

BMET Geometry

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Cifarelli Fiori Gloag Greenberg Jordan Sconyers

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Contents

1	Basics of Geometry	1
1.1	Points, Lines, and Planes	1
1.2	Segments and Distance	9
1.3	Angles and Measurement	18
1.4	Midpoints and Bisectors	29
1.5	Angle Pairs	38
1.6	Classifying Polygons	45
1.7	Chapter 1 Review	54
1.8	Study Guide	57
2	Coordinate Geometry	61
2.1	Vocabulary Self-Rating	61
2.2	Distance and Midpoint	62
2.3	Parallel and Perpendicular	69
2.4	Equation of a Circle	81
2.5	Translating and Reflecting	86
2.6	Rotating	92
3	Triangles and Congruence	99
3.1	Triangle Sums	99
3.2	Congruent Figures	112
3.3	Triangle Congruence using SSS and SAS	119
3.4	Triangle Congruence using ASA, AAS, and HL	133
3.5	Isosceles and Equilateral Triangles	146
3.6	Chapter 4 Review	157
3.7	Study Guide	159
4	Right Triangle Trigonometry	162
4.1	The Pythagorean Theorem	162

4.2	Converse of the Pythagorean Theorem	171
4.3	Using Similar Right Triangles	176
4.4	Special Right Triangles	185
4.5	Tangent, Sine and Cosine	194
4.6	Inverse Trigonometric Ratios	202
4.7	Chapter 8 Review	210
4.8	Study Guide	214
5	Circles	217
5.1	Parts of Circles & Tangent Lines	217
5.2	Properties of Arcs	229
5.3	Properties of Chords	237
5.4	Inscribed Angles	248
5.5	Angles of Chords, Secants, and Tangents	259
5.6	Segments of Chords, Secants, and Tangents	270
5.7	Extension: Writing and Graphing the Equations of Circles	282
5.8	Chapter 9 Review	286
5.9	Study Guide	288
6	Perimeter and Area	293
6.1	Triangles and Parallelograms	293
6.2	Trapezoids, Rhombi, and Kites	302
6.3	Areas of Similar Polygons	311
6.4	Circumference and Arc Length	315
6.5	Areas of Circles and Sectors	322
6.6	Chapter 10 Review	331
6.7	Study Guide	334
7	Surface Area and Volume	337
7.1	Exploring Solids	337
7.2	Surface Area of Prisms and Cylinders	347
7.3	Surface Area of Pyramids and Cones	355
7.4	Volume of Prisms and Cylinders	363
7.5	Volume of Pyramids and Cones	371
7.6	Surface Area and Volume of Spheres	380
7.7	Extension: Exploring Similar Solids	386
7.8	Chapter 11 Review	392
7.9	Study Guide	394

8 Analyzing Conic Sections	398
8.1 Introduction to Conic Sections	398
8.2 Circles and Ellipses	406
8.3 Parabolas	425
8.4 Hyperbolas	441
8.5 General Algebraic Forms	454

Chapter 1

Basics of Geometry

In this chapter, students will learn about the building blocks of geometry. We will start with what the basic terms: point, line and plane. From here, students will learn about segments, midpoints, angles, bisectors, angle relationships, and how to classify polygons.

1.1 Points, Lines, and Planes

Learning Objectives

- Understand the terms *point*, *line*, and *plane*.
- Draw and label terms in a diagram.

Review Queue

1. List and draw pictures of five geometric figures you are familiar with.
2. What shape is a yield sign?



3. Solve the algebraic equations.

(a) $4x - 7 = 29$

(b) $-3x + 5 = 17$

Know What? Geometry is everywhere. Remember these wooden blocks that you played with as a kid? If you played with these blocks, then you have been “studying” geometry since you were a child.

How many sides does the octagon have? What is something in-real life that is an octagon?

Geometry: The study of shapes and their spatial properties.



Building Blocks

Point: An exact location in space.

A point describes a **location**, but has no size. Examples:



Table 1.1:

<i>Label It</i>	<i>Say It</i>
A	point A

Line: Infinitely many points that extend forever in both directions.

A line has **direction** and **location** is always straight.



Table 1.2:

<i>Label It</i>	<i>Say It</i>
line g	line g
\overleftrightarrow{PQ}	line PQ
\overleftrightarrow{QP}	line QP

Plane: Infinitely many intersecting lines that extend forever in all directions.

Think of a plane as a huge sheet of paper that goes on forever.

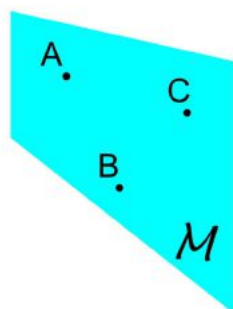


Table 1.3:

<i>Label It</i>	<i>Say It</i>
Plane M	Plane M
Plane ABC	Plane ABC

Example 1: What best describes San Diego, California on a globe?

- A. point
- B. line
- C. plane

Solution: A city is usually labeled with a dot, or point, on a globe.

Example 2: What best describes the surface of a movie screen?

- A. point
- B. line
- C. plane

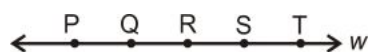
Solution: The surface of a movie screen is most like a plane.

Beyond the Basics Now we can use **point**, **line**, and **plane** to define new terms.

Space: The set of all points expanding in *three* dimensions.

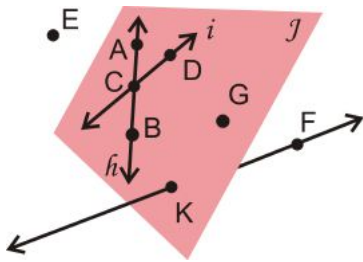
Think back to the plane. It goes up and down, and side to side. If we add a third direction, we have space, something three-dimensional.

Collinear: Points that lie on the same line.



P, Q, R, S , and T are collinear because they are all on line w . If a point U was above or below line w , it would be **non-collinear**.

Coplanar: Points and/or lines within the same plane.



Lines h and i and points A, B, C, D, G , and K are **coplanar** in Plane \mathcal{J} .

Line \overleftrightarrow{KF} and point E are **non-coplanar** with Plane \mathcal{J} .

Example 3: Use the picture above to answer these questions.

- List another way to label Plane \mathcal{J} .
- List another way to label line h .
- Are K and F collinear?
- Are E, B and F coplanar?

Solution:

- Plane BDG . Any combination of three coplanar points that are not collinear would be correct.
- \overleftrightarrow{AB} . Any combination of two of the letters A, C or B would also work.
- Yes
- Yes

Endpoint: A point at the end of a line.

Line Segment: A line with two endpoints. Or, a line that stops at both ends.

Line segments are labeled by their endpoints. Order does not matter.



Table 1.4:

<i>Label It</i>	<i>Say It</i>
\overline{AB}	Segment AB
\overline{BA}	Segment BA

Ray: A line with one endpoint and extends forever in the other direction.

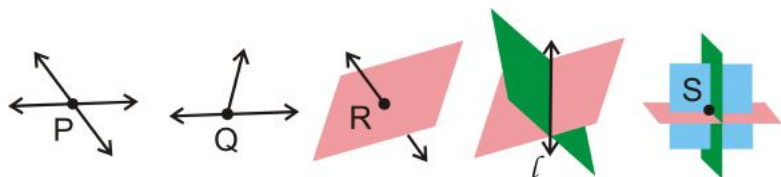
A ray is labeled by its endpoint and one other point on the line. For rays, order matters. When labeling, put endpoint under the side **WITHOUT** an arrow.



Table 1.5:

<i>Label It</i>	<i>Say It</i>
\overrightarrow{CD}	Ray CD
\overleftarrow{DC}	Ray CD

Intersection: A point or line where lines, planes, segments or rays cross.

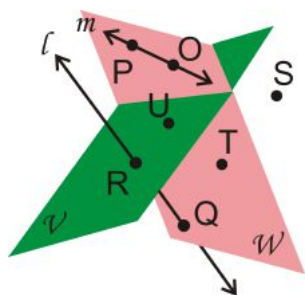


Example 4: What best describes a straight road connecting two cities?

- A. ray
- B. line
- C. segment
- D. plane

Solution: The straight road connects two cities, which are like endpoints. The best term is segment, or C.

Example 5: Answer the following questions about the picture to the right.



- a) Is line l coplanar with Plane \mathcal{V} or \mathcal{W} ?
- b) Are R and Q collinear?
- c) What point is non-coplanar with either plane?
- d) List three coplanar points in Plane \mathcal{W} .

Solution:

- a) No.
- b) Yes.
- c) S
- d) Any combination of P, O, T and Q would work.

Further Beyond This section introduces a few basic postulates.

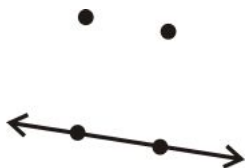
Postulates: Basic rules of geometry. *We can assume that all postulates are true.*

Theorem: A statement that is *proven true* using postulates, definitions, and previously proven theorems.

Postulate 1-1: There is exactly one (straight) line through any two points.

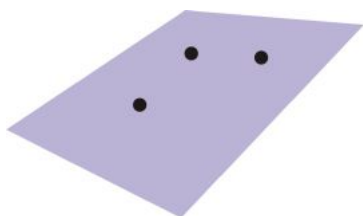


Investigation 1-1: Line Investigation

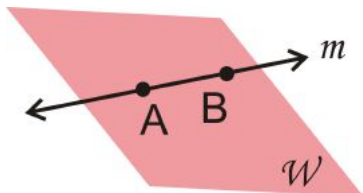


1. Draw two points anywhere on a piece of paper.
2. Use a ruler to connect these two points.
3. How many lines can you draw to go through these two points?

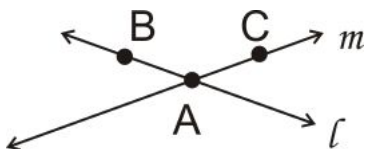
Postulate 1-2: One plane contains any three non-collinear points.



Postulate 1-3: A line with points in a plane is also in that plane.

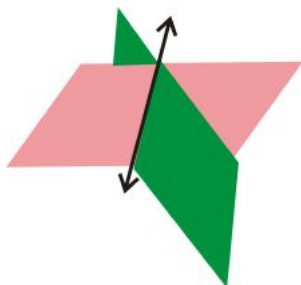


Postulate 1-4: The intersection of two lines will be one point.



Lines l and m intersect at point A.

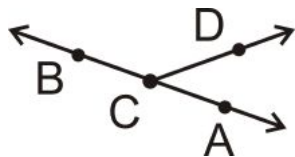
Postulate 1-5: The intersection of two planes is a line.



When making geometric drawings, you need to be clear and label all points and lines.

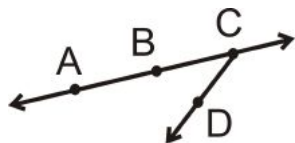
Example 6a: Draw and label the intersection of line \overleftrightarrow{AB} and ray \overrightarrow{CD} at point C .

Solution: It does not matter where you put A or B on the line, nor the direction that \overrightarrow{CD} points.



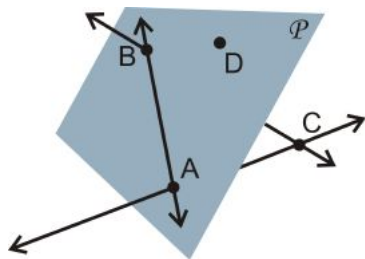
Example 6b: Redraw Example 6a, so that it looks different but is still true.

Solution:



Example 7: Describe the picture below using the geometric terms you have learned.

Solution: \overleftrightarrow{AB} and D are coplanar in Plane \mathcal{P} , while \overleftrightarrow{BC} and \overleftrightarrow{AC} intersect at point C which is non-coplanar.



Know What? Revisited The octagon has 8 sides. In Latin, “octo” or “octa” means 8, so octagon, literally means “8-sided figure.” An octagon in real-life would be a stop sign.

Review Questions

- Questions 1-5 are similar to Examples 6a and 6b.
- Questions 6-8 are similar to Examples 3 and 5.
- Questions 9-12 are similar to Examples 1, 2, and 4.
- Questions 13-16 are similar to Example 7.
- Questions 17-25 use the definitions and postulates learned in this lesson.

For questions 1-5, draw and label an image to fit the descriptions.

