) = \frac{2}{(x-2)(x+3)}$.

Solution: Using the Zero-Product Property, there are two cases for asymptotes; where each parenthesis equals zero.

$$x - 2 = 0 \rightarrow x = 2$$

$$x + 3 = 0 \rightarrow x = -3$$

The two asymptotes for this function are $x = 2$ and $x = -3$.

Check your solution by graphing the function.

The domain of the rational function above has two points of discontinuity. Therefore, its domain cannot include the numbers 2 or -3. The *domain*: $x \neq 2, x \neq -3$.

Horizontal Asymptotes

Rational functions can also have horizontal asymptotes. The equation of a horizontal asymptote is $y = c$, where c represents the vertical shift of the rational function.

Example: Identify the vertical and horizontal asymptotes of $f(x) = \frac{3}{(x-4)(x+8)} - 5$

Solution: The vertical asymptotes occur where the denominator is equal to zero.

$$x - 4 = 0 \rightarrow x = 4$$

$$x + 8 = 0 \rightarrow x = -8$$

The vertical asymptotes are $x = 4$ and $x = -8$

The rational function has been shifted down five units $f(x) = \frac{3}{(x-4)(x+8)} - 5$.

Therefore, the horizontal asymptote is $y = -5$.

Multimedia Link: For further explanation about asymptotes, read through this [PowerPoint](#) presentation presented by North Virginia Community College or watch this [CK-12 Basic Algebra: Finding Vertical Asymptotes of Rational Functions](#)] - You Tube video.



Figure 8.2: Finding Vertical Asymptotes of Rational Functions. In this video, I show what to look for, in order to find vertical asymptotes of rational functions, and do 4 examples of finding vertical asymptotes. ([Watch Youtube Video](#))

http://www.youtube.com/v/_qEOZNPce60?f=videos&c=ytapi-CK12Foundation-Flexrwikiimport-fg5akohk-0&d=AT8BNcsNZiISDLhsoSt-gqIO88HsQjpE1a8d1GxQnGDm&app=youtube_gdata

Real World Rational Functions

Electrical circuits are commonplace in everyday life. For instance, they are present in all electrical appliances in your home. The figure below shows an example of a simple electrical circuit. It consists of a battery

which provides a voltage (V , measured in Volts), a resistor (R , measured in ohms, Ω) which resists the flow of electricity, and an ammeter that measures the current (I , measured in amperes, A) in the circuit. Your light bulb, toaster and hairdryer are all basically simple resistors. In addition, resistors are used in an electrical circuit to control the amount of current flowing through a circuit and to regulate voltage levels. One important reason to do this is to prevent sensitive electrical components from burning out due to too much current or too high a voltage level. Resistors can be arranged in series or in parallel.

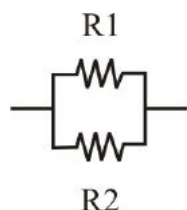
For resistors placed in a series, the total resistance is just the sum of the resistances of the individual resistors.

$$R_{tot} = R_1 + R_2$$



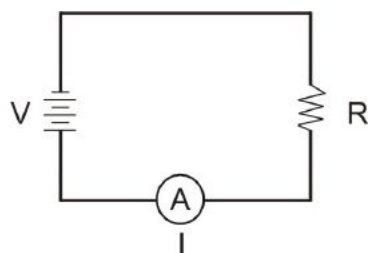
For resistors placed in parallel, the reciprocal of the total resistance is the sum of the reciprocals of the resistance of the individual resistors.

$$\frac{1}{R_c} = \frac{1}{R_1} + \frac{1}{R_2}$$

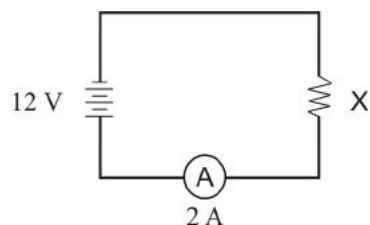


Ohm's Law gives a relationship between current, voltage and resistance. It states that

$$I = \frac{V}{R}$$



Example: Find the value of x marked in the diagram.



Solution: Using Ohm's Law, $I = \frac{V}{R}$, and substituting the appropriate information

$$2 = \frac{12}{R}$$

Using the cross multiplication of a proportion,

$$2R = 12 \rightarrow R = 6 \Omega$$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following videos. Note that there is not always a match between the number of the practice exercise in the videos and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. [CK-12 Basic Algebra: Asymptotes](#) (21:06)



Figure 8.3: Asymptotes of Rational Functions ([Watch Youtube Video](#))

http://www.youtube.com/v/2N62v_63SBo?f=videos&c=ytapi-CK12Foundation-Flexrwikiimport-fg5akohk-0&d=AT8BNcsNZiISDLhsoSt-gqIO88HsQjpE1a8d1GxQnGDm&app=youtube_gdata

[CK-12 Basic Algebra: Another Rational Function Graph Example](#) (8:20)



Figure 8.4: Another Rational Function Graph Example ([Watch Youtube Video](#))

http://www.youtube.com/v/ReEMqdZEEEX0?f=videos&c=ytapi-CK12Foundation-Flexrwikiimport-fg5akohk-0&d=AT8BNcsNZiISDLhsoSt-gqIO88HsQjpE1a8d1GxQnGDm&app=youtube_gdata

[CK-12 Basic Algebra: A Third Example of Graphing a Rational Function](#) (11:31)

1. What is a rational function?
2. Define *asymptote*. How does an asymptote relate algebraically to an rational equation?
3. Which asymptotes are described in this lesson. What is the general equation for these asymptotes?

Identify the vertical and horizontal asymptotes of each rational function.

4. $y = \frac{4}{x+2}$



Figure 8.5: A Third Example of Graphing a Rational Function ([Watch Youtube Video](#))

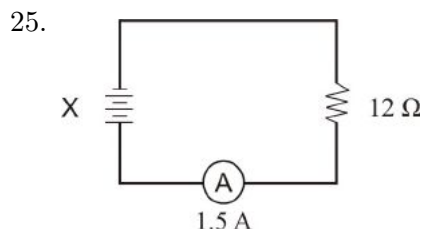
http://www.youtube.com/v/p7ycTWq6BFk?f=videosamp;c=ytapi-CK12Foundation-Flexrwikiimport-fg5akohk-0&id=AT8BNcsNZiISDLhsoSt-gqIO88HsQjpE1a8d1GxQnGDm&app=youtube_gdata

5. $f(x) = \frac{5}{2x-6} + 3$
6. $y = \frac{10}{x}$
7. $g(x) = \frac{4}{4x^2+1} - 2$
8. $h(x) = \frac{2}{x^2-9}$
9. $y = \frac{1}{x^2+4x+3} + \frac{1}{2}$
10. $y = \frac{3}{x^2-4} - 8$
11. $f(x) = \frac{-3}{x^2-2x-8}$

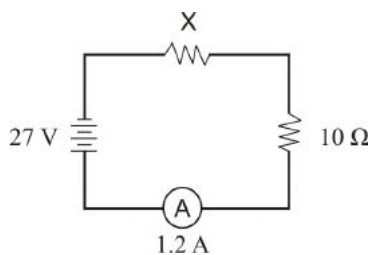
Graph each rational function. Show the vertical asymptote and horizontal asymptote as a dotted line.

12. $y = -\frac{6}{x}$
13. $y = \frac{x}{2-x^2} - 3$
14. $f(x) = \frac{3}{x^2}$
15. $g(x) = \frac{1}{x-1} + 5$
16. $y = \frac{2}{x+2} - 6$
17. $f(x) = \frac{-1}{x^2+2}$
18. $h(x) = \frac{4}{x^2+9}$
19. $y = \frac{-2}{x^2+1}$
20. $j(x) = \frac{1}{x^2-1} + 1$
21. $y = \frac{2}{x^2-9}$
22. $f(x) = \frac{8}{x^2-16}$
23. $g(x) = \frac{3}{x^2-4x+4}$
24. $h(x) = \frac{1}{x^2-x-6} - 2$

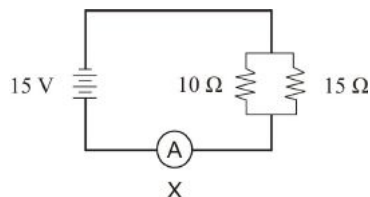
Find the quantity labeled x in the following circuit.



26.



27.



Mixed Review

28. A building 350-foot tall casts a shadow $\frac{1}{2}$ mile long. How long is the shadow of a person five-feet tall?
29. State the Cross Product Property.
30. Find the slope between (1, 1) and (-4, 5).
31. The amount of refund from soda cans in Michigan is directly proportional to the number of returned cans. If you earn \$12.00 refund for 120 cans, how much do you get per can?
32. You put the letters from VACATION into a hat. If you reach in randomly, what is the probability you will pick the letter A?
33. Give an example of a sixth degree binomial.

8.3 Division of Polynomials

We will begin with a property that is the converse of the Adding Fractions Property presented in chapter 2.

For all real numbers a, b , and c , and $c \neq 0$, if $\frac{a+b}{c}$, then $\frac{a}{c} + \frac{b}{c}$.

This property allows you to separate the numerator into its individual fractions. This property is used when dividing a polynomial by a monomial.