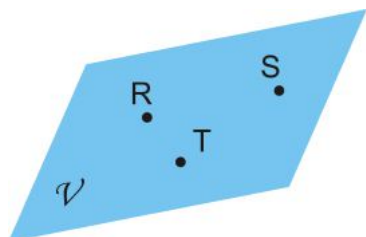
1. \overrightarrow{CD} intersecting \overline{AB} and Plane \mathcal{P} containing \overline{AB} but not \overrightarrow{CD} .
2. Three collinear points A , B , and C and B is also collinear with points D and E .
3. \overrightarrow{XY} , \overrightarrow{XZ} , and \overrightarrow{XW} such that \overrightarrow{XY} and \overrightarrow{XZ} are coplanar, but \overrightarrow{XW} is not.
4. Two intersecting planes, \mathcal{P} and \mathcal{Q} , with \overline{GH} where G is in plane \mathcal{P} and H is in plane \mathcal{Q} .
5. Four non-collinear points, I , J , K , and L , with line segments connecting all points to each other.
6. Name this line in five ways.



7. Name the geometric figure in three different ways.

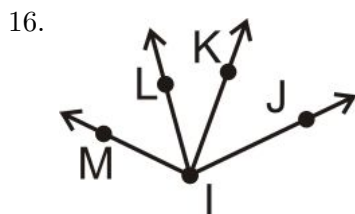
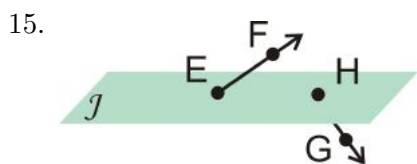
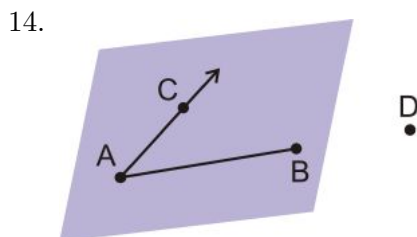
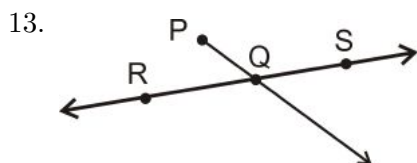


8. Name the geometric figure below in two different ways.



9. What is the best possible geometric model for a soccer field? Explain your answer.
10. List two examples of where you see rays in real life.
11. What type of geometric object is the intersection of a line and a plane? Draw your answer.
12. What is the difference between a postulate and a theorem?

For 13-16, use geometric notation to explain each picture in as much detail as possible.



For 17-25, determine if the following statements are true or false.

17. Any two points are collinear.

18. Any three points determine a plane.
19. A line is to two rays with a common endpoint.
20. A line segment is infinitely many points between two endpoints.
21. A point takes up space.
22. A line is one-dimensional.
23. Any four points are coplanar.
24. \overrightarrow{AB} could be read “ray AB ” or “ray BA .”
25. \overleftrightarrow{AB} could be read “line AB ” or “line BA .”

Review Queue Answers

1. Examples could be triangles, squares, rectangles, lines, circles, points, pentagons, stop signs (octagons), boxes (prisms), or dice (cubes).
2. A yield sign is a triangle with equal sides.
3. (a) $4x - 7 = 29$
 $4x = 36$
 $x = 9$
 (b) $-3x + 5 = 17$
 $-3x = 12$
 $x = -4$

1.2 Segments and Distance

Learning Objectives

- Use the ruler postulate.
- Use the segment addition postulate.
- Plot line segments on the $x - y$ plane.

Review Queue

1. Draw a line segment with endpoints C and D .
2. How would you label the following figure? List 2 different ways.

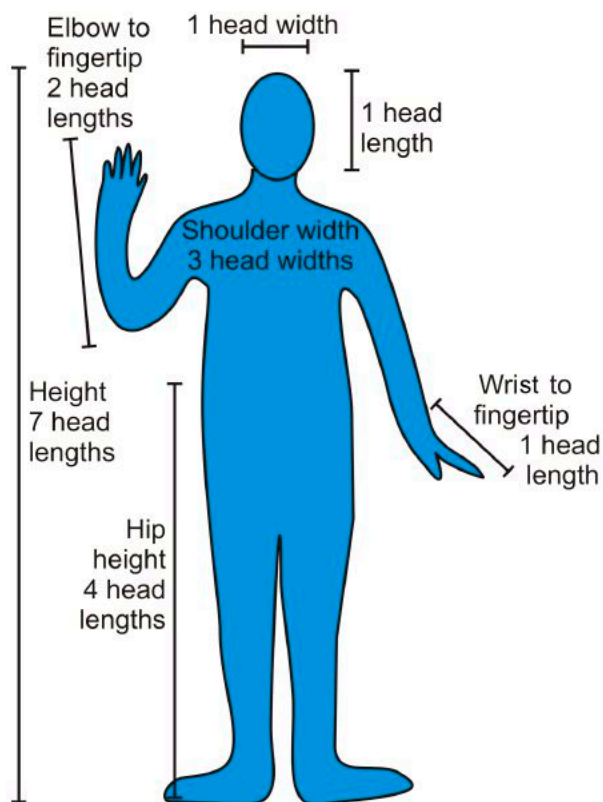


3. Draw three collinear points and a fourth that is coplanar.
4. Plot the following points on the $x - y$ plane.
 - (a) $(3, -3)$
 - (b) $(-4, 2)$
 - (c) $(0, -7)$
 - (d) $(6, 0)$

Know What? The average adult human body can be measured in “heads.” For example, the average human is 7-8 heads tall. When doing this, each person uses their own head to measure their own body. Other measurements are in the picture to the right.

See if you can find the following measurements:

- The length from the wrist to the elbow
- The length from the top of the neck to the hip
- The width of each shoulder



Measuring Distances

Distance: The length between two points.

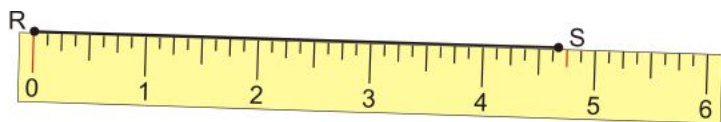
Measure: To determine how far apart two geometric objects are.

The most common way to measure distance is with a ruler. In this text we will use inches and centimeters.

Example 1: Determine how long the line segment is, in inches. Round to the nearest quarter-inch.

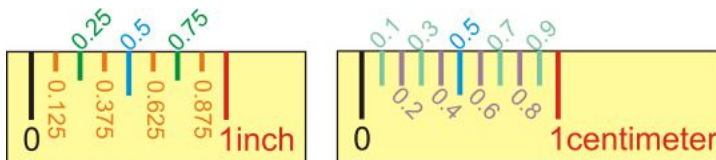


Solution: To measure this line segment, it is very important to line up the “0” with the one of the endpoints. DO NOT USE THE EDGE OF THE RULER.



From this ruler, it looks like the segment is 4.75 inches (in) long.

Inch-rulers are usually divided up by eighth-inch (or 0.125 in) segments. Centimeter rulers are divided up by tenth-centimeter (or 0.1 cm) segments.



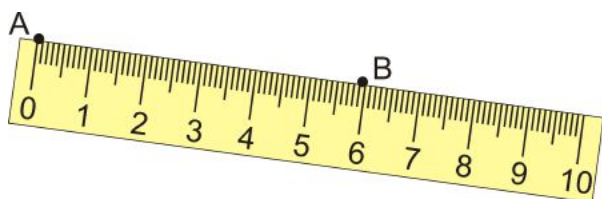
The two rulers above are **NOT DRAWN TO SCALE**, which means that the measured length is not the distance apart that it is labeled.

Example 2: Determine the measurement between the two points to the nearest tenth of a centimeter.

A.

B.

Solution: Even though there is no line segment between the two points, we can still measure the distance using a ruler.



It looks like the two points are 6 centimeters (cm) apart.

NOTE: We label a line segment, \overline{AB} and the **distance** between A and B is shown below. m means measure. The two can be used interchangeably.

Table 1.6:

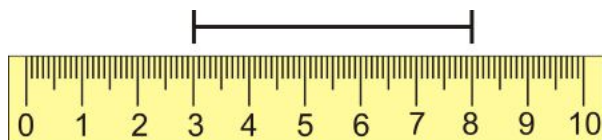
Label It	Say It
\overline{AB}	The distance between A and B
$m\overline{AB}$	The measure of \overline{AB}

Ruler Postulate

Ruler Postulate: The distance between two points is the absolute value of the difference between the numbers shown on the ruler.

The ruler postulate implies that you do not need to start measuring at “0”, as long as you subtract the first number from the second. “Absolute value” is used because **distance is always positive**.

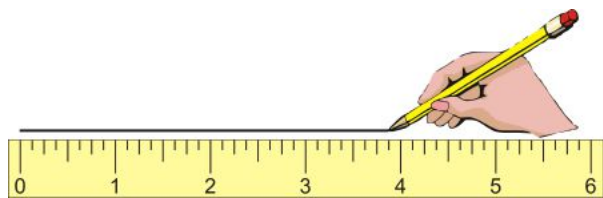
Example 3: What is the distance marked on the ruler below? The ruler is in centimeters.



Solution: Subtract one endpoint from the other. The line segment spans from 3 cm to 8 cm. $|8-3| = |5| = 5$. The line segment is 5 cm long. Notice that you also could have done $|3-8| = |-5| = 5$.

Example 4: Draw \overline{CD} , such that $CD = 3.825$ in.

Solution: To draw a line segment, start at “0” and draw a segment to 3.825 in.



Put points at each end and label.



Segment Addition Postulate

First, in the picture below, B is between A and C . As long as B is *anywhere on the segment*, it can be considered to be *between* the endpoints.



Segment Addition Postulate: If A , B , and C are collinear and B is between A and C , then $AB + BC = AC$.

For example, if $AB = 5\text{ cm}$ and $BC = 12\text{ cm}$, then AC must equal $5 + 12$ or 17 cm (in the picture above).

Example 5: Make a sketch of \overline{OP} , where Q is between O and P .

Solution: Draw \overline{OP} first, then place Q on the segment.



Example 6: In the picture from Example 5, if $OP = 17$ and $QP = 6$, what is OQ ?

Solution: Use the Segment Additional Postulate.

$$\begin{aligned} OQ + QP &= OP \\ OQ + 6 &= 17 \\ OQ &= 17 - 6 \\ OQ &= 11 \end{aligned}$$

Example 7: Make a sketch of: S is between T and V . R is between S and T . $TR = 6$, $RV = 23$, and $TR = SV$.

Solution: Interpret the first sentence first: S is between T and V .



Then add in what we know about R : It is between S and T . Put markings for $TR = SV$.



Example 8: Find SV , TS , RS and TV from Example 7.

Solution:

For SV : It is equal to TR , so $SV = 6\text{ cm}$.

$$\begin{array}{lll}\text{For } RS : & RV = RS + SV & \text{For } TS : TS = TR + RS & \text{For } TV : TV = TR + RS + SV \\ & 23 = RS + 6 & TS = 6 + 17 & TV = 6 + 17 + 6 \\ & RS = 17\text{ cm} & TS = 23\text{ cm} & TV = 29\text{ cm}\end{array}$$

Example 9: Algebra Connection For \overline{HK} , suppose that J is between H and K . If $HJ = 2x + 4$, $JK = 3x + 3$, and $KH = 22$, find x .

Solution: Use the Segment Addition Postulate.

$$\begin{aligned}HJ + JK &= KH \\ (2x + 4) + (3x + 3) &= 22 \\ 5x + 7 &= 22 \\ 5x &= 15 \\ x &= 3\end{aligned}$$

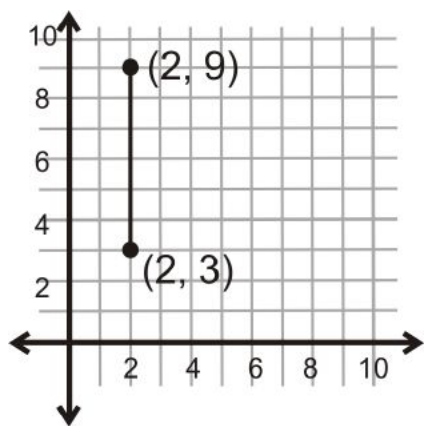
Distances on a Grid

You can now find the distances between points in the $x - y$ plane if the lines are horizontal or vertical.