Example: Simplify $\frac{8x^2-4x+16}{2}$

Solution: Using the property above, separate the polynomial into its individual fractions.

$$\begin{array}{r} \text{Reduce} \end{array} \quad \frac{8x^2}{2} - \frac{4x}{2} + \frac{16}{2}$$

$$4x^2 - 2x + 8$$

Example 1: Simplify $\frac{-3m^2-18m+6}{9m}$

Solution: Separate the trinomial into its individual fractions and reduce.

$$\begin{array}{r} -\frac{3m^2}{9m} - \frac{18m}{9m} + \frac{6}{9m} \\ -\frac{m}{3} - 2 + \frac{2}{3m} \end{array}$$

Polynomials can also be divided by binomials. However, instead of separating into its individual fractions, we use a process called long division.

Example: Simplify $\frac{x^2+4x+5}{x+3}$

Solution: When we perform division, the expression in the numerator is called the **dividend** and the expression in the denominator is called the **divisor**.

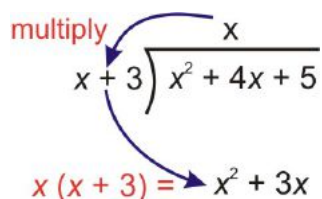
To start the division we rewrite the problem in the following form.

$$x + 3 \overline{) x^2 + 4x + 5}$$

Start by dividing the first term in the dividend by the first term in the divisor $\frac{x^2}{x} = x$. Place the answer on the line above the x term.

$$\begin{array}{r} x \\ x + 3 \overline{) x^2 + 4x + 5} \end{array}$$

Next, multiply the x term in the answer by each of the $x + 3$ in the divisor and place the result under the dividend matching like terms.



$$\begin{array}{r} x \\ x + 3 \overline{) x^2 + 4x + 5} \end{array}$$

$$x(x + 3) = x^2 + 3x$$

Now subtract $x^2 + 3x$ from $x^2 + 4x + 5$. It is useful to change the signs of the terms of $x^2 + 3x$ to $-x^2 - 3x$ and add like terms vertically.

$$\begin{array}{r} x \\ x + 3 \overline{) x^2 + 4x + 5} \\ \underline{-x^2 - 3x} \\ x \end{array}$$

Now, bring down 5, the next term in the dividend.

$$\begin{array}{r} x \\ x + 3 \overline{) x^2 + 4x + 5} \\ \underline{-x^2 - 3x} \\ x + 5 \end{array}$$

Repeat the process. First divide the first term of $x + 5$ by the first term of the divisor $\left(\frac{x}{x}\right) = 1$. Place this answer on the line above the constant term of the dividend,

$$\begin{array}{r}
 x + 1 \\
 x + 3 \overline{) x^2 + 4x + 5} \\
 \underline{-x^2 - 3x} \\
 x + 5
 \end{array}$$

Multiply 1 by the divisor $x + 3$ and write the answer below $x + 5$ matching like terms.

multiply

$$\begin{array}{r}
 x + 1 \\
 x + 3 \overline{) x^2 + 4x + 5} \\
 \underline{-x^2 - 3x} \\
 x + 5
 \end{array}$$

$x + 3$

Subtract $x + 3$ from $x + 5$ by changing the signs of $x + 3$ to $-x - 3$ and adding like terms.

$$\begin{array}{r}
 x + 1 \quad \text{quotient} \\
 x + 3 \overline{) x^2 + 4x + 5} \\
 \underline{-x^2 - 3x} \\
 x + 5 \\
 \underline{-x - 3} \quad \text{remainder} \\
 2
 \end{array}$$

Since there are no more terms from the dividend to bring down, we are done.

The answer is $x + 1$ with a remainder of 2.

Multimedia Link: For more help with using long division to simplify rational expressions, visit this <http://www.purplemath.com/modules/polydiv2.htm> - website or watch this [CK-12 Basic Algebra: 6 7 Polynomial long division with Mr. Nystrom](#) - You Tube video.



Figure 8.6: 6-7 Polynomial long division ([Watch Youtube Video](#))

http://www.youtube.com/v/UtfkblWGpsY?f=videos&c=ytapi-CK12Foundation-Flexrwikiimport-fg5akohk-0&d=AT8BNcsNZiISDLhsoSt-gqIO88HsQjpE1a8d1GxQnGDm&app=youtube_gdata

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. [CK-12 Basic Algebra: Polynomial Division](#) (12:09)



Figure 8.7: Polynomial Division ([Watch Youtube Video](#))

http://www.youtube.com/v/FXgV9ySNusc?f=videosamp;c=ytapi-CK12Foundation-Flexrwikiimport-fg5akohk-0&d=AT8BNcsNZiISDLhsoSt-gqIO88HsQjpE1a8d1GxQnGDm&app=youtube_gdata

Divide the following polynomials:

1. $\frac{2x+4}{2}$
2. $\frac{x-4}{x-4}$
3. $\frac{5x-35}{5x}$
4. $\frac{x^2+2x-5}{x^2+2x-5}$
5. $\frac{4x^2+12x-36}{-4x}$
6. $\frac{2x^2+10x+7}{2x^2}$
7. $\frac{x^3-x}{-2x^2}$
8. $\frac{5x^4-9}{3x}$
9. $\frac{x^3-12x^2+3x-4}{12x^2}$
10. $\frac{3-6x+x^3}{-9x^3}$
11. $\frac{x^2+3x+6}{x+1}$
12. $\frac{x^2-9x+6}{x-1}$
13. $\frac{x^2+5x+4}{x+4}$
14. $\frac{x^2-10x+25}{x-5}$
15. $\frac{x^2-20x+12}{x-3}$
16. $\frac{3x^2-x+5}{x-2}$
17. $\frac{9x^2+2x-8}{x+4}$
18. $\frac{3x^2-4}{3x+1}$
19. $\frac{5x^2+2x-9}{2x-1}$
20. $\frac{x^2-6x-12}{5x+4}$
21. $\frac{x^4-2x}{8x+24}$
22. $\frac{x^3+1}{4x-1}$

Mixed Review

23. Boyle's Law states the pressure of a compressed gas varies inversely as its pressure. If the pressure of a 200-pound gas is 16.75 psi, find the pressure if the amount of gas is 60 pounds.
24. Is $5x^3 + x^2 - x^{-1} + 8$ an example of a polynomial? Explain your answer.
25. Find the slope of the line perpendicular to $y = -\frac{3}{4}x + 5$.
26. How many two-person teams can be made from a group of nine individuals?
27. What is a problem with face to face interviews? What do you think is a potential solution to this problem?
28. Solve for m : $-4 = \frac{\sqrt{m-3}}{-2}$

8.4 Rational Expressions

You have gained experience working with rational functions so far this chapter. In this lesson, you will continue simplifying rational expressions by factoring.

To **simplify** a rational expression means to reduce the fraction into its lowest terms.

To do this, you will need to remember a property about multiplication.

For all real values a, b , and $b \neq 0$, $\frac{ab}{b} = a$.

Example: Simplify $\frac{4x-2}{2x^2+x-1}$.

Solution:

Both the numerator and denominator can be factored using methods learned in Chapter 9.

$$\frac{4x-2}{2x^2+x-1} \rightarrow \frac{2(2x-1)}{(2x-1)(x+1)}$$

The expression $(2x-1)$ appears in both the numerator and denominator and can be canceled. The expression becomes

$$\frac{4x-2}{2x^2+x-1} = \frac{2}{x+1}$$

Example 1: Simplify $\frac{x^2-2x+1}{8x-8}$

Solution: Factor both pieces of the rational expression and reduce.

$$\begin{aligned} \frac{x^2-2x+1}{8x-8} &\rightarrow \frac{(x-1)(x-1)}{8(x-1)} \\ \frac{x^2-2x+1}{8x-8} &= \frac{x-1}{8} \end{aligned}$$

Finding Excluded Values of Rational Expressions

As stated in lesson two of this chapter, excluded values are also called **points of discontinuity**. These are the values that make the denominator equal to zero and are not part of the domain.

Example 2: Find the excluded values of $\frac{2x+1}{x^2-x-6}$

Solution: Factor the denominator of the rational expression.

$$\frac{2x+1}{x^2-x-6} = \frac{2x+1}{(x+2)(x-3)}$$

Find the value that makes each factor equal zero.

$$x = -2, x = 3$$

These are excluded values of the domain of the rational expression.

Real Life Rational Expressions

The gravitational force between two objects is given by the formula $F = \frac{G(m_1m_2)}{(d^2)}$. if the gravitation constant is given by $G = 6.67 \times 10^{-11} (N \cdot m^2/kg^2)$. The force of attraction between the Earth and the Moon is $F = 2.0 \times 10^{20} N$ (with masses of $m_1 = 5.97 \times 10^{24} kg$ for the Earth and $m_2 = 7.36 \times 10^{22} kg$ for the Moon).

What is the distance between the Earth and the Moon?

Lets start with the Law of Gravitation formula. $F = G \frac{m_1m_2}{d^2}$

Now plug in the known values. $2.0 \times 10^{20} N = 6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2} \cdot \frac{(5.97 \times 10^{24} kg)(7.36 \times 10^{22} kg)}{d^2}$

Multiply the masses together. $2.0 \times 10^{20} N = 6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2} \cdot \frac{4.39 \times 10^{47} kg^2}{d^2}$

Cancel the kg^2 units. $2.0 \times 10^{20} N = 6.67 \times 10^{-11} \frac{N \cdot m^2}{\cancel{kg^2}} \cdot \frac{4.39 \times 10^{47} \cancel{kg^2}}{d^2}$

Multiply the numbers in the numerator. $2.0 \times 10^{20} N \frac{2.93 \times 10^{37}}{d^2} N \cdot m^2$

Multiply both sides by d^2 . $2.0 \times 10^{20} N \cdot d^2 = \frac{2.93 \times 10^{37}}{d^2} \cdot d^2 \cdot N \cdot m^2$

Cancel common factors. $2.0 \times 10^{20} N \cdot d^2 = \frac{2.93 \times 10^{37}}{\cancel{d^2}} \cdot \cancel{d^2} \cdot N \cdot m^2$

Simplify. $2.0 \times 10^{20} N \cdot d^2 = 2.93 \times 10^{37} N \cdot m^2$

Divide both sides by $2.0 \times 10^{20} N$. $d^2 = \frac{2.93 \times 10^{37} N \cdot m^2}{2.0 \times 10^{20} N}$

Simplify. $d^2 = 1.465 \times 10^{17} m^2$

Take the square root of both sides. $d = 3.84 \times 10^8 m$ Answer

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. [CK-12 Basic Algebra: Simplifying Rational Expressions](#) (15:22)

Reduce each fraction to lowest terms.

1. $\frac{4}{2x-8}$
2. $\frac{x^2+2x}{x^2+2x}$
3. $\frac{9x+3}{12x+4}$
4. $\frac{6x^2+2x}{4x}$



Figure 8.8: Simplifying Rational Expressions ([Watch Youtube Video](#))

http://www.youtube.com/v/7Uos1ED3KHI?f=videosamp;c=ytapi-CK12Foundation-Flexrwikiimport-fg5akohk-0&d=AT8BNcsNZiISDLhsoSt-gqIO88HsQjpE1a8d1GxQnGDm&app=youtube_gdata

5. $\frac{x-2}{x^2-4x+4}$
6. $\frac{x^2-9}{5x+15}$
7. $\frac{x^2+6x+8}{x^2+4x}$
8. $\frac{2x^2+10x}{x^2+10x+25}$
9. $\frac{x^2+6x+5}{x^2-x-2}$
10. $\frac{x^2-16}{x^2+2x-8}$
11. $\frac{3x^2+3x-18}{2x^2+5x-3}$
12. $\frac{x^3+x^2-20x}{6x^2+6x-120}$

Find the excluded values for each rational expression.

13. $\frac{2}{x}$
14. $\frac{4}{x+2}$
15. $\frac{2x-1}{(x-1)^2}$
16. $\frac{3x+1}{x^2-4}$
17. $\frac{x^2}{x^2+9}$
18. $\frac{2x^2+3x-1}{x^2-3x-28}$
19. $\frac{5x^3-4}{x^2+3x}$
20. $\frac{9}{x^3+11x^2+30x}$
21. $\frac{4x-1}{x^2+3x-5}$
22. $\frac{5x+11}{3x^2-2x-4}$
23. $\frac{x^2-1}{2x^3+x+3}$
24. $\frac{12}{x^2+6x+1}$
25. In an electrical circuit with resistors placed in parallel, the reciprocal of the total resistance is equal to the sum of the reciprocals of each resistance. $\frac{1}{R_c} = \frac{1}{R_1} + \frac{1}{R_2}$. If $R_1 = 25\Omega$ and the total resistance is $R_c = 10\Omega$, what is the resistance R_2 ?
26. Suppose that two objects attract each other with a gravitational force of 20 Newtons. If the distance between the two objects is doubled, what is the new force of attraction between the two objects?
27. Suppose that two objects attract each other with a gravitational force of 36 Newtons. If the mass of both objects was doubled, and if the distance between the objects was doubled, then what would be the new force of attraction between the two objects?
28. A sphere with radius r has a volume of $\frac{4}{3}\pi r^3$ and a surface area of $4\pi r^2$. Find the ratio the surface

area to the volume of a sphere.

29. The side of a cube is increased by a factor of two. Find the ratio of the old volume to the new volume.
30. The radius of a sphere is decreased by four units. Find the ratio of the old volume to the new volume.

Mixed Review

31. Name $4p^6 + 7p^3 - 9$
32. Simplify $(4b^2 + b + 7b^3) + (5b^2 - 6b^4 + b^3)$. Write the answer in standard form.
33. State the Zero Product Property.
34. Why can't the Zero Product Property be used in this situation? $(5x + 1)(x - 4) = 2$
35. Shelly earns \$4.85 an hour plus \$15 in tips. Graph her possible total earnings for one day of work.
36. Multiply and simplify $(-4x^2 + 8x - 1)(-7x^2 + 6x + 8)$
37. A rectangle's perimeter is 65 yards. The length is 7 more yards than its width. What dimensions would give the largest area?

8.5 Multiplication and Division of Rational Expressions

Because a rational expression is really a fraction, two (or more) rational expressions can be combined through multiplication and/or division in the same manner as numerical fractions. A reminder of how to multiply fractions is below.

For any rational expressions $a \neq 0, b \neq 0, c \neq 0, d \neq 0$,

$$\begin{aligned}\frac{a}{b} \cdot \frac{c}{d} &= \frac{ac}{bd} \\ \frac{a}{b} \div \frac{c}{d} &\rightarrow \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}\end{aligned}$$

Example: Multiply the following $\frac{a}{16b^8} \cdot \frac{4b^3}{5a^2}$

Solution:

$$\frac{a}{16b^8} \cdot \frac{4b^3}{5a^2} \rightarrow \frac{4ab^3}{80a^2b^8}$$

Simplify exponents using methods learned in chapter 8.

$$\frac{4ab^3}{80a^2b^8} = \frac{1}{20ab^5}$$

Example 1: Simplify $9c^2 \cdot \frac{4y^2}{21c^4}$

Solution:

$$\begin{aligned}9c^2 \cdot \frac{4y^2}{21c^4} &\rightarrow \frac{9c^2}{1} \cdot \frac{4y^2}{21c^4} \\ \frac{9c^2}{1} \cdot \frac{4y^2}{21c^4} &= \frac{36c^2y^2}{21c^4} \\ \frac{36c^2y^2}{21c^4} &= \frac{12y^2}{7c^2}\end{aligned}$$

Multiplying Rational Expressions Involving Polynomials

When rational expressions become complex it is usually easier to factor all expressions and reduce **before** attempting to multiply the expressions.

Example: Multiply $\frac{4x+12}{3x^2} \cdot \frac{x}{x^2-9}$.

Solution: Factor all pieces of these rational expressions and reduce before multiplying.

$$\begin{aligned}\frac{4x+12}{3x^2} \cdot \frac{x}{x^2-9} &\rightarrow \frac{4(x+3)}{3x^2} \cdot \frac{x}{(x+3)(x-3)} \\&\frac{\cancel{4(x+3)}}{3x^{\cancel{2}}} \cdot \frac{\cancel{x}}{\cancel{(x+3)}(x-3)} \\&\frac{4}{3x} \cdot \frac{1}{x-3} \rightarrow \frac{4}{3x^2-9x}\end{aligned}$$

Example 1: Multiply $\frac{12x^2-x-6}{x^2-1} \cdot \frac{x^2+7x+6}{4x^2-27x+18}$

Solution: Factor all pieces, reduce, and then multiply.

$$\begin{aligned}\frac{12x^2-x-6}{x^2-1} \cdot \frac{x^2+7x+6}{4x^2-27x+18} &\rightarrow \frac{(3x+2)(4x-3)}{(x+1)(x-1)} \cdot \frac{(x+1)(x+6)}{(4x-3)(x-6)} \\&\frac{(3x+2)\cancel{(4x-3)}}{\cancel{(x+1)}(x-1)} \cdot \frac{\cancel{(x+1)}(x+6)}{\cancel{(4x-3)}(x-6)} \rightarrow \frac{(3x+2)(x+6)}{(x-1)(x-6)} \\&\frac{12x^2-x-6}{x^2-1} \cdot \frac{x^2+7x+6}{4x^2-27x+18} = \frac{3x^2+10x+12}{x^2-7x+6}\end{aligned}$$

Dividing Rational Expressions Involving Polynomials

Division of rational expressions works in the same manner as multiplication. A reminder of how to divide fractions is below.

For any rational expressions $a \neq 0, b \neq 0, c \neq 0, d \neq 0$,

$$\frac{a}{b} \div \frac{c}{d} \rightarrow \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Example: Simplify $\frac{9x^2-4}{2x-2} \div \frac{21x^2-2x-8}{1}$.