If the line is vertical, find the change in the y -coordinates.

If the line is horizontal, find the change in the x -coordinates.

Example 10: What is the distance between the two points shown below?

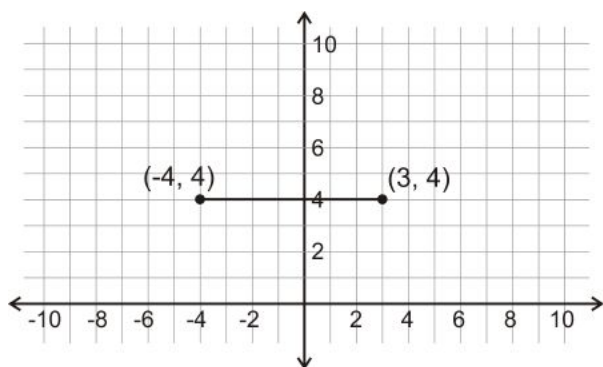


Solution: Because this line is vertical, look at the change in the y -coordinates.

$$|9 - 3| = |6| = 6$$

The distance between the two points is 6 units.

Example 11: What is the distance between the two points shown below?



Solution: Because this line is horizontal, look at the change in the x -coordinates.

$$|(-4) - 3| = |-7| = 7$$

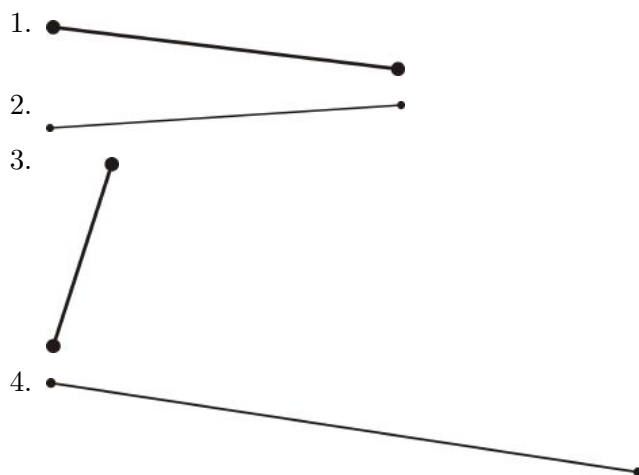
The distance between the two points is 7 units.

Know What? Revisited The length from the wrist to the elbow is one head, the length from the top of the neck to the hip is two heads, and the width of each shoulder one head width.

Review Questions

- Questions 1-8 are similar to Examples 1 and 2.
- Questions 9-12 are similar to Example 3.
- Questions 13-17 are similar to Examples 5 and 6.
- Questions 18 and 19 are similar to Example 7 and 8.
- Questions 20 and 21 are similar to Example 9.
- Questions 22-26 are similar to Examples 10 and 11.

For 1-4, find the length of each line segment in inches. Round to the nearest $\frac{1}{8}$ of an inch.



For 5-8, find the distance between each pair of points in centimeters. Round to the nearest tenth.

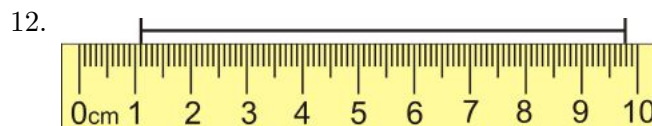
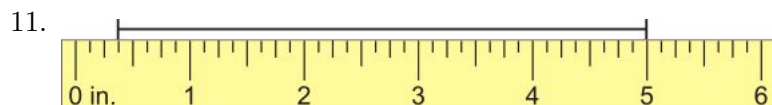
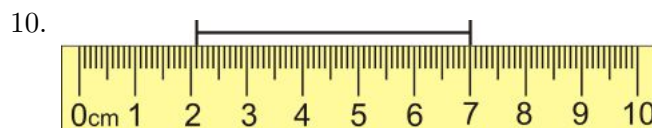
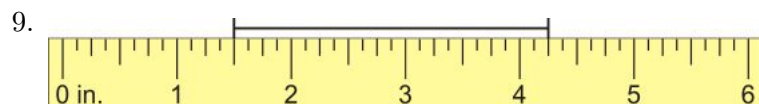
5. ●

6. ● ●

7. ● ●

8. ● ●

For 9-12, use the ruler in each picture to determine the length of the line segment.



13. Make a sketch of \overline{BT} , with A between B and T .
14. If O is in the middle of \overline{LT} , where exactly is it located? If $LT = 16\text{ cm}$, what is LO and OT ?
15. For three collinear points, A between T and Q .
 - (a) Draw a sketch.
 - (b) Write the Segment Addition Postulate.
 - (c) If $AT = 10\text{ in}$ and $AQ = 5\text{ in}$, what is TQ ?
16. For three collinear points, M between H and A .
 - (a) Draw a sketch.

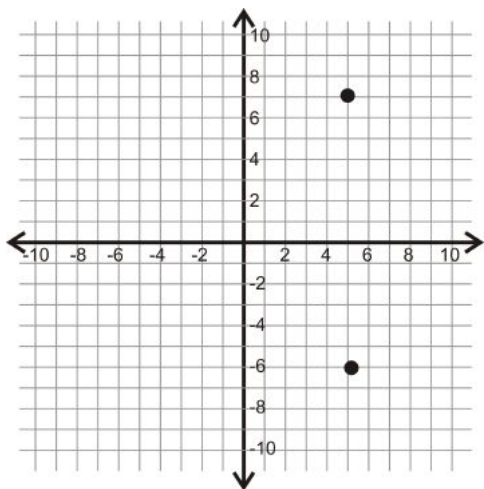
- (b) Write the Segment Addition Postulate.
 (c) If $HM = 18\text{ cm}$ and $HA = 29\text{ cm}$, what is AM ?
17. For three collinear points, I between M and T .
- (a) Draw a sketch.
 (b) Write the Segment Addition Postulate.
 (c) If $IT = 6\text{ cm}$ and $MT = 25\text{ cm}$, what is AM ?
18. Make a sketch that matches the description: B is between A and D . C is between B and D . $AB = 7\text{ cm}$, $AC = 15\text{ cm}$, and $AD = 32\text{ cm}$. Find BC , BD , and CD .
19. Make a sketch that matches the description: E is between F and G . H is between F and E . $FH = 4\text{ in}$, $EG = 9\text{ in}$, and $FH = HE$. Find FE , HG , and FG .

For 20 and 21, Suppose J is between H and K . Use the Segment Addition Postulate to solve for x . Then find the length of each segment.

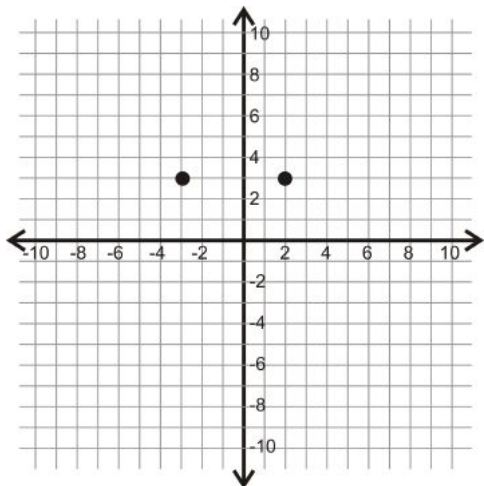
20. $HJ = 4x + 9$, $JK = 3x + 3$, $KH = 33$
 21. $HJ = 5x - 3$, $JK = 8x - 9$, $KH = 131$

For 23-26, determine the vertical or horizontal distance between the two points.

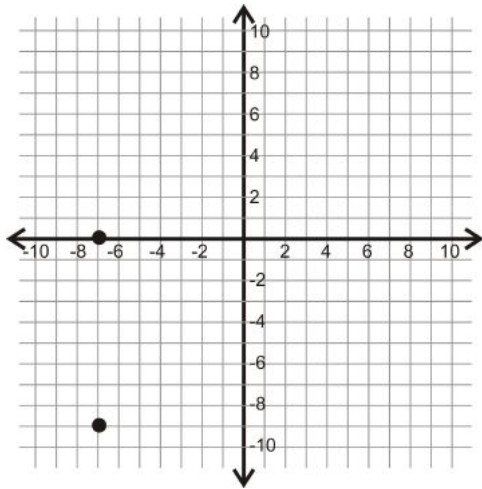
23.



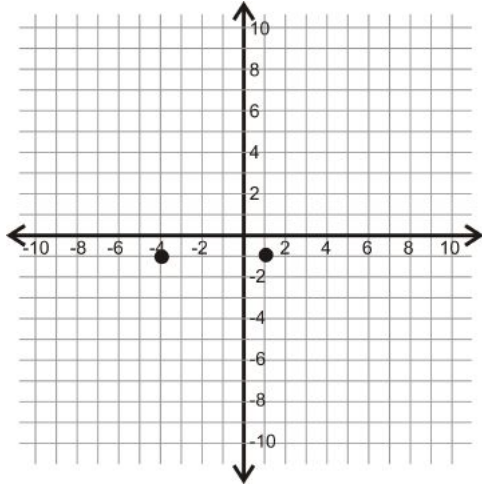
24.



25.



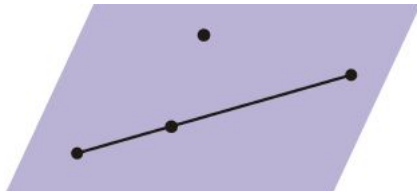
26.



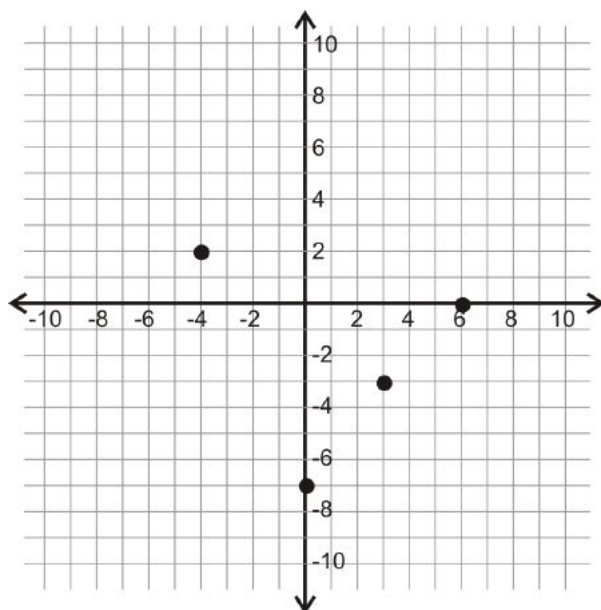
Review Queue Answers

1. 

2. line l , \overline{MN}

3. 

4.



1.3 Angles and Measurement

Learning Objectives

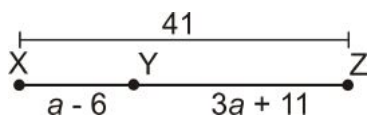
- Classify angles.
- Apply the Protractor Postulate and the Angle Addition Postulate.

Review Queue

1. Label the following geometric figure. What is it called?



2. Find a , XY and YZ .



3. B is between A and C on \overline{AC} . If $AB = 4$ and $BC = 9$, what is AC ?

Know What? Back to the building blocks. Every block has its own dimensions, angles and measurements. Using a protractor, find the measure of the three outlined angles in the “castle” to the right.



Two Rays = One Angle

In #1 above, the figure was a ray. It is labeled \overrightarrow{AB} , with the arrow over the point that is NOT the endpoint. When two rays have the same endpoint, an angle is created.

Angle: When two rays have the same endpoint.

Vertex: The common endpoint of the two rays that form an angle.

Sides: The two rays that form an angle.