Solution:

$$\frac{9x^2-4}{2x-2} \div \frac{21x^2-2x-8}{1} \rightarrow \frac{9x^2-4}{2x-2} \cdot \frac{1}{21x^2-2x-8}$$

Repeat the process for multiplying rational expressions.

$$\begin{aligned}\frac{9x^2-4}{2x-2} \cdot \frac{1}{21x^2-2x-8} &\rightarrow \frac{(3x-2)\cancel{(3x-2)}}{2(x-1)} \cdot \frac{1}{\cancel{(3x-2)}(7x+4)} \\&\frac{9x^2-4}{2x-2} \div \frac{21x^2-2x-8}{1} = \frac{3x-2}{14x^2-6x-8}\end{aligned}$$

Real Life Application

Suppose Marciel is training for a running race. Marciel's speed (in miles per hour) of his training run each morning is given by the function $x^3 - 9x$, where x is the number of bowls of cereal he had for breakfast

($1 \leq x \leq 6$). Marciel's training distance (in miles), if he eats x bowls of cereal, is $3x^2 - 9x$. What is the function for Marciel's time and how long does it take Marciel to do his training run if he eats five bowls of cereal on Tuesday morning?

$$\begin{aligned}\text{time} &= \frac{\text{distance}}{\text{speed}} \\ \text{time} &= \frac{3x^2 - 9x}{x^3 - 9x} = \frac{3x(x-3)}{x(x^2-9)} = \frac{3x\cancel{(x-3)}}{x(x+3)\cancel{(x-3)}} \\ \text{time} &= \frac{3}{x+3} \\ \text{If } x &= 5, \text{ then} \\ \text{time} &= \frac{3}{5+3} = \frac{3}{8}\end{aligned}$$

Marciel will run for $\frac{3}{8}$ of an hour.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. [CK-12 Basic Algebra: Multiplying and Dividing Rational Expressions \(9:19\)](#)



Figure 8.9: Multiplying and Dividing Rational Expressions ([Watch Youtube Video](#))

http://www.youtube.com/v/x_5hDLe8UL0?f=videosamp;c=ytapi-CK12Foundation-Flexrwikiimport-fg5akohk-0amp;d=AT8BNcsNZiISDLhsoSt-gqIO88HsQjpE1a8d1GxQnGDmamp;app=youtube_gdata

Perform the indicated operation and reduce the answer to lowest terms

- $\frac{x^3}{2y^3} \cdot \frac{2y^2}{x}$
- $2xy \div \frac{2x^2}{y}$
- $\frac{2x}{y^2} \cdot \frac{4y}{5x}$
- $2xy \cdot \frac{2y^2}{x^3}$
- $\frac{4y^2-1}{y^2-9} \cdot \frac{y-3}{2y-1}$
- $\frac{6ab}{a^2} \cdot \frac{a^3b}{3b^2}$
- $\frac{x^2}{x-1} \div \frac{x}{x^2+x-2}$
- $\frac{33a^2}{-5} \cdot \frac{20}{11a^3}$

9. $\frac{a^2+2ab+b^2}{ab^2-a^2b} \div (a+b)$
10. $\frac{2x^2+2x-24}{x^2+3x} \cdot \frac{x^2+x-6}{x+4}$
11. $\frac{3-x}{3x-5} \div \frac{x^2-9}{2x^2-8x-10}$
12. $\frac{x^2-25}{x+3} \div (x-5)$
13. $\frac{2x+1}{2x-1} \div \frac{4x^2-1}{1-2x}$
14. $\frac{x}{x-5} \cdot \frac{x^2-8x+15}{x^2-3x}$
15. $\frac{3x^2+5x-12}{x^2-9} \div \frac{3x-4}{3x+4}$
16. $\frac{5x^2+16x+3}{36x^2-25} \cdot (6x^2+5x)$
17. $\frac{x^2+7x+10}{x^2-9} \cdot \frac{x^2-3x}{3x^2+4x-4}$
18. $\frac{x^2+x-12}{x^2+4x+4} \div \frac{x-3}{x+2}$
19. $\frac{x^4-16}{x^2-9} \div \frac{x^2+4}{x^2+6x+9}$
20. $\frac{x^2+8x+16}{7x^2+9x+2} \div \frac{7x+2}{x^2+4x}$
21. Maria's recipe asks for $2\frac{1}{2}$ times more flour than sugar. How many cups of flour should she mix in if she uses $3\frac{1}{3}$ cups of sugar?
22. George drives from San Diego to Los Angeles. On the return trip, he increases his driving speed by 15 miles per hour. In terms of his initial speed, by what factor is the driving time decreased on the return trip?
23. Ohm's Law states that in an electrical circuit $I = \frac{V}{R_c}$. The total resistance for resistors placed in parallel is given by $\frac{1}{R_{tot}} = \frac{1}{R_1} + \frac{1}{R_2}$. Write the formula for the electric current in term of the component resistances: R_1 and R_2 .

Mixed Review

24. The time it takes to reach a destination varies inversely as the speed in which you travel. It takes 3.6 hours to reach your destination traveling 65 miles per hour. How long would it take to reach your destination traveling 78 miles per hour?
25. A local nursery makes two types of fall arrangements. One arrangement uses eight mums and five black-eyed susans. The other arrangement uses six mums and 9 black-eyed susans. The nursery can use no more than 144 mums and 135 black-eyed susans. The first arrangement sells for \$49.99 and the second arrangement sells for 38.95. how many of each type should be sold to maximize revenue?
26. Solve for r and graph the solution on a number line: $-24 \geq |2r + 3|$
27. What is true of any line parallel to $5x + 9y = -36$?
28. Solve for d : $3 + 5d = -d - (3x - 3)$
29. Graph and determine the domain and range: $y - 9 = -x^2 - 5x$
30. Rewrite in vertex form by completing the square. Identify the vertex: $y^2 - 16y = 3 = 4$

Quick Quiz

1. h is inversely proportional to t . If $t = -0.05153$ when $h = -16$, find t when $h = 1.45$.
2. Use for $f(x) = \frac{-5}{x^2-25}$ the following questions
 1. Find the excluded values.
 2. Determine the vertical asymptotes.
 3. Sketch a graph of this function.
 4. Determine its domain and range.
3. Simplify $\frac{8c^4+12c^2-22c+1}{4}$
4. Simplify $\frac{10a^2-30a}{a-3}$. What are its excluded values?

5. Fill the blank with *directly*, *inversely*, or *neither*. “The amount of time it takes to mow the lawn varies _____ with the size of the lawn mower.”

8.6 Addition and Subtraction of Rational Expressions

Like numerical fractions, rational expressions represent a part of a whole quantity. Remember when adding or subtracting fractions, the denominators must be the same. Once the denominators are identical, the numerators are combined by adding or subtracting like terms.

Example 1: Simplify $\frac{4x^2-3}{x+5} + \frac{2x^2-1}{x+5}$

Solution: The denominators are identical; therefore we can add the like terms of the numerator to simplify.

$$\frac{4x^2-3}{x+5} + \frac{2x^2-1}{x+5} = \frac{6x^2-4}{x+5}$$

Not all denominators are the same however. In the case of **unlike denominators**, common denominators must be created through multiplication by finding the **least common multiple**.

The **least common multiple (LCM)** is the smallest number that is evenly divisible by every member of the set.

What is the least common multiple of 2, $4x$, and $6x^2$? The smallest number 2, 4, and 6 can divide into evenly is six. The largest exponent of x is 2. Therefore, the LCM of 2, $4x$, and $6x^2$ is $6x^2$.

Example 2: Find the least common multiple of $2x^2 + 8x + 8$ and $x^3 - 4x^2 - 12x$.

Solution: Factor the polynomials completely.

$$\begin{aligned} 2x^2 + 8x + 8 &= 2(x^2 + 4x + 4) = 2(x+2)^2 \\ x^3 - 4x^2 - 12x &= x(x^2 - 4x - 12) = x(x-6)(x+2) \end{aligned}$$

The LCM is found by taking each factor to the highest power that it appears in either expression. $LCM = 2x(x+2)^2(x-6)$

Use this add rational expressions with unlike denominators.

Example: Add $\frac{2}{x+2} - \frac{3}{2x-5}$

Solution: The denominators cannot be factored any further, so the LCD is just the product of the separate denominators.

$$LCD = (x+2)(2x-5)$$

The first fraction needs to be multiplied by the factor $(2x-5)$ and the second fraction needs to be multiplied by the factor $(x+2)$.

$$\frac{2}{x+2} \cdot \frac{(2x-5)}{(2x-5)} - \frac{3}{2x-5} \cdot \frac{(x+2)}{(x+2)}$$

We combine the numerators and simplify.

$$\frac{2(2x-5) - 3(x+2)}{(x+2)(2x-5)} = \frac{4x-10-3x-6}{(x+2)(2x-5)}$$

Combine like terms in the numerator.

$$\frac{x - 16}{(x + 2)(2x - 5)} \text{ Answer}$$

Work Problems

These are problems where two objects work together to complete a job. Work problems often contain rational expressions. Typically we set up such problems by looking at the part of the task completed by each person or machine. The completed task is the sum of the parts of the tasks completed by each individual or each machine.

Part of task completed by first person + Part of task completed by second person = One completed task

To determine the part of the task completed by each person or machine we use the following fact.

Part of the task completed = rate of work \times time spent on the task

In general, it is very useful to set up a table where we can list all the known and unknown variables for each person or machine and then combine the parts of the task completed by each person or machine at the end.

Example: *Mary can paint a house by herself in 12 hours. John can paint a house by himself in 16 hours. How long would it take them to paint the house if they worked together?*

Solution: Let t = the time it takes Mary and John to paint the house together.

Since Mary takes 12 hours to paint the house by herself, in one hour she paints $\frac{1}{12}$ of the house.

Since John takes 16 hours to pain the house by himself, in one hour he paints $\frac{1}{16}$ of the house.

Mary and John work together for t hours to paint the house together. Using the formula

Part of the task completed = rate of work \cdot time spent on the task

We can write that Mary completed $\frac{t}{12}$ of the house and John completed $\frac{t}{16}$ of the house in this time and summarize the data in the following table.

Table 8.2:

| Painter | Rate of work (per hour) | Time worked | Part of Task |
|---------|-------------------------|-------------|----------------|
| Mary | $\frac{1}{12}$ | t | $\frac{t}{12}$ |
| John | $\frac{1}{16}$ | t | $\frac{t}{16}$ |

Using the formula

Part of task completed by first person + Part of task completed by second person = One completed task

Write an equation to model the situation,

$$\frac{t}{12} + \frac{t}{16} = 1.$$

Solve the equation by finding the least common multiple.

$$\begin{aligned}
 LCM &= 48 \\
 48 \cdot \frac{t}{12} + 48 \cdot \frac{t}{16} &= 48 \cdot 1 \\
 \cancel{48^4} \cdot \frac{t}{\cancel{12}} + \cancel{48^3} \cdot \frac{t}{\cancel{16}} &= 48 \cdot 1 \\
 4t + 3t &= 48 \\
 7t = 48 \Rightarrow t &= \frac{48}{7} = 6.86 \text{ hours Answer}
 \end{aligned}$$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following videos. Note that there is not always a match between the number of the practice exercise in the videos and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. [CK-12 Basic Algebra: Adding Rational Expressions Example 1](#) (3:47)



Figure 8.10: Adding Rational Expressions Example 1 ([Watch Youtube Video](#))

http://www.youtube.com/v/nWEso_M7By4?f=videosamp;c=ytapi-CK12Foundation-Flexrwikiimport-fg5akohk-0amp;d=AT8BNcsNZiISDLhsoSt-gqIO88HsQjpE1a8d1GxQnGDmamp;app=youtube_gdata

[CK-12 Basic Algebra: Adding Rational Expressions Example 2](#) (6:40)



Figure 8.11: Adding Rational Expressions Example 2 ([Watch Youtube Video](#))

http://www.youtube.com/v/euyg3SwS2lU?f=videosamp;c=ytapi-CK12Foundation-Flexrwikiimport-fg5akohk-0amp;d=AT8BNcsNZiISDLhsoSt-gqIO88HsQjpE1a8d1GxQnGDmamp;app=youtube_gdata

[CK-12 Basic Algebra: Adding Rational Expressions Example 3](#) (6:23)

Perform the indicated operation and simplify. Leave the denominator in factored form.



Figure 8.12: Adding Rational Expressions Example 3 ([Watch Youtube Video](#))

http://www.youtube.com/v/4VO-BgDgAHE?f=videosamp;c=ytapi-CK12Foundation-Flexrwikiimport-fg5akohk-0&id=AT8BNcsNZiISDLhsoSt-gqIO88HsQjpE1a8d1GxQnGDm&app=youtube_gdata

1. $\frac{5}{24} - \frac{7}{24}$
2. $\frac{10}{21} + \frac{9}{35}$
3. $\frac{5}{2x+3} + \frac{3}{2x+3}$
4. $\frac{3x-1}{x+9} - \frac{4x+3}{x+9}$
5. $\frac{4x+7}{2x^2} - \frac{3x-4}{2x^2}$
6. $\frac{x^2}{x+5} - \frac{25}{x+5}$
7. $\frac{2x}{x-4} + \frac{x}{4-x}$
8. $\frac{10}{3x-1} - \frac{7}{1-3x}$
9. $\frac{5}{2x+3} - 3$
10. $\frac{5x+1}{x+4} + 2$
11. $\frac{1}{x} + \frac{2}{3x}$
12. $\frac{4}{5x^2} - \frac{2}{7x^3}$
13. $\frac{4x}{x+1} - \frac{2}{2(x+1)}$
14. $\frac{10}{x+5} + \frac{2}{x+2}$
15. $\frac{2x}{x-3} - \frac{3x}{x+4}$
16. $\frac{4x-3}{2x+1} + \frac{x+2}{x-9}$
17. $\frac{x^2}{x+4} - \frac{3x^2}{4x-1}$
18. $\frac{2}{5x+2} - \frac{x+1}{x^2}$
19. $\frac{x+4}{2x} + \frac{2}{9x}$
20. $\frac{5x+3}{x^2+x} + \frac{2x+1}{x}$
21. $\frac{4}{(x+1)(x-1)} - \frac{5}{(x+1)(x+2)}$
22. $\frac{2x}{(x+2)(3x-4)} + \frac{7x}{(3x-4)^2}$
23. $\frac{3x+5}{x(x-1)} - \frac{9x-1}{(x-1)^2}$
24. $\frac{1}{(x-2)(x-3)} + \frac{4}{(2x+5)(x-6)}$
25. $\frac{3x-2}{x-2} + \frac{1}{x^2-4x+4}$
26. $\frac{-x^2}{x^2-7x+6} + x - 4$
27. $\frac{2x}{x^2+10x+25} - \frac{3x}{2x^2+7x-15}$
28. $\frac{1}{x^2-9} + \frac{2}{x^2+5x+6}$
29. $\frac{-x+4}{2x^2-x-15} + \frac{x}{4x^2+8x-5}$
30. $\frac{4}{9x^2-49} - \frac{1}{3x^2+5x-28}$
31. One number is 5 less than another. The sum of their reciprocals is $\frac{13}{36}$. Find the two numbers.
32. One number is 8 times more than another. The difference in their reciprocals is $\frac{21}{20}$. Find the two

numbers.

33. A pipe can fill a tank full of oil in 4 hours and another pipe can empty the tank in 8 hours. If the valves to both pipes are open, how long would it take to fill the tank?
34. Stefan could wash the cars by himself in 6 hours and Misha could wash the cars by himself in 5 hours. Stefan starts washing the cars by himself, but he needs to go to his football game after 2.5 hours. Misha continues the task. How long did it take Misha to finish washing the cars?
35. Amanda and her sister Chyna are shoveling snow to clear their driveway. Amanda can clear the snow by herself in three hours and Chyna can clear the snow by herself in four hours. After Amanda has been working by herself for one hour, Chyna joins her and they finish the job together. How long does it take to clear the snow from the driveway?
36. At a soda bottling plant one bottling machine can fulfill the daily quota in ten hours and a second machine can fill the daily quota in 14 hours. The two machines start working together but after four hours the slower machine broke and the faster machine had to complete the job by itself. How many hours does the fast machine works by itself?

Mixed Review

37. Explain the difference between these two situations. Write an equation to model each situation. Assume the town started with 10,000 people. When will statement 2 become larger than statement 1?
 1. For the past seven years, the population grew by 500 people every year.
 2. For the past seven years, the population grew by 5% every year.
38. Simplify. Your answer should have only positive exponents. $\frac{16x^2y^7}{-2x^8y} \cdot \frac{1}{2}x^{-10}$
39. Solve for j : $-12 = j^2 - 8j$. Which method did you use? Why did you choose this method?
40. Jimmy shot a basketball from a height of four feet with an upward velocity of 12 *feet/sec*.
 1. Write an equation to model this situation.
 2. Will Jimmy's ball make it to the ten-foot tall hoop?
41. The distance you travel varies directly as the speed in which you drive. If you can drive 245 miles in five hours, how long will it take you to drive 90 miles?
42. Two cities are 3.78 centimeters apart on an atlas. The atlas has a scale of $\frac{1}{2}cm = 14 \text{ miles}$. What is the true distance between these cities?

8.7 Solution of Rational Equations

You are now ready to solve rational equations! There are two main methods you will learn in this lesson to solve rational equations

- Cross products
- Lowest common denominators

Solving a Rational Proportion

When two rational expressions are equal, a **proportion** is created and can be solved using its cross products. For example, to solve $\frac{x}{5} = \frac{(x+1)}{2}$, cross multiply and the products are equal.

$$\frac{x}{5} = \frac{(x+1)}{2} \rightarrow 2(x) = 5(x+1)$$

Solve for x :

$$\begin{aligned}
 2(x) &= 5(x+1) \rightarrow 2x = 5x+5 \\
 2x-5x &= 5x-5x+5 \\
 -3x &= 5 \\
 x &= -\frac{5}{3}
 \end{aligned}$$

Example 1: Solve $\frac{2x}{x+4} = \frac{5}{x}$

Solution:

$$\begin{aligned}
 \frac{2x}{x+4} &= \frac{5}{x} \rightarrow 2x^2 = 5(x+4) \\
 2x^2 &= 5(x+4) \rightarrow 2x^2 = 5x+20 \\
 2x^2-5x-20 &= 0
 \end{aligned}$$

Notice that this equation has a degree of two, that is, it is a *quadratic equation*. We can solve it using the quadratic formula.

$$x = \frac{5 \pm \sqrt{185}}{4} \Rightarrow x \approx -2.15 \text{ or } x \approx 4.65$$

Solving by Clearing Denominators

When a rational equation has several terms, it may not be possible to use the method of cross products. A second method to solve rational equations is to **clear the fractions** by multiplying the entire equation by the least common multiple of the denominators.

Example: Solve $\frac{3}{x+2} - \frac{4}{x-5} = \frac{2}{x^2-3x-10}$.

Solution: Factor all denominators and find the least common multiple.

$$\begin{aligned}
 \frac{3}{x+2} - \frac{4}{x-5} - \frac{2}{(x+2)(x-5)} \\
 LCM = (x+2)(x-5)
 \end{aligned}$$

Multiply all terms in the equation by the LCM and cancel the common terms.

$$\begin{aligned}
 (x+2)(x-5) \cdot \frac{3}{x+2} - (x+2)(x-5) \cdot \frac{4}{x-5} &= (x+2)(x-5) \cdot \frac{2}{(x+2)(x-5)} \\
 \cancel{(x+2)}(x-5) \cdot \frac{3}{\cancel{x+2}} - (x+2)\cancel{(x-5)} \cdot \frac{4}{\cancel{x-5}} &= \cancel{(x+2)}\cancel{(x-5)} \cdot \frac{2}{\cancel{(x+2)}\cancel{(x-5)}}
 \end{aligned}$$

Now solve and simplify

$$\begin{aligned}
 3(x-5) - 4(x+2) &= 2 \\
 3x-15-4x-8 &= 2 \\
 x &= -25 \text{ Answer}
 \end{aligned}$$

Check your answer.

$$\frac{\frac{3}{x+2} - \frac{4}{x-5}}{2} = \frac{\frac{3}{-25+2} - \frac{4}{-25-5}}{2} = 0.003$$

$$\frac{3}{x^2 - 3x - 10} = \frac{4}{(-25)^2 - 3(-25) - 10} = 0.003$$

Example: A group of friends decided to pool together and buy a birthday gift that cost \$200. Later 12 of the friends decided not to participate any more. This meant that each person paid \$15 more than the original share. How many people were in the group to start?