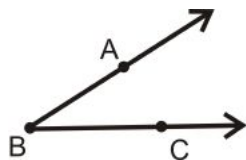
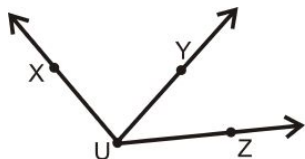


Table 1.7:

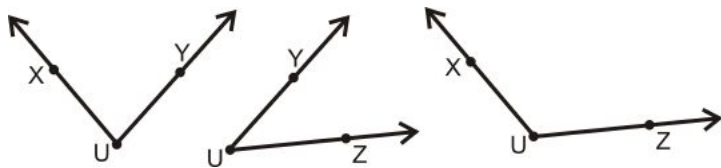
<i>Label It</i>	<i>Say It</i>
$\angle ABC$	Angle ABC
$\angle CBA$	Angle CBA

The vertex is B and the sides are \overrightarrow{BA} and \overrightarrow{BC} . **Always use three letters to name an angle, \angle SIDE-VERTEX-SIDE.**

Example 1: How many angles are in the picture below? Label each one.



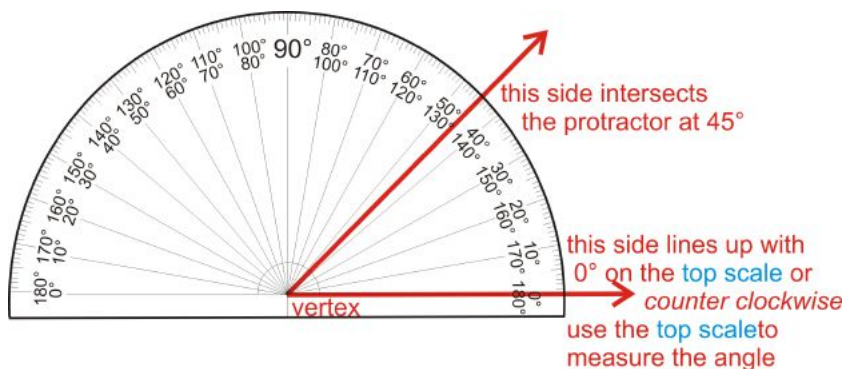
Solution: There are three angles with vertex U . It might be easier to see them all if we separate them.



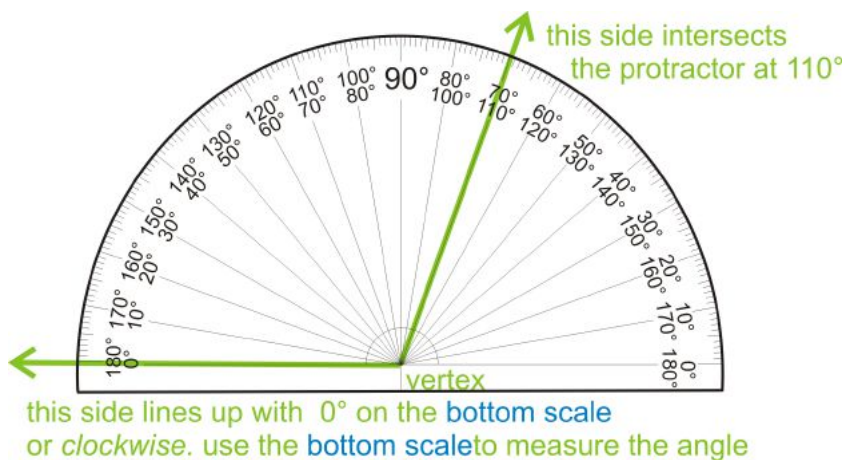
So, the three angles can be labeled, $\angle XUY$ (or $\angle YUX$), $\angle YUZ$ (or $\angle ZUY$), and $\angle XUZ$ (or $\angle ZUX$).

Protractor Postulate

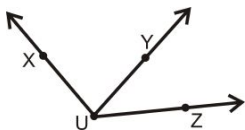
We measure a line segment's *length* with a ruler. Angles are measured with something called a **protractor**. A protractor is a measuring device that measures how “open” an angle is. Angles are measured in degrees, and labeled with a $^\circ$ symbol.



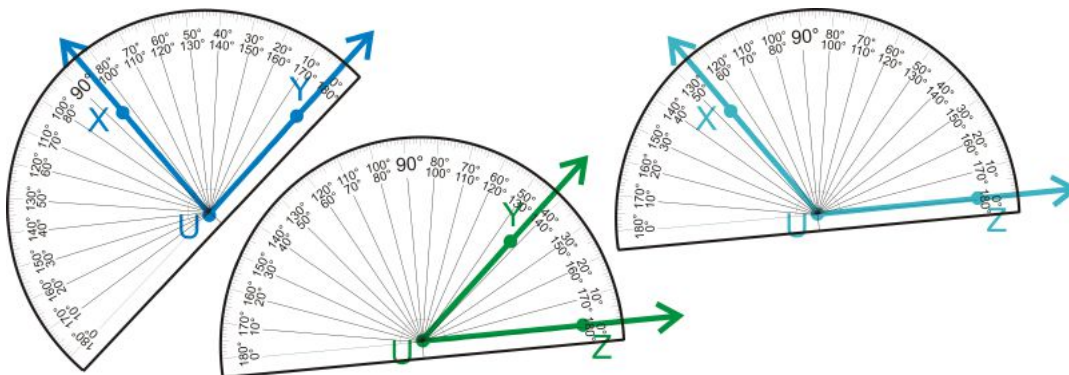
There are two sets of measurements, one starting on the left and the other on the right side of the protractor. Both go around from 0° to 180° . When measuring angles, always line up one side with 0° , and see where the other side hits the protractor. The vertex lines up in the middle of the bottom line.



Example 2: Measure the three angles from Example 1, using a protractor.



Solution: Just like in Example 1, it might be easier to measure these three angles if we separate them.



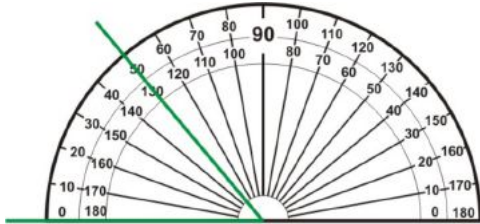
With measurement, we put an m in front of the \angle sign to indicate measure. So, $m\angle XUY = 84^\circ$, $m\angle YUZ = 42^\circ$ and $m\angle XUZ = 126^\circ$.

Just like the Ruler Postulate for line segments, there is a Protractor Postulate for angles.

Protractor Postulate: For every angle there is a number between 0° and 180° that is the measure of the angle. The angle's measure is the difference of the degrees where the sides of the angle intersect the protractor. *For now, angles are always positive.*

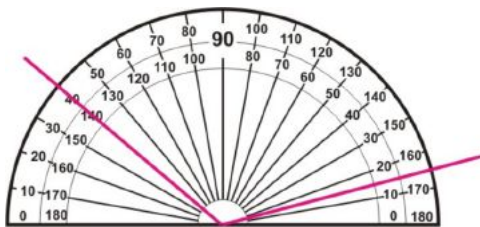
In other words, you do not have to start measuring an angle at 0° , as long as you subtract one measurement from the other.

Example 3: What is the measure of the angle shown below?



Solution: This angle is lined up with 0° , so where the second side intersects the protractor is the angle measure, which is 50° .

Example 4: What is the measure of the angle shown below?

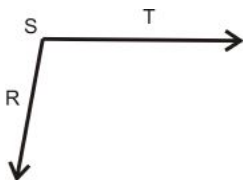


Solution: This angle is not lined up with 0° , so use subtraction to find its measure. It does not matter which scale you use.

Inner scale: $140^\circ - 25^\circ = 125^\circ$

Outer scale: $165^\circ - 40^\circ = 125^\circ$

Example 5: Use a protractor to measure $\angle RST$ below.



Solution: Lining up one side with 0° on the protractor, the other side hits 100° .

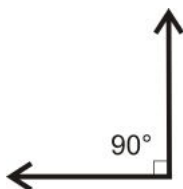
Classifying Angles

Angles can be grouped into four different categories.

Straight Angle: An angle that measures exactly 180° .



Right Angle: An angle that measures exactly 90° .



This half-square marks right, or 90° , angles.

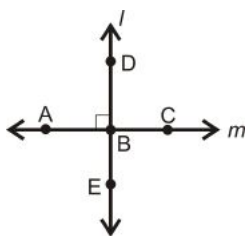
Acute Angles: Angles that measure between 0° and 90° .



Obtuse Angles: Angles that measure between 90° and 180° .



Perpendicular: When two lines intersect to form four right angles.



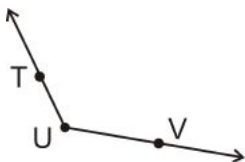
Even though all four angles are 90° , only one needs to be marked with the half-square.

The symbol for perpendicular is \perp .

Table 1.8:

<i>Label It</i>	<i>Say It</i>
$l \perp m$	Line l is perpendicular to line m .
$\overleftrightarrow{AC} \perp \overleftrightarrow{DE}$	Line AC is perpendicular to line DE .

Example 6: Name the angle and determine what type of angle it is.



Solution: The vertex is U . So, the angle can be $\angle TUV$ or $\angle VUT$. To determine what type of angle it is, compare it to a right angle.

Because it opens wider than a right angle, and less than a straight angle it is **obtuse**.

Example 7: What type of angle is 84° ? What about 165° ?

Solution: 84° is less than 90° , so it is **acute**. 165° is greater than 90° , but less than 180° , so it is **obtuse**.

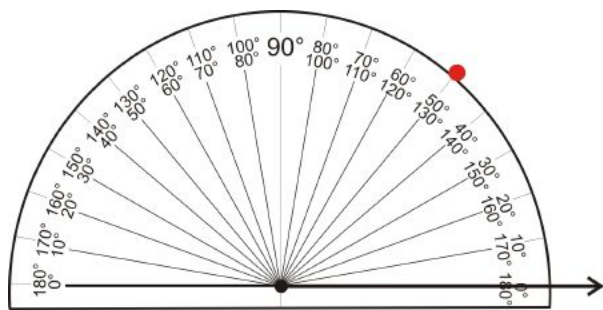
Drawing an Angle

Investigation 1-2: Drawing a 50° Angle with a Protractor

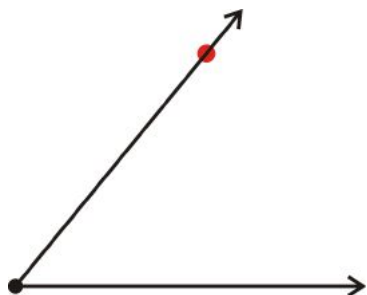
1. Start by drawing a horizontal line across the page, 2 in long.



2. Place an endpoint at the left side of your line.
3. Place the protractor on this point, such that the bottom line of the protractor is on the line and the endpoint is at the center. Mark 50° on the appropriate scale.



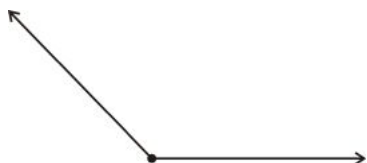
4. Remove the protractor and connect the vertex and the 50° mark.



This process can be used to draw any angle between 0° and 180° . See <http://www.mathsisfun.com/geometry/protractor-using.html> for an **animation** of this investigation.

Example 8: Draw a 135° angle.

Solution: Following the steps from above, your angle should look like this:



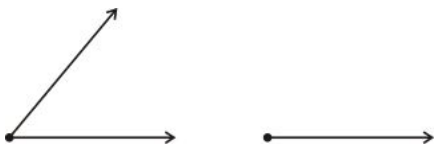
Now that we know how to draw an angle, we can also copy that angle with a compass and a ruler. Anytime we use a compass and ruler to draw geometric figures, it is called a **construction**.



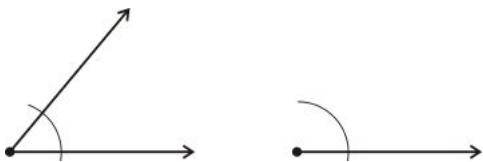
Compass: A tool used to draw circles and arcs.

Investigation 1-3: Copying an Angle with a Compass and Ruler

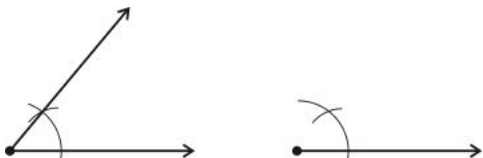
1. We are going to copy the 50° angle from Investigation 1-2. First, draw a straight line, 2 inches long, and place an endpoint at one end.