Solution: Let x = the number of friends in the original group

Table 8.3:

| | Number of People | Gift Price | Share Amount |
|----------------|------------------|------------|-----------------|
| Original group | x | 200 | $\frac{200}{x}$ |
| Later group | $x - 12$ | 200 | $\frac{200}{x}$ |

Since each person's share went up by \$15 after 2 people refused to pay, we write the equation:

$$\frac{200}{x-12} = \frac{200}{x} + 15$$

Solve by clearing the fractions. Don't forget to check!

$$\begin{aligned} LCM &= x(x-12) \\ x(x-12) \cdot \frac{200}{x-12} &= x(x-12) \cdot \frac{200}{x} + x(x-12) \cdot 15 \\ \cancel{x(x-12)} \cdot \frac{200}{\cancel{x-12}} &= \cancel{x(x-12)} \cdot \frac{200}{\cancel{x}} + x(x-12) \cdot 15 \\ 200x &= 200(x-12) + 15x(x-12) \\ 200x &= 200x - 2400 + 15x^2 - 180x \\ 0 &= 15x^2 - 180x - 2400 \\ x &= 20, x = -8 \end{aligned}$$

The answer is 20 people. We discard the negative solution since it does not make sense in the context of this problem.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following videos. Note that there is not always a match between the number of the practice exercise in the videos and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. [CK-12 Basic Algebra: Solving Rational Equations](#) (12:57)

[CK-12 Basic Algebra: Two More Examples of Solving Rational Equations](#) (9:58)

Solve the following equations.

- $\frac{2x+1}{4} = \frac{x-3}{10}$



Figure 8.13: Solving Rational Equations ([Watch Youtube Video](#))

http://www.youtube.com/v/6eqgIZyXgK8?f=videosamp;c=ytapi-CK12Foundation-Flexrwikiimport-fg5akohk-0&d=AT8BNcsNZiISDLhsoSt-gqIO88HsQjpE1a8d1GxQnGDm&app=youtube_gdata



Figure 8.14: Two more examples of solving rational equations ([Watch Youtube Video](#))

http://www.youtube.com/v/u28w46QrE0Q?f=videosamp;c=ytapi-CK12Foundation-Flexrwikiimport-fg5akohk-0&d=AT8BNcsNZiISDLhsoSt-gqIO88HsQjpE1a8d1GxQnGDm&app=youtube_gdata

2. $\frac{4x}{x+2} = \frac{5}{9}$
3. $\frac{5}{3x-4} = \frac{2}{x+1}$
4. $\frac{7x}{x-5} = \frac{x+3}{x}$
5. $\frac{x}{x+3} - \frac{1}{x+4} = 0$
6. $\frac{3x^2+2x-1}{x^2-1} = -2$
7. $x + \frac{1}{x} = 2$
8. $-3 + \frac{1}{x+1} = \frac{2}{x}$
9. $\frac{1}{x} - \frac{x}{x-2} = 2$
10. $\frac{3}{2x-1} + \frac{2}{x+4} = 2$
11. $\frac{2x}{x-1} - \frac{x}{3x+4} = 3$
12. $\frac{x+1}{x-1} + \frac{x-4}{x+4} = 3$
13. $\frac{x}{x-2} + \frac{x}{x+3} = \frac{1}{x^2+x-6}$
14. $\frac{2}{x^2+4x+3} = 2 + \frac{x-2}{x+3}$
15. $\frac{1}{x+5} - \frac{1}{x-5} = \frac{1-x}{x+5}$
16. $\frac{x}{x^2-36} + \frac{1}{x-6} = \frac{1}{x+6}$
17. $\frac{2x}{3x+3} - \frac{1}{4x+4} = \frac{2}{x+1}$
18. $\frac{-x}{x-2} + \frac{3x-1}{x+4} = \frac{1}{x^2+2x-8}$
19. Juan jogs a certain distance and then walks a certain distance. When he jogs he averages seven miles per hour. When he walks, he averages 3.5 miles per hour. If he walks and jogs a total of six miles in a total of seven hours, how far does he jog and how far does he walk?
20. A boat travels 60 miles downstream in the same time as it takes it to travel 40 miles upstream. The boat's speed in still water is 20 miles/hour. Find the speed of the current.
21. Paul leaves San Diego driving at 50 miles/hour. Two hours later, his mother realizes that he forgot something and drives in the same direction at 70 miles/hour. How long does it take her to catch up to Paul?
22. On a trip, an airplane flies at a steady speed against the wind. On the return trip the airplane flies with the wind. The airplane takes the same amount of time to fly 300 miles against the wind as it takes to fly 420 miles with the wind. The wind is blowing at 30 miles/hour. What is the speed of the airplane when there is no wind?
23. A debt of \$420 is shared equally by a group of friends. When five of the friends decide not to pay, the share of the other friends goes up by \$25. How many friends were in the group originally?
24. A non-profit organization collected \$2,250 in equal donations from their members to share the cost of improving a park. If there were thirty more members, then each member could contribute \$20 less. How many members does this organization have?

Mixed Review

25. Divide $-2\frac{9}{10} \div -\frac{15}{8}$
26. Solve for g : $-1.5(-3\frac{4}{5} + g) = \frac{201}{20}$
27. Find the discriminant of $6x^2 + 3x + 4 = 0$ and determine the nature of the roots.
28. Simplify $\frac{6b}{2b+2} + 3$
29. Simplify $\frac{8}{2x-4} - \frac{5x}{x-5}$
30. Divide $(7x^2 + 16x - 10) \div (x + 3)$
31. Simplify $(n - 1) * 3n + 2)(n - 4)$

8.8 Surveys and Samples

One of the most important applications of statistics is collecting information. Statistical studies are done for many purposes:

- To find out more about animal behaviors;
- To determine which Presidential candidate is favored;
- To figure out what type of chip product is most popular;
- To determine the gas consumption of cars

In most cases except the Census, it is not possible to survey everyone in the population. So a **sample** is taken. It is essential that the sample is a **representative sample** of the population being studied. For example, if we are trying to determine the effect of a drug on teenage girls, it would make no sense to include males in our sample population, nor would it make sense to include women that are not teenagers.

The two types of sampling methods studied in this book are:

- Random Sampling
- Stratified Sampling

Random Samples

Random sampling is a method in which people are chosen “out of the blue.” In a true random sample, everyone in the population must have the same chance of being chosen. It is important that each person in the population has a chance of being picked.

Stratified Samples

Stratified sampling is a method actively seeking to poll people from many different backgrounds. The population is first divided into different categories (or **strata**) and the number of members in each category is determined.

Sample Size

In order to get lessen the chance of a biased result the sample size must be large enough. The larger the sample size is, the more precise the estimate is. However, the larger the sample size, the more expensive and time consuming the statistical study becomes.

Example 1: *For a class assignment you have been asked to find if students in your school are planning to attend university after graduating high-school. Students can respond with “yes”, “no” or “undecided”. How will you choose those you wish to interview if you want your results to be reliable?*

Solution:

The stratified sampling method would be the best option. By randomly picking a certain number of students in each grade, you will get the most accurate results.

Biased Samples

If the sample ends up with one or more sub-groups that are either over-represented or under-represented, then we say the sample is **biased**. We would not expect the results of a **biased sample** to represent the entire population, so it is important to avoid selecting a biased sample.

Some samples may deliberately seek a biased sample in order to obtain a particular viewpoint. For example, if a group of students were trying to petition the school to allow eating candy in the classroom, they may only survey students immediately before lunchtime when students are hungry. The practice of polling only those who you believe will support your cause is sometimes referred to as **cherry picking**.

Many surveys may have a **non-response bias**. In this case, a survey that is simply handed out gains few responses when compared to the number of surveys given out. People who are either too busy or simply not interested will be excluded from the results. Non-response bias may be reduced by conducting face-to-face interviews.

Self-selected respondents who tend to have stronger opinions on subjects than others and are more motivated to respond. For this reason *phone-in* and *online* polls also tend to be poor representations of the overall population. Even though it appears that both sides are responding, the poll may disproportionately represent extreme viewpoints from both sides, while ignoring more moderate opinions which may, in fact, be the majority view. Self selected polls are generally regarded as unscientific.

Example 2: *Determine whether the following survey is biased. Explain your reasoning.*

“Asking people shopping at a farmer’s market if they think locally grown fruit and vegetables are healthier than supermarket fruits and vegetables.”

Solution: This would be a biased sample because people shopping a farmer’s market are generally interested in buying fresher fruits and vegetables than a regular supermarket provides. The study can be improved by interviewing an equal number of people coming out of a supermarket, or by interviewing people in a more neutral environment such as the post office.

Biased Questions

Although your sample may be a good representation of the population, the way questions are worded in the survey can still provoke a biased result. There are several ways to identify a biased question.

1. They may use polarizing language, words and phrases that people associate with emotions.
 - (a) How much of your time do you waste on TV every week?
2. They may refer to a majority or to a supposed authority.
 - (a) Would you agree with the American Heart and Lung Association that smoking is bad for your health?
3. The question may be phrased so as to suggest the person asking the question already knows the answer to be true, or to be false.
 - (a) You wouldn’t want criminals free to roam the streets, would you?
4. The question may be phrased in ambiguous way (often with double negatives) which may confuse people.
 - (a) Do you disagree with people who oppose the ban on smoking in public places?