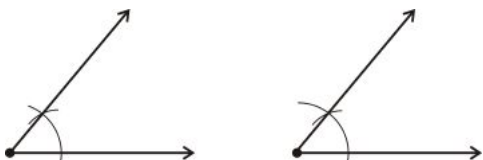
2. With the point (non-pencil side) of the compass on the vertex, draw an arc that passes through both sides of the angle. Repeat this arc with the line we drew in #1.



3. Move the point of the compass to the horizontal side of the angle we are copying. Place the point where the arc intersects this side. Open (or close) the “mouth” of the compass so that you can draw an arc that intersects the other side and the arc drawn in #2. Repeat this on the line we drew in #1.



4. Draw a line from the new vertex to the arc intersections.

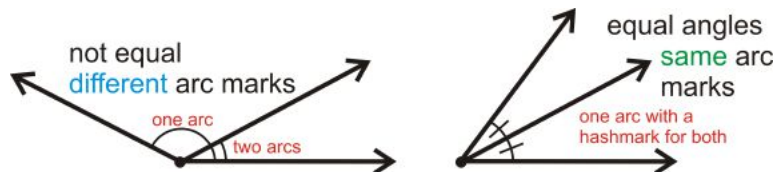


To watch an **animation** of this construction, see <http://www.mathsisfun.com/geometry/construct-anglesame.html>

Marking Angles and Segments in a Diagram

With all these segments and angles, we need to have different ways to label equal angles and segments.

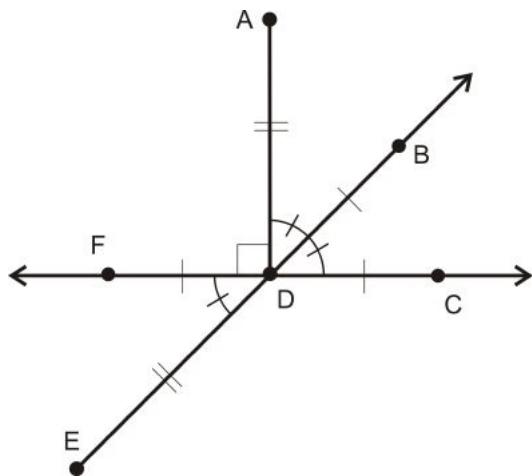
Angle Markings



Segment Markings



Example 9: Write all equal angle and segment statements.



Solution: $\overline{AD} \perp \overleftrightarrow{FC}$

$$m\angle ADB = m\angle BDC = m\angle FDE = 45^\circ$$

$$AD = DE$$

$$FD = DB = DC$$

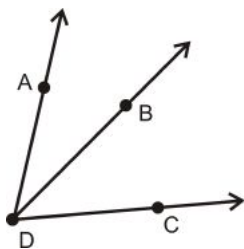
$$m\angle ADF = m\angle ADC = 90^\circ$$

Angle Addition Postulate

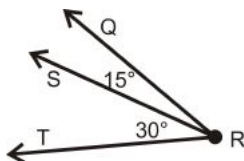
Like the Segment Addition Postulate, there is an Angle Addition Postulate.

Angle Addition Postulate: If B is on the interior of $\angle ADC$, then

$$m\angle ADC = m\angle ADB + m\angle BDC$$

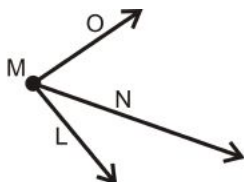


Example 10: What is $m\angle QRT$ in the diagram below?



Solution: Using the Angle Addition Postulate, $m\angle QRT = 15^\circ + 30^\circ = 45^\circ$.

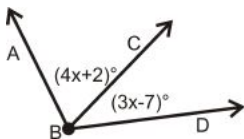
Example 11: What is $m\angle LMN$ if $m\angle LMO = 85^\circ$ and $m\angle NMO = 53^\circ$?



Solution: $m\angle LMO = m\angle NMO + m\angle LMN$, so $85^\circ = 53^\circ + m\angle LMN$.

$$m\angle LMN = 32^\circ.$$

Example 12: Algebra Connection If $m\angle ABD = 100^\circ$, find x .



Solution: $m\angle ABD = m\angle ABC + m\angle CBD$. Write an equation.

$$100^\circ = (4x + 2)^\circ + (3x - 7)^\circ$$

$$100^\circ = 7x^\circ - 5^\circ$$

$$105^\circ = 7x^\circ$$

$$15^\circ = x$$

Know What? Revisited Using a protractor, the measurement marked in the red triangle is 90° , the measurement in the blue triangle is 45° and the measurement in the orange square is 90° .

Review Questions

- Questions 1-10 use the definitions, postulates and theorems from this section.
- Questions 11-16 are similar to Investigation 1-2 and Examples 7 and 8.
- Questions 17 and 18 are similar to Investigation 1-3.
- Questions 19-22 are similar to Examples 2-5.

- Question 23 is similar to Example 9.
- Questions 24-28 are similar to Examples 10 and 11.
- Questions 29 and 30 are similar to Example 12.

For questions 1-10, determine if the statement is true or false.

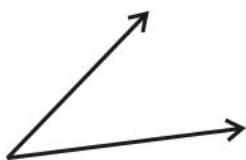
1. Two angles always add up to be greater than 90° .
2. 180° is an obtuse angle.
3. 180° is a straight angle.
4. Two perpendicular lines intersect to form four right angles.
5. A construction uses a protractor and a ruler.
6. For an angle $\angle ABC$, C is the vertex.
7. For an angle $\angle ABC$, \overline{AB} and \overline{BC} are the sides.
8. The m in front of $m\angle ABC$ means measure.
9. Angles are always measured in degrees.
10. The Angle Addition Postulate says that an angle is equal to the sum of the smaller angles around it.

For 11-16, draw the angle with the given degree, using a protractor and a ruler. Also, state what type of angle it is.

11. 55°
12. 92°
13. 178°
14. 5°
15. 120°
16. 73°
17. **Construction** Copy the angle you made from #12, using a compass and a ruler.
18. **Construction** Copy the angle you made from #16, using a compass and a ruler.

For 19-22, use a protractor to determine the measure of each angle.

19.



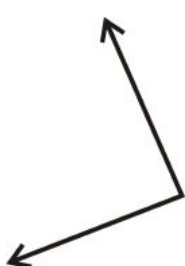
20.



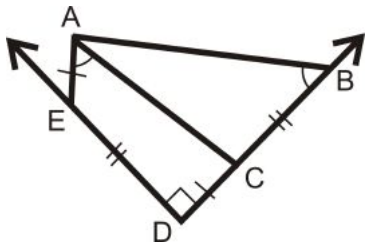
21.



22.



23. Interpret the picture to the right. Write down all equal angles, segments and if any lines are perpendicular.

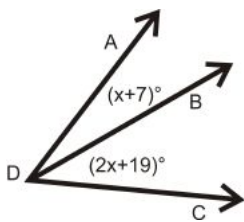


In Exercises 24-29, use the following information: Q is in the interior of $\angle ROS$. S is in the interior of $\angle QOP$. P is in the interior of $\angle SOT$. S is in the interior of $\angle ROT$ and $m\angle ROT = 160^\circ$, $m\angle SOT = 100^\circ$, and $m\angle ROQ = m\angle QOS = m\angle POT$.

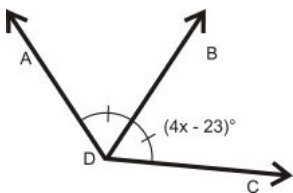
24. Make a sketch.
25. Find $m\angle QOP$
26. Find $m\angle QOT$
27. Find $m\angle ROQ$
28. Find $m\angle SOP$

Algebra Connection Solve for x .

29. $m\angle ADC = 56^\circ$



30. $m\angle ADC = 130^\circ$



Review Queue Answers

1. \overrightarrow{AB} , a ray
2. $XY = 3$, $YZ = 38$
 $a - 6 + 3a + 11 = 41$
 $4a + 5 = 41$

$$4a = 36$$

$$a = 9$$

3. Use the Segment Addition Postulate, $AC = 13$.

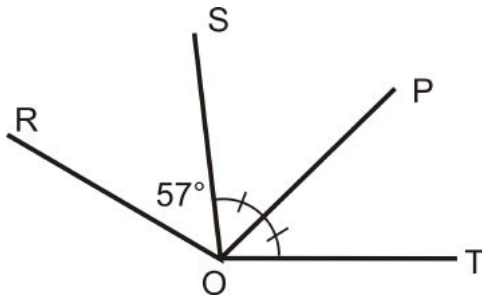
1.4 Midpoints and Bisectors

Learning Objectives

- Identify the midpoint of line segments.
- Identify the bisector of a line segment.
- Understand and use the Angle Bisector Postulate.

Review Queue

1. $m\angle SOP = 38^\circ$, find $m\angle POT$ and $m\angle ROT$.



2. Find the slope between the two numbers.

- (a) $(-4, 1)$ and $(-1, 7)$
 (b) $(5, -6)$ and $(-3, -4)$

3. Find the average of these numbers: 23, 30, 18, 27, and 32.

Know What? The building to the right is the Transamerica Building in San Francisco. This building was completed in 1972 and, at that time was one of the tallest buildings in the world. In order to make this building as tall as it is and still abide by the building codes, the designer used this pyramid shape.

It is very important in designing buildings that the angles and parts of the building are equal. What components of this building look equal? Analyze angles, windows, and the sides of the building.



Congruence

You could argue that another word for *equal* is *congruent*. But, the two are a little different.

Congruent: When two geometric figures have the same shape and size.

Table 1.9:

<i>Label It</i>	<i>Say It</i>
$\overline{AB} \cong \overline{BA}$	\overline{AB} is congruent to \overline{BA}

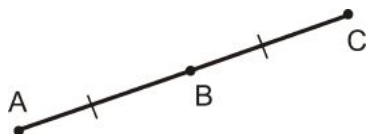
Table 1.10:

<i>Equal</i>	<i>Congruent</i>
=	\cong
used with <i>measurement</i>	used with <i>figures</i>
$m\overline{AB} = AB = 5cm$	$\overline{AB} \cong \overline{BA}$
$m\angle ABC = 60^\circ$	$\angle ABC \cong \angle CBA$

If two segments or angles are congruent, then they are also equal.

Midpoints

Midpoint: A point on a line segment that divides it into two congruent segments.



Because $AB = BC$, B is the midpoint of \overline{AC} .

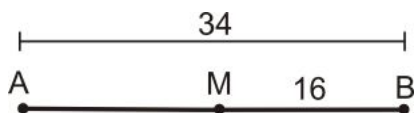
Midpoint Postulate: Any line segment will have exactly one midpoint.

This postulate is referring to the *midpoint*, not the lines that pass through the midpoint.



There are infinitely many lines that pass through the midpoint.

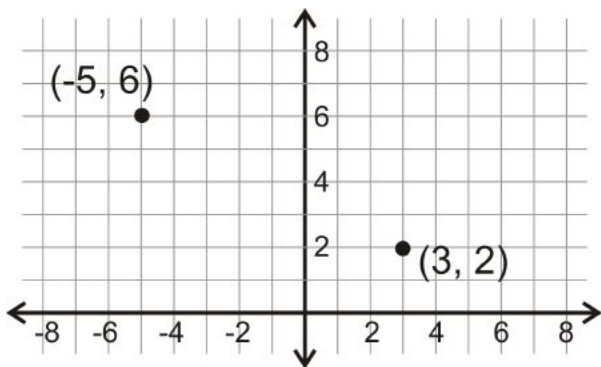
Example 1: Is M a midpoint of \overline{AB} ?



Solution: No, it is not $MB = 16$ and $AM = 34 - 16 = 18$. AM must equal MB in order for M to be the midpoint of \overline{AB} .

Midpoint Formula

When points are plotted in the coordinate plane, we can use a formula to find the midpoint between them. Here are two points, $(-5, 6)$ and $(3, 4)$.



It follows that the midpoint should be halfway between the points on the line. Just by looking, it seems like the midpoint is $(-1, 4)$.

Midpoint Formula: For two points, (x_1, y_1) and x_2, y_2 , the midpoint is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Let's use the formula to make sure $(-1, 4)$ is the midpoint between $(-5, 6)$ and $(3, 2)$.

$$\left(\frac{-5 + 3}{2}, \frac{6 + 2}{2}\right) = \left(\frac{-2}{2}, \frac{8}{2}\right) = (-1, 4)$$

Always use this formula to determine the midpoint.