Design and Conduct a Survey

The method in which you design and conduct the survey is crucial to its accuracy. Surveys are a set of questions in which the sample answers. The data is compiled to form results, or findings. When designing a survey, be aware of the following recommendations.

1. Determine the goal of your survey, What question do you want to answer?
2. Identify the sample population. Who will you interview?
3. Choose an interviewing method, face-to-face interview, phone interview or self-administered paper survey or internet survey.
4. Conduct the interview and collect the information.
5. Analyze the results by making graphs and drawing conclusions.

Surveys can be conducted in several ways.

Face to face interviews

- ✓ Fewer misunderstood questions
- ✓ High response rate
- ✓ Additional information can be collected from respondents
 - Time consuming
 - Expensive
 - Can be biased based upon the attitude or appearance of surveyor

Self administered surveys

- ✓ Respondent can complete on their free time
- ✓ Less expensive than face to face interviews
- ✓ Anonymity causes more honest results
 - Lower response rate

Example: *Martha wants to construct a survey that shows which sports students at her school like to play the most.*

1. *List the goal of the survey.*
2. *What population sample should she interview?*
3. *How should she administer the survey?*
4. *Create a data collection sheet that she can use to record your results.*

Solution: The goal of the survey is to find the answer to the question: “Which sports do students at Martha’s school like to play the most?”

1. A sample of the population would include a random sample of the student population in Martha’s school. A good strategy would be to randomly select students (using dice or a random number generator) as they walk into an all school assembly.
2. Face-to-face interviews are a good choice in this case since the survey consists of only one question which can be quickly answered and recorded.
3. In order to collect the data to this simple survey Martha can design a data collection sheet such as the one below:

Display, Analyze, and Interpret Survey Data

This textbook has shown you several ways to display data. These graphs are also useful when displaying survey results. Survey data can be displayed as:

- A bar graph
- A histogram
- A pie chart
- A tally sheet
- A box-and-whisker plot
- A stem-and-leaf plot

The method in which you choose to display your data will depend upon your survey results and who you plan to present the data.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. [CK-12 Basic Algebra: Surveys and Samples](#) (12:09)



Figure 8.15: Surveys and Samples ([Watch Youtube Video](#))

http://www.youtube.com/v/nabw6v93Xns?f=videosamp;c=ytapi-CK12Foundation-Flexrwikiimport-fg5akohk-0&d=AT8BNcsNZiISDLhsoSt-gqIO88HsQjpE1a8d1GxQnGDm&app=youtube_gdata

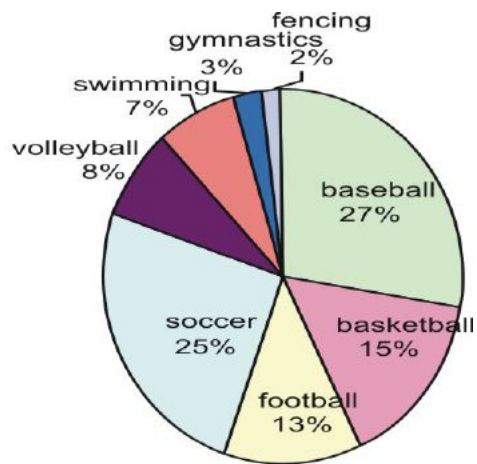
1. Explain the most common types of sampling methods. If you needed to survey a city about a new road project, which sampling method would you choose? Explain.
2. What is a *biased survey*? How can bias be avoided?
3. How are surveys conducted, according to this text? List one advantage and one disadvantage of each? List one additional method that can be used to conduct surveys.
4. What are some keys to recognizing biased questions? What could you do if you were presented with a biased question?
5. For a class assignment, you have been asked to find out how students get to school. Do they take public transportation, drive themselves, their parents drive them, use carpool or walk/bike. You decide to interview a sample of students. How will you choose those you wish to interview if you want your results to be reliable?
6. Comment on the way the following samples have been chosen. For the unsatisfactory cases, suggest a way to improve the sample choice.

- (a) You want to find whether wealthier people have more nutritious diets by interviewing people coming out of a five-star restaurant.
 - (b) You want to find if there is there a pedestrian crossing needed at a certain intersection by interviewing people walking by that intersection.
 - (c) You want to find out if women talk more than men by interviewing an equal number of men and women.
 - (d) You want to find whether students in your school get too much homework by interviewing a stratified sample of students from each grade level.
 - (e) You want to find out whether there should be more public busses running during rush hour by interviewing people getting off the bus.
 - (f) You want to find out whether children should be allowed to listen to music while doing their homework by interviewing a stratified sample of male and female students in your school.
7. Raoul wants to construct a survey that shows how many hours per week the average student at his school works.
- (a) List the goal of the survey.
 - (b) What population sample will he interview?
 - (c) How would he administer the survey?
 - (d) Create a data collection sheet that Raoul can use the record your results.
8. Raoul found that 30% of the students at his school are in 9th grade, 26% of the students are in the 10th grade, 24% of the students are in 11th grade and 20% of the students are in the 12th grade. He surveyed a total of 60 students using these proportions as a guide for the number of students he interviewed from each grade. Raoul recorded the following data.

Table 8.4:

| Grade Level | Record Number of Hours Worked | Total Number of Students |
|------------------------|---|--------------------------|
| 9 th grade | 0, 5, 4, 0, 0, 10, 5, 6, 0, 0, 2, 4, 0, 8, 0, 5, 7, 0 | 18 |
| 10 th grade | 6, 10, 12, 0, 10, 15, 0, 0, 8, 5, 0, 7, 10, 12, 0, 0 | 16 |
| 11 th grade | 0, 12, 15, 18, 10, 0, 0, 20, 8, 15, 10, 15, 0, 5 | 14 |
| 12 th grade | 22, 15, 12, 15, 10, 0, 18, 20, 10, 0, 12, 16 | 12 |

- (a) Construct a stem-and-leaf plot of the collected data.
 - (b) Construct a frequency table with bin size of 5.
 - (c) Draw a histogram of the data.
 - (d) Find the five number summary of the data and draw a box-and-whisker plot
9. The following pie chart displays data to a survey asking students the type of sports they enjoyed playing most. Make five conclusions regarding the survey results.










10. Melissa conducted a survey to answer the question “What sport do high school students like to watch on TV the most?” She collected the following information on her data collection sheet.

Table 8.5:

| Sport | Tally | |
|----------------|-------|-----------|
| Baseball | | 32 |
| Basketball | | 28 |
| Football | | 24 |
| Soccer | | 18 |
| Gymnastics | | 19 |
| Figure Skating | | 8 |
| Hockey | | 18 |
| | | Total 147 |

- Make a pie-chart of the results showing the percentage of people in each category.
 - Make a bar-graph of the results.
11. Samuel conducted a survey to answer the following question: “What is the favorite kind of pie of the people living in my town?” By standing in front of his grocery store, he collected the following information on his data collection sheet:

Table 8.6:

| Type of Pie | Tally | |
|------------------|--|-----------|
| Apple |  | 37 |
| Pumpkin |  | 13 |
| Lemon Meringue |  | 7 |
| Chocolate Mousse |  | 23 |
| Cherry |  | 4 |
| Chicken Pot Pie |  | 31 |
| Other |  | 7 |
| | | Total 122 |

1. Make a pie chart of the results showing the percentage of people in each category.
 2. Make a bar graph of the results.
12. Myra conducted a survey of people at her school to see “In which month does a person’s birthday fall?” She collected the following information in her data collection sheet:

Table 8.7:








| Month | Tally | |
|----------|---|----|
| January |  | 16 |
| February |  | 13 |
| March |  | 12 |
| April |  | 11 |
| May |  | 13 |
| June |  | 12 |
| July |  | 9 |

Table 8.7: (continued)

| Month | Tally | |
|-----------|-------|------------|
| August | | 7 |
| September | | 9 |
| October | | 8 |
| November | | 13 |
| December | | 13 |
| | | Total: 136 |

1. Make a pie chart of the results showing the percentage of people whose birthday falls in each month.
 2. Make a bar graph of the results.
13. Nam-Ling conducted a survey that answers the question “Which student would you vote for in your school’s elections?” She collected the following information:

Table 8.8:

| Candidate | 9 th graders | 10 th graders | 11 th graders | 12 th graders | Total |
|--------------------|-------------------------|--------------------------|--------------------------|--------------------------|-------|
| Susan Cho | | | | | 19 |
| Margarita Martinez | | | | | 31 |
| Steve Coogan | | | | | 16 |
| Solomon Dunning | | | | | 26 |
| Juan Rios | | | | | 28 |
| Total | 36 | 30 | 30 | 24 | 120 |

1. Make a pie chart of the results showing the percentage of people planning to vote for each candidate.
 2. Make a bar graph of the results.
14. Graham conducted a survey to find how many hours of TV teenagers watch each week in the United States. He collaborated with three friends that lived in different parts of the US and found the following information:

Table 8.9:

| Part of the country | Number of hours of TV watched per week | Total number of teens |
|---------------------|--|-----------------------|
| West Coast | 10, 12, 8, 20, 6, 0, 15, 18, 12, 22, 9, 5, 16, 12, 10, 18, 10, 20, 24, 8 | 20 |
| Mid West | 20, 12, 24, 10, 8, 26, 34, 15, 18, 6, 22, 16, 10, 20, 15, 25, 32, 12, 18, 22 | 20 |
| New England | 16, 9, 12, 0, 6, 10, 15, 24, 20, 30, 15, 10, 12, 8, 28, 32, 24, 12, 10, 10 | 20 |
| South | 24, 22, 12, 32, 30, 20, 25, 15, 10, 14, 10, 12, 24, 28, 32, 38, 20, 25, 15, 12 | 20 |