Example 2: Find the midpoint between $(9, -2)$ and $(-5, 14)$.

Solution: Plug the points into the formula.

$$\left(\frac{9 + (-5)}{2}, \frac{-2 + 14}{2}\right) = \left(\frac{4}{2}, \frac{12}{2}\right) = (2, 6)$$

Example 3: If $M(3, -1)$ is the midpoint of \overline{AB} and $B(7, -6)$, find A .

Solution: Plug in what you know into the midpoint formula.

$$\begin{aligned}\left(\frac{7 + x_A}{2}, \frac{-6 + y_A}{2}\right) &= (3, -1) \\ \frac{7 + x_A}{2} &= 3 \text{ and } \frac{-6 + y_A}{2} = -1 \\ 7 + x_A &= 6 \text{ and } -6 + y_A = -2 \\ x_A &= -1 \text{ and } y_A = 4 \\ \text{So, } A &\text{ is } (-1, 4).\end{aligned}$$

Segment Bisectors

Segment Bisector: A bisector cuts a line segment into two congruent parts and passes through the midpoint.

Example 4: Use a ruler to draw a bisector of the segment.



Solution: First, find the midpoint. Measure the line segment. It is 4 cm long. To find the midpoint, divide 4 cm by 2 because we want 2 equal pieces. Measure 2 cm from one endpoint and draw the midpoint.



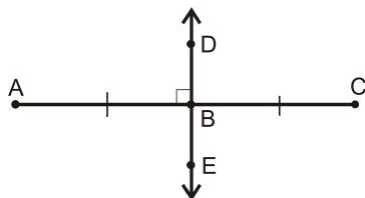
To finish, draw a line that passes through Z.



A specific type of segment bisector is called a perpendicular bisector.

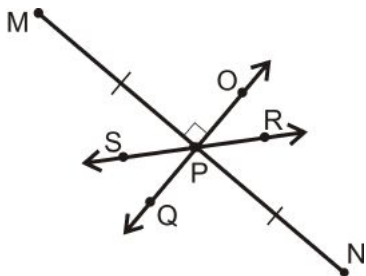
Perpendicular Bisector: A line, ray or segment that passes through the midpoint of another segment and intersects the segment at a right angle.

$$\begin{aligned}\overline{AB} &\cong \overline{BC} \\ \overline{AC} &\perp \overleftrightarrow{DE}\end{aligned}$$



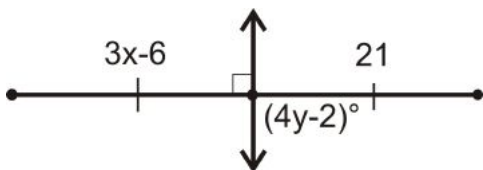
Perpendicular Bisector Postulate: For every line segment, there is one perpendicular bisector.

Example 5: Which line is the perpendicular bisector of \overline{MN} ?



Solution: The perpendicular bisector must bisect \overline{MN} and be perpendicular to it. Only \overleftrightarrow{OQ} fits this description. \overleftrightarrow{SR} is a bisector, but is not perpendicular.

Example 6: Algebra Connection Find x and y .



Solution: The line shown is the perpendicular bisector.

$$\text{So, } 3x - 6 = 21$$

$$3x = 27$$

$$x = 9$$

$$\text{And, } (4y - 2)^\circ = 90^\circ$$

$$4y = 92^\circ$$

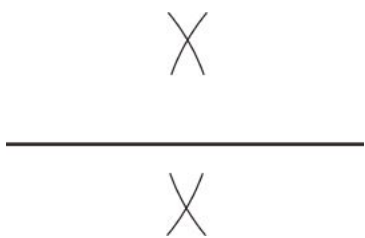
$$y = 23^\circ$$

Investigation 1-4: Constructing a Perpendicular Bisector

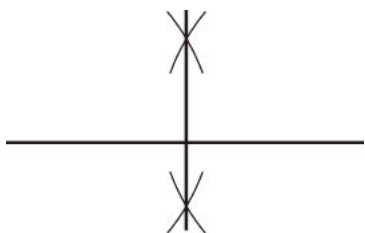
1. Draw a line that is 6 cm long, halfway down your page.



2. Place the pointer of the compass at an endpoint. Open the compass to be greater than half of the segment. Make arc marks above and below the segment. Repeat on the other endpoint. Make sure the arc marks intersect.



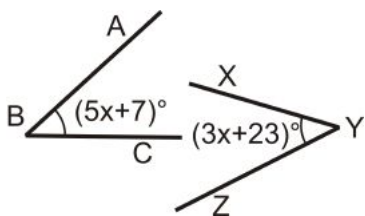
3. Use your straightedge to draw a line connecting the arc intersections.



This constructed line bisects the line you drew in #1 **and** intersects it at 90° . To see an animation of this investigation, go to <http://www.mathsisfun.com/geometry/construct-linebisect.html>.

Congruent Angles

Example 7: Algebra Connection What is the measure of each angle?



Solution: From the picture, we see that the angles are equal.

Set the angles equal to each other and solve.

$$(5x + 7)^\circ = (3x + 23)^\circ$$

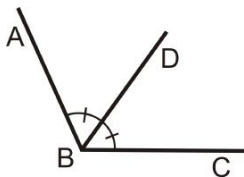
$$2x^\circ = 16^\circ$$

$$x = 8^\circ$$

To find the measure of $\angle ABC$, plug in $x = 8^\circ$ to $(5x + 7)^\circ \rightarrow (5(8) + 7)^\circ = (40 + 7)^\circ = 47^\circ$. Because $m\angle ABC = m\angle XYZ$, $m\angle XYZ = 47^\circ$ too.

Angle Bisectors

Angle Bisector: A ray that divides an angle into two congruent angles, each having a measure exactly half of the original angle.



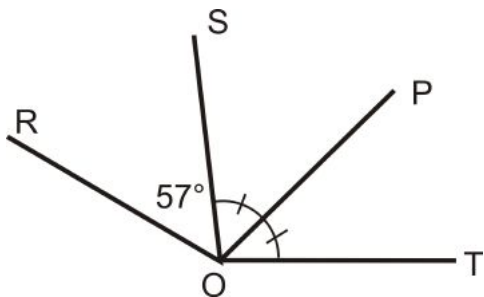
\overline{BD} is the angle bisector of $\angle ABC$

$$\angle ABD \cong \angle DBC$$

$$m\angle ABD = \frac{1}{2}m\angle ABC$$

Angle Bisector Postulate: Every angle has exactly one angle bisector.

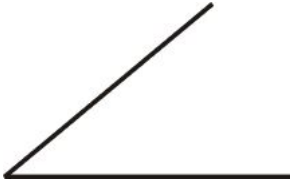
Example 8: Let's take a look at Review Queue #1 again. Is \overline{OP} the angle bisector of $\angle SOT$?



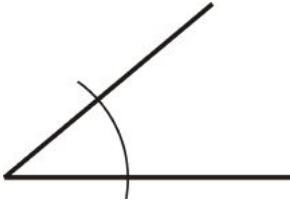
Solution: Yes, \overline{OP} is the angle bisector of $\angle SOT$ from the markings in the picture.

Investigation 1-5: Constructing an Angle Bisector

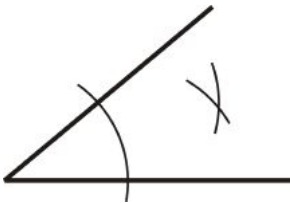
1. Draw an angle on your paper. Make sure one side is horizontal.



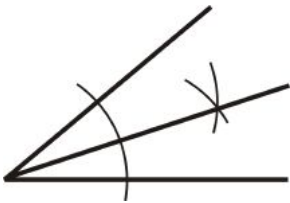
2. Place the pointer on the vertex. Draw an arc that intersects both sides.



3. Move the pointer to the arc intersection with the horizontal side. Make a second arc mark on the interior of the angle. Repeat on the other side. Make sure they intersect.



4. Connect the arc intersections from #3 with the vertex of the angle.

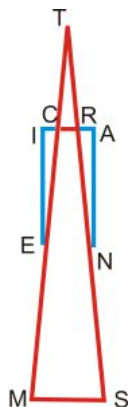


To see an animation of this construction, view <http://www.mathsisfun.com/geometry/construct-anglebisect.html>.

Know What? Revisited The image to the right is an outline of the Transamerica Building from earlier in the lesson. From this outline, we can see the following parts are congruent:

$$\begin{array}{ll}
 \overline{TR} \cong \overline{TC} & \angle TCR \cong \angle TRC \\
 \overline{RS} \cong \overline{CM} & \angle CIE \cong \angle RAN \\
 \overline{CI} \cong \overline{RA} & \text{and } \angle TMS \cong \angle TSM \\
 \overline{AN} \cong \overline{IE} & \angle IEC \cong \angle ANR \\
 \overline{TS} \cong \overline{TM} & \angle TCI \cong \angle TRA
 \end{array}$$

All the four triangular sides of the building are congruent to each other as well.



Review Questions

- Questions 1-18 are similar to Examples 1, 4, 5 and 8.
- Questions 19-22 are similar to Examples 6 and 7.
- Question 23 is similar to Investigation 1-5.
- Question 24 is similar to Investigation 1-4.
- Questions 25-28 are similar to Example 2.
- Question 29 and 30 are similar to Example 3.

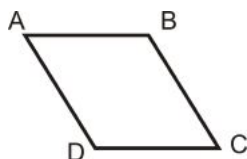
1. Copy the figure below and label it with the following information:

$$\angle A \cong \angle C$$

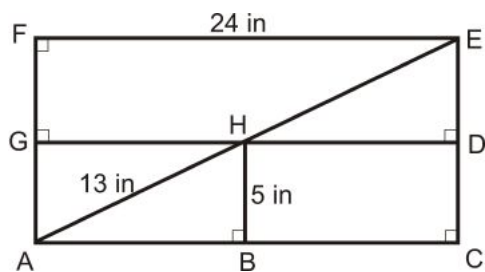
$$\angle B \cong \angle D$$

$$\overline{AB} \cong \overline{CD}$$

$$\overline{AD} \cong \overline{BC}$$



H is the midpoint of \overline{AE} and \overline{DG} , B is the midpoint of \overline{AC} , \overline{GD} is the perpendicular bisector of \overline{FA} and \overline{EC}
 $\overline{AC} \cong \overline{FE}$ and $\overline{FA} \cong \overline{EC}$

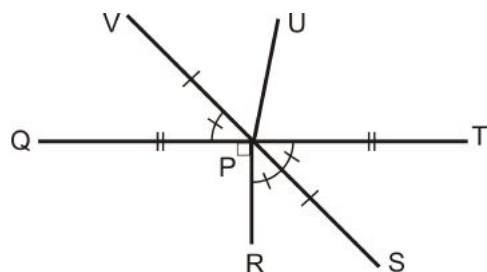


Find:

2. AB

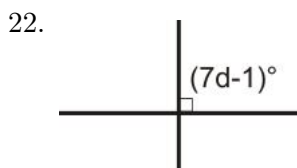
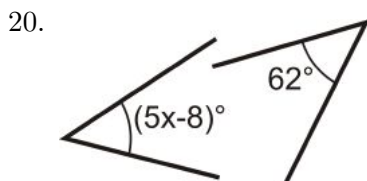
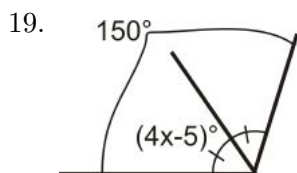
3. GA
4. ED
5. HE
6. $m\angle HDC$
7. FA
8. GD
9. $m\angle FED$
10. How many copies of triangle AHB can fit inside rectangle $FECA$ without overlapping?

For 11-18, use the following picture to answer the questions.



11. What is the angle bisector of $\angle TPR$?
12. P is the midpoint of what two segments?
13. What is $m\angle QPR$?
14. What is $m\angle TPS$?
15. How does \overline{VS} relate to \overline{QT} ?
16. How does \overline{QT} relate to \overline{VS} ?
17. What is $m\angle QPV$?
18. Is \overline{PU} a bisector? If so, of what?

Algebra Connection For 19-22, use algebra to determine the value of variable in each problem.



23. **Construction** Using your protractor, draw an angle that is 110° . Then, use your compass to construct the angle bisector. What is the measure of each angle?
24. **Construction** Using your ruler, draw a line segment that is 7 cm long. Then use your compass to construct the perpendicular bisector, What is the measure of each segment?

For questions 25-28, find the midpoint between each pair of points.

25. $(-2, -3)$ and $(8, -7)$
26. $(9, -1)$ and $(-6, -11)$
27. $(-4, 10)$ and $(14, 0)$
28. $(0, -5)$ and $(-9, 9)$

Given the midpoint (M) and either endpoint of \overline{AB} , find the other endpoint.

29. $A(-1, 2)$ and $M(3, 6)$
30. $B(-10, -7)$ and $M(-2, 1)$

Review Queue Answers

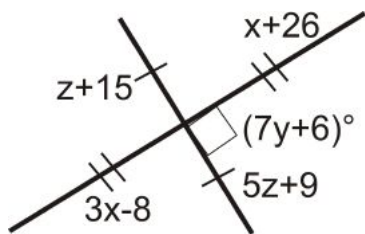
1. $m\angle POT = 38^\circ$, $m\angle ROT = 57^\circ + 38^\circ + 38^\circ = 133^\circ$
2. (a) $\frac{7-1}{-1+4} = \frac{6}{3} = 2$
 (b) $\frac{-4+6}{3-5} = \frac{2}{-2} = -1$
3. $\frac{23+30+18+27+32}{5} = \frac{130}{5} = 26$

1.5 Angle Pairs

Learning Objectives

- Recognize complementary angles supplementary angles, linear pairs, and vertical angles.
- Apply the Linear Pair Postulate and the Vertical Angles Theorem.

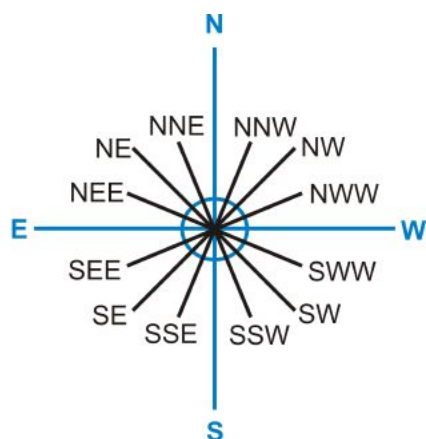
Review Queue



1. Find x .
2. Find y .
3. Find z .

Know What? A compass (as seen to the right) is used to determine the direction a person is traveling. The angles between each direction are very important because they enable someone to be more specific with their direction. A direction of 45° NW, would be straight out along that northwest line.

What headings have the same angle measure? What is the angle measure between each compass line?



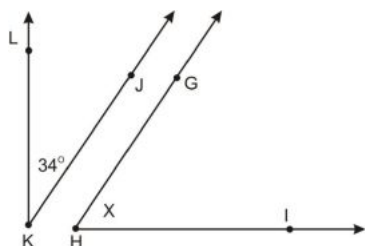
Complementary Angles

Complementary: Two angles that add up to 90° .

Complementary angles do not have to be:

- congruent
- next to each other

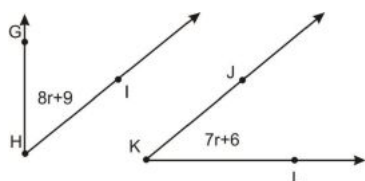
Example 1: The two angles below are complementary. $m\angle GHI = x$. What is x ?



Solution: Because the two angles are complementary, they add up to 90° . Make an equation.

$$\begin{aligned} x + 34^\circ &= 90^\circ \\ x &= 56^\circ \end{aligned}$$

Example 2: The two angles below are complementary. Find the measure of each angle.



Solution: The two angles add up to 90° . Make an equation.

$$8r + 9^\circ + 7r + 6^\circ = 90^\circ$$

$$15r + 15^\circ = 90^\circ$$

$$15r = 75^\circ$$

$$r = 5^\circ$$

However, you need to find each angle. Plug r back into each expression.

$$m\angle GHI = 8(5^\circ) + 9^\circ = 49^\circ$$

$$m\angle JKL = 7(5^\circ) + 6^\circ = 41^\circ$$

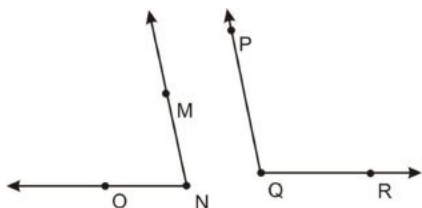
Supplementary Angles

Supplementary: Two angles that add up to 180° .

Supplementary angles do not have to be:

- congruent
- next to each other

Example 3: The two angles below are supplementary. If $m\angle MNO = 78^\circ$ what is $m\angle PQR$?



Solution: Set up an equation. However, instead of equaling 90° , now it is 180° .

$$78^\circ + m\angle PQR = 180^\circ$$

$$m\angle PQR = 102^\circ$$

Example 4: What is the measure of two congruent, supplementary angles?

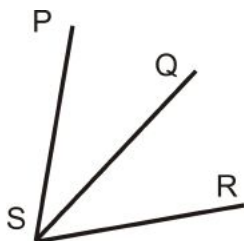
Solution: Supplementary angles add up to 180° . Congruent angles have the same measure. So, $180^\circ \div 2 = 90^\circ$, which means two congruent, supplementary angles are right angles, or 90° .

Linear Pairs

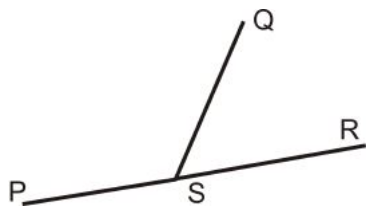
Adjacent Angles: Two angles that have the same vertex, share a side, and do not overlap.

$\angle PSQ$ and $\angle QSR$ are adjacent.

$\angle PQR$ and $\angle PQS$ are NOT adjacent because they overlap.



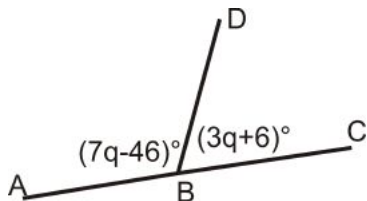
Linear Pair: Two angles that are adjacent and the non-common sides form a straight line.



$\angle PSQ$ and $\angle QSR$ are a linear pair.

Linear Pair Postulate: If two angles are a linear pair, then they are supplementary.

Example 5: Algebra Connection What is the measure of each angle?



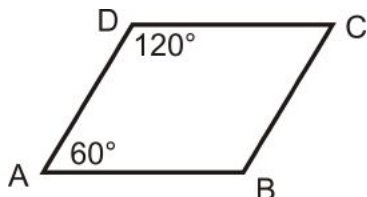
Solution: These two angles are a linear pair, so they add up to 180° .

$$\begin{aligned}(7q - 46)^\circ + (3q + 6)^\circ &= 180^\circ \\ 10q - 40^\circ &= 180^\circ \\ 10q &= 220^\circ \\ q &= 22^\circ\end{aligned}$$

Plug in q to get the measure of each angle. $m\angle ABD = 7(22^\circ) - 46^\circ = 108^\circ$ $m\angle DBC = 180^\circ - 108^\circ = 72^\circ$

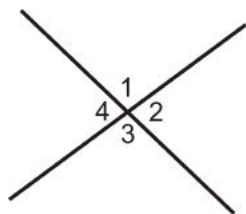
Example 6: Are $\angle CDA$ and $\angle DAB$ a linear pair? Are they supplementary?

Solution: The two angles are not a linear pair because they do not have the same vertex. They are supplementary, $120^\circ + 60^\circ = 180^\circ$.



Vertical Angles

Vertical Angles: Two non-adjacent angles formed by intersecting lines.



$\angle 1$ and $\angle 3$ are vertical angles

$\angle 2$ and $\angle 4$ are vertical angles

These angles are labeled with numbers. You can tell that these are labels because they do not have a degree symbol.

Investigation 1-6: Vertical Angle Relationships

1. Draw two intersecting lines on your paper. Label the four angles created $\angle 1$, $\angle 2$, $\angle 3$, and $\angle 4$, just like the picture above.
2. Use your protractor to find $m\angle 1$.
3. What is the angle relationship between $\angle 1$ and $\angle 2$ called? Find $m\angle 2$.
4. What is the angle relationship between $\angle 1$ and $\angle 4$ called? Find $m\angle 4$.
5. What is the angle relationship between $\angle 2$ and $\angle 3$ called? Find $m\angle 3$.
6. Are any angles congruent? If so, write them down.

From this investigation, you should find that $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$.

Vertical Angles Theorem: If two angles are vertical angles, then they are congruent.

We can prove the Vertical Angles Theorem using the same process we used in the investigation. We will not use any specific values for the angles.

From the picture above:

$$\angle 1 \text{ and } \angle 2 \text{ are a linear pair} \rightarrow m\angle 1 + m\angle 2 = 180^\circ \quad \text{Equation 1}$$

$$\angle 2 \text{ and } \angle 3 \text{ are a linear pair} \rightarrow m\angle 2 + m\angle 3 = 180^\circ \quad \text{Equation 2}$$

$$\angle 3 \text{ and } \angle 4 \text{ are a linear pair} \rightarrow m\angle 3 + m\angle 4 = 180^\circ \quad \text{Equation 3}$$

All of the equations = 180° , so Equation 1 = Equation 2 and Equation 2 = Equation 3.

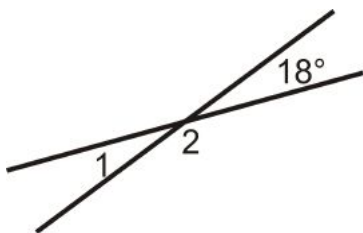
$$m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3 \quad \text{and} \quad m\angle 2 + m\angle 3 = m\angle 3 + m\angle 4$$

Cancel out the like terms

$$m\angle 1 = m\angle 3 \quad \text{and} \quad m\angle 2 = m\angle 4$$

Recall that anytime the measures of two angles are equal, the angles are also congruent. So, $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$ too.

Example 7: Find $m\angle 1$ and $m\angle 2$.



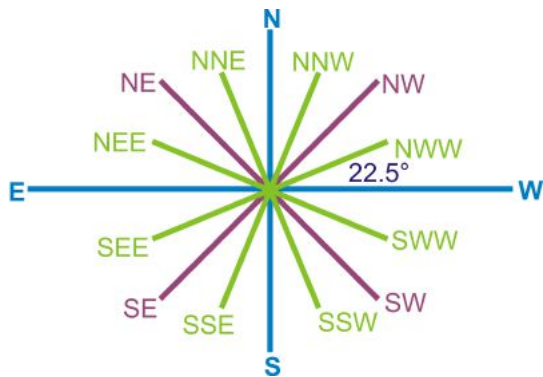
Solution: $\angle 1$ is vertical angles with 18° , so $m\angle 1 = 18^\circ$.

$\angle 2$ is a linear pair with $\angle 1$ or 18° , so $18^\circ + m\angle 2 = 180^\circ$.

$$m\angle 2 = 180^\circ - 18^\circ = 162^\circ.$$

Know What? Revisited The compass has several vertical angles and all of the smaller angles are 22.5° , $180^\circ \div 8$. Directions that are opposite each other have the same angle measure, but of course, a

different direction. All of the green directions have the same angle measure, 22.5° , and the purple have the same angle measure, 45° . N , S , E and W all have different measures, even though they are all 90° apart.



Review Questions

- Questions 1 and 2 are similar to Examples 1, 2, and 3.
- Questions 3-8 are similar to Examples 3, 4, 6 and 7.
- Questions 9-16 use the definitions, postulates and theorems from this section.
- Questions 17-25 are similar to Example 5.

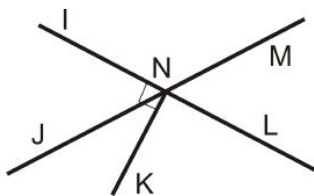
1. Find the measure of an angle that is complementary to $\angle ABC$ if $m\angle ABC$ is

- (a) 45°
- (b) 82°
- (c) 19°
- (d) z°

2. Find the measure of an angle that is supplementary to $\angle ABC$ if $m\angle ABC$ is

- (a) 45°
- (b) 118°
- (c) 32°
- (d) x°

Use the diagram below for exercises 3-7. Note that $\overline{NK} \perp \overleftrightarrow{IL}$.