1. Make a stem-and-leaf plot of the data.
 2. Decide on an appropriate bin size and construct a frequency table.
 3. Make a histogram of the results.
 4. Find the five-number summary of the data and construct a box-and-whisker plot.
15. “What do students in your high-school like to spend their money on?”
 - (a) Which categories would you include on your data collection sheet?
 - (b) Design the data collection sheet that can be used to collect this information.
 - (c) Conduct the survey. This activity is best done as a group with each person contributing at least 20 results.
 - (d) Make a pie chart of the results showing the percentage of people in each category.
 - (e) Make a bar graph of the results.
 16. “What is the height of students in your class?”
 1. Design the data collection sheet that can be used to collect this information.
 2. Conduct the survey. This activity is best done as a group with each person contributing at least 20 results.
 3. Make a stem-and-leaf plot of the data.
 4. Decide on an appropriate bin size and construct a frequency table.
 5. Make a histogram of the results.
 6. Find the five-number summary of the data and construct a box-and-whisker plot.
 17. “How much allowance money do students in your school get per week?”
 1. Design the data collection sheet that can be used to collect this information,
 2. Conduct the survey. This activity is best done as a group with each person contributing at least 20 results.
 3. Make a stem-and-leaf plot of the data.
 4. Decide on an appropriate bin size and construct a frequency table.
 5. Make a histogram of the results.
 6. Find the five-number summary of the data and construct a box-and-whisker plot.
 18. Are the following statements biased?
 - (a) You want to find out public opinion on whether teachers get paid a sufficient salary by interviewing the teachers in your school.

- (b) You want to find out if your school needs to improve its communications with parents by sending home a survey written in English.
19. “What time do students in your school get up in the morning during the school week?”
1. Design the data collection sheet that can be used to collect this information.
 2. Conduct the survey. This activity is best done as a group with each person contributing at least 20 results.
 3. Make a stem-and-leaf plot of the data.
 4. Decide on an appropriate bin size and construct a frequency table.
 5. Make a histogram of the results.
 6. Find the five-number summary of the data and construct a box-and-whisker plot.

Mixed Review

20. Write the equation containing (8, 1) and (4, -6) in point-slope form.
- (a) What is the equation for the line perpendicular to this containing (0, 0)?
 - (b) What is the equation for the line parallel to this containing (4, 0)?
21. Classify $\sqrt[3]{64}$ according to the real number hierarchy.
22. A ferry traveled to its destination, 22 miles across the harbor. On the first voyage, the ferry took 45 minutes. On the return trip, the ferry encountered a head wind and its trip took one hour, ten minutes. Find the speed of the ferry and the speed of the wind.
23. Solve for a : $\frac{6a}{a-1} = \frac{7}{a+7}$
24. Simplify $\frac{7}{18x^2} \cdot \frac{9x}{14}$
25. Use long division to simplify $\frac{3w^3-6w^2-27w+54}{2w^2-4w-30}$
26. A hot air balloon rises 16 meters ever second.
- (a) Is this an example of a linear function, a quadratic function, or an exponential function? Explain.
 - (b) At four seconds the balloon is 68.5 meters from the ground. What was its beginning height?

8.9 Review

Define the following terms used in this chapter.

1. Inverse variation
2. Asymptotes
3. Hyperbola
4. Points of discontinuity
5. Least common multiple
6. Random sampling
7. Stratified sampling
8. Biased
9. Cherry picking
10. What quadrants are the branches of the hyperbola located if $k < 0$?

Are the following examples of *direct variation* or *inverse variation*?

11. The number of slices n people get from sharing one pizza.
12. The thickness of a phone book given n telephone numbers.

13. The amount of coffee n people receive from a single pot.
14. The total cost of pears given the nectarines cost \$0.99 per pound.

For each variation equation,

1. Translate the sentence into an inverse variation equation.
 2. Find k , the constant of variation.
 3. Find the unknown value.
15. y varies inversely as x . When $x = 5, y = \frac{2}{15}$. Find y when $x = -\frac{1}{2}$.
 16. y is inversely proportional to the square root of y . When $x = 16, y = 0.5625$. Find y when $x = \frac{1}{8}$.
 17. Habitat for Humanity uses volunteers to build houses. The number of days it takes to build a house varies inversely as the number of volunteers. It takes eight days to build a house with twenty volunteers. How many days will it take sixteen volunteers to complete the same job?
 18. The Law of the Fulcrum states the distance you sit to balance a seesaw varies inversely as your weight. If Gary weighs 20.43 kg sits 1.8 meters from the fulcrum, how far would Shelley sit, assuming she weighs 36.32 kilograms?

For each function,

1. Graph it on a Cartesian plane.
2. State its domain and range.
3. Determine any horizontal and/or vertical asymptotes the function may have.

19. $y = \frac{4}{x}$
20. $f(x) = \frac{2}{4-x}$
21. $g(x) = \frac{-1}{x+1}$
22. $y = \frac{6}{3x+1} - 2$
23. $f(x) = \frac{3}{x} - 5$

Perform the indicated operation.

24. $\frac{5a}{6} - \frac{5b}{4b}$
25. $\frac{4}{3m} + \frac{4m}{5}$
26. $\frac{3x}{2xy} + \frac{4}{3}$
27. $\frac{2}{5n-2} + \frac{2n}{2}$
28. $\frac{2x+1}{3x+9} - \frac{x+5}{3x+9}$
29. $\frac{5m+n}{30n^4} - \frac{4m+n}{30n^4}$
30. $\frac{r-6}{4r^2-12r+8} - \frac{r+6}{4r^2-12r+8}$
31. $\frac{2}{16x^3y^2} + \frac{x-2y}{16x^3y^2}$
32. $\frac{n-6}{n+2} + \frac{2n}{5}$
33. $\frac{8}{4} - \frac{x+5}{x+8}$
34. $\frac{3x}{2(x+1)} + \frac{6}{7x-6}$
35. $\frac{11}{8} \cdot \frac{20x^2}{2}$
36. $\frac{17r}{16} \cdot \frac{7r^4}{2}$
37. $\frac{15}{18} \cdot \frac{14}{17t}$
38. $\frac{2(b-11)}{14b} \cdot \frac{b+5}{(b+5)(b-11)}$

39. $\frac{17w^2}{w+4} \cdot \frac{18(w+4)}{17w^2(w-9)}$
40. $\frac{10s^3-30s^2}{30s^2-10s^3} \cdot \frac{s-3}{8}$
41. $\frac{1}{f-5} \div \frac{f+3}{f^2+6f+9}$
42. $\frac{(a+8)(a+3)}{4(a+3)} \div \frac{10a^2(a+10)}{4}$
43. $\frac{1}{(h-10)(h+7)} \div \frac{(h-4)}{4h(h-10)}$
44. $\frac{2(5x-8)}{4x^2(8-5x)} \div \frac{6}{4x^2}$
45. $\frac{2(q-7)}{40q(q+1)} \div \frac{1}{40q(q+1)}$

Solve each equation.

46. $\frac{3}{3x^2} = \frac{1}{x} + \frac{1}{3x^2}$
47. $\frac{2}{5x^2} = -\frac{12}{x-3}$
48. $\frac{7x}{x-6} = \frac{3}{4x+16}$
49. $\frac{4}{c-2} = \frac{3}{c+4}$
50. $\frac{d-4}{4d^2} = \frac{1}{4d^2} + \frac{1}{4d}$
51. $\frac{1}{2} = \frac{2z-12}{z} - \frac{z+1}{4z}$
52. $\frac{1}{n} = \frac{1}{n^2} + \frac{6}{n}$
53. $\frac{1}{2a} = \frac{1}{2a^2} + \frac{1}{a}$
54. $\frac{k+4}{k^2} = \frac{5k-30}{3k^2} + \frac{1}{3k^2}$
55. It takes Jayden seven hours to paint a room. Andie can do it in five hours. How long will it take to paint the room if Jayden and Andie work together?
56. Kiefer can mow the lawn in 4.5 hours. Brad can do it in two hours. How long will it take if they worked together?
57. Melissa can mop the floor in 1.75 hours. With Brad's help, it took only 50 minutes. How long would it take Brad to mop it alone?
58. Working together, it took Frankie and Ricky eight hours to frame a room. It would take Frankie fifteen hours doing it alone. How long would it take Ricky to do it alone?
59. A parallel circuit has $R_1 = 50\Omega$ and $R_t = 16\Omega$. Find R_2 .
60. A parallel circuit has $R_1 = 6\Omega$ and $R_2 = 9\Omega$. Find R_T .
61. A series circuit has $R_1 = 200\Omega$ and $R_t = 300\Omega$. Find R_2 .
62. A series circuit has $R_1 = 11\Omega$ and $R_2 = 25\Omega$. Find R_2 .
63. Write the formula for the total resistance for a parallel circuit with three individual resistors.
64. What would be the bias in this situation? To determine the popularity of a new snack chip, a survey is conducted by asking seventy-five people walking down the chip aisle in a supermarket which chip they prefer.
65. Describe the steps necessary to design and conduct a survey.
66. You need to survey potential voters for an upcoming school board election. Design a survey with at least three questions you could ask. How will you plan to conduct the survey?
67. What is a *stratified sample*? Name one case that a stratified sample would be more beneficial.