- 3. Name one pair of vertical angles.
- 4. Name one linear pair of angles.
- 5. Name two complementary angles.
- 6. Name two supplementary angles.

7. What is:

- (a) $m\angle INL$

(b) $m\angle LNK$

8. If $m\angle INJ = 63^\circ$, find:

(a) $m\angle JNL$

(b) $m\angle KNJ$

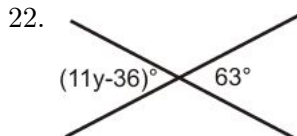
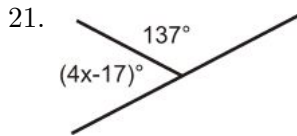
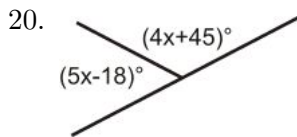
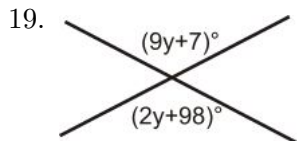
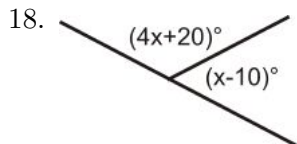
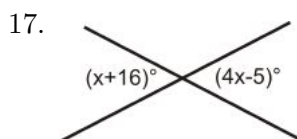
(c) $m\angle MNL$

(d) $m\angle MNI$

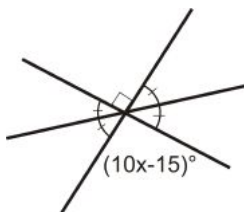
For 9-16, determine if the statement is true or false.

9. Vertical angles are congruent.
10. Linear pairs are congruent.
11. Complementary angles add up to 180° .
12. Supplementary angles add up to 180°
13. Adjacent angles share a vertex.
14. Adjacent angles overlap.
15. Complementary angles are always 45° .
16. Vertical angles have the same vertex.

For 17-25, find the value of x or y .

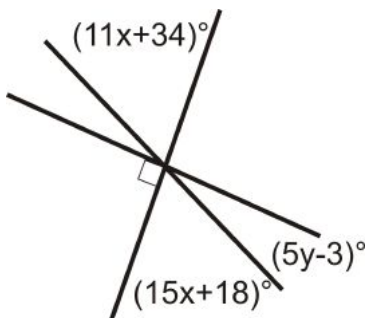


23.



24. Find x .

25. Find y .



Review Queue Answers

1. $x + 26 = 3x - 8$

$$34 = 2x$$

$$17 = x$$

2. $(7y + 6)^\circ = 90^\circ$

$$7y = 84^\circ$$

$$y = 12^\circ$$

3. $z + 15 = 5z + 9$

$$6 = 4z$$

$$1.5 = z$$

1.6 Classifying Polygons

Learning Objectives

- Define triangle and polygon.
- Classify triangles by their sides and angles.
- Understand the difference between convex and concave polygons.
- Classify polygons by number of sides.

Review Queue

1. Draw a triangle.
2. Where have you seen 4, 5, 6 or 8 - sided polygons in real life? List 3 examples.
3. Fill in the blank.
 - (a) Vertical angles are always _____.
 - (b) Linear pairs are _____.
 - (c) The parts of an angle are called _____ and a _____.

Know What? The pentagon in Washington DC is a pentagon with congruent sides and angles. There is a smaller pentagon inside of the building that houses an outdoor courtyard. Looking at the picture, the building is divided up into 10 smaller sections. What are the shapes of these sections? Are any of these division lines diagonals? How do you know?



Triangles

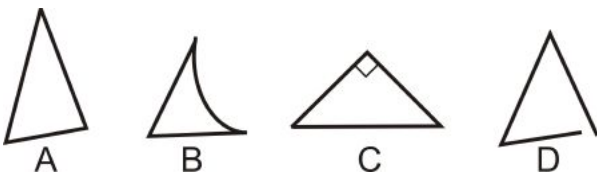
Triangle: Any closed figure made by three line segments intersecting at their endpoints.

Every triangle has three **vertices** (the points where the segments meet), three **sides** (the segments), and three **interior angles** (formed at each vertex). All of the following shapes are triangles.



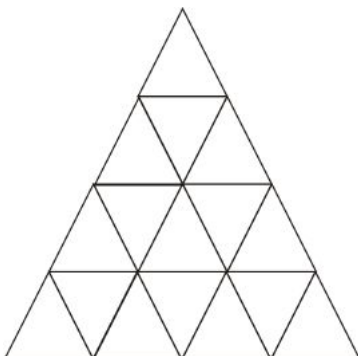
You might have also learned that the sum of the interior angles in a triangle is 180° . Later we will prove this, but for now you can use this fact to find missing angles.

Example 1: Which of the figures below are not triangles?



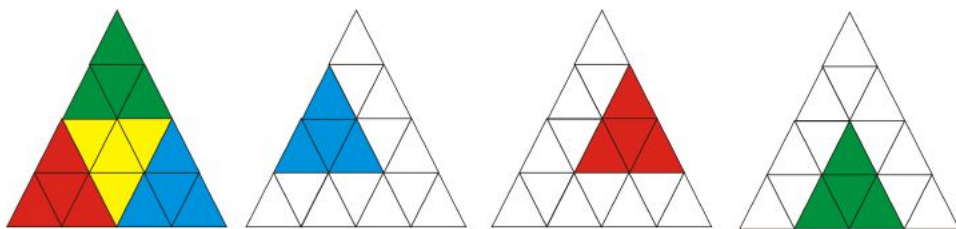
Solution: *B* is not a triangle because it has one curved side. *D* is not closed, so it is not a triangle either.

Example 2: How many triangles are in the diagram below?

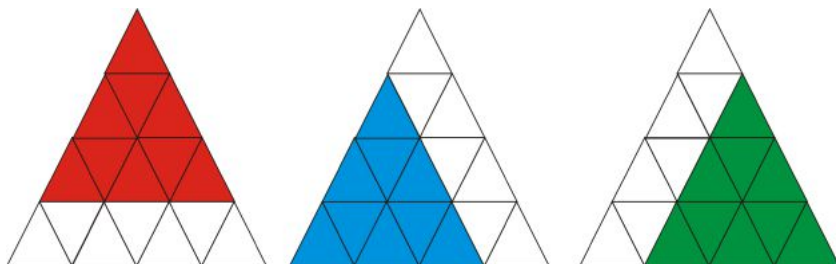


Solution: Start by counting the smallest triangles, 16.

Now count the triangles that are formed by 4 of the smaller triangles, 7.



Next, count the triangles that are formed by 9 of the smaller triangles, 3.

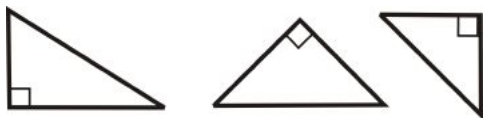


Finally, there is the one triangle formed by all 16 smaller triangles. Adding these numbers together, we get $16 + 7 + 3 + 1 = 27$.

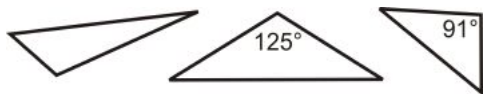
Classifying by Angles

Angles can be grouped by their angles; acute, obtuse or right. In any triangle, two of the angles will always be acute. The third angle can be acute, obtuse, or right. ***We classify each triangle by this angle.***

Right Triangle: A triangle with one right angle.



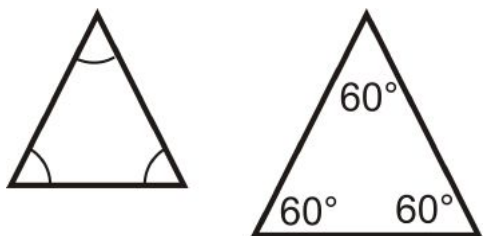
Obtuse Triangle: A triangle with one obtuse angle.



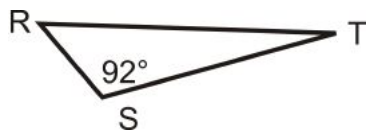
Acute Triangle: A triangle where all three angles are acute.



Equiangular Triangle: When all the angles in a triangle are congruent.



Example 3: Which term best describes $\triangle RST$ below?

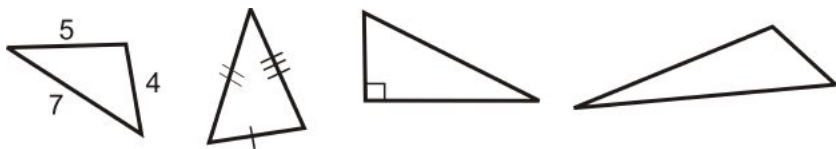


Solution: This triangle has one labeled obtuse angle of 92° . Triangles can only have one obtuse angle, so it is an obtuse triangle.

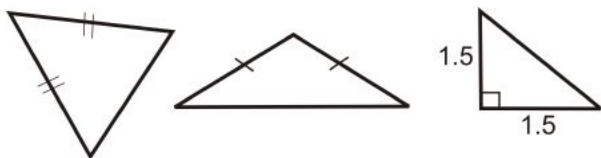
Classifying by Sides

You can also group triangles by their sides.

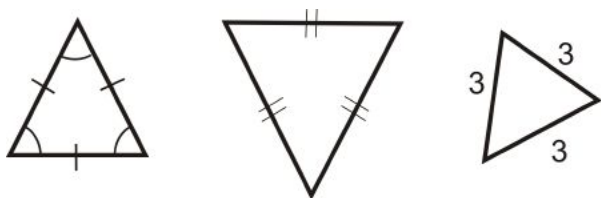
Scalene Triangle: A triangles where all three sides are different lengths.



Isosceles Triangle: A triangle with *at least* two congruent sides.

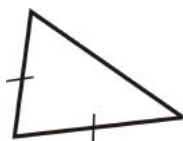


Equilateral Triangle: A triangle with three congruent sides.



From the definitions, an equilateral triangle is also an isosceles triangle.

Example 4: Classify the triangle by its sides and angles.



Solution: We see that there are two congruent sides, so it is isosceles. By the angles, they all look acute.

We say this is an acute isosceles triangle.

Example 5: Classify the triangle by its sides and angles.



Solution: This triangle has a right angle and no sides are marked congruent. So, it is a right scalene triangle.

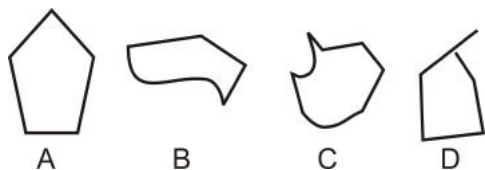
Polygons

Polygon: Any closed, 2-dimensional figure that is made entirely of line segments that intersect at their endpoints.

Polygons can have any number of sides and angles, but the sides can never be curved.

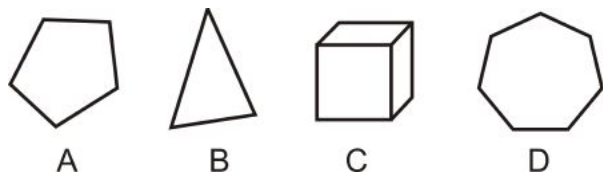
The segments are called the **sides** of the polygons, and the points where the segments intersect are called **vertices**.

Example 6: Which of the figures below is a polygon?



Solution: The easiest way to identify the polygon is to identify which shapes are **not** polygons. *B* and *C* each have at least one curved side, so they are not polygons. *D* has all straight sides, but one of the vertices is not at the endpoint, so it is not a polygon. *A* is the only polygon.

Example 7: Which of the figures below is not a polygon?



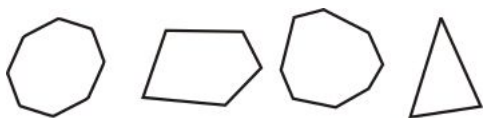
Solution: *C* is a three-dimensional shape, so it does not lie within one plane, so it is not a polygon.

Convex and Concave Polygons

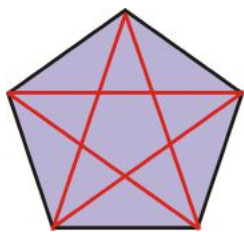
Polygons can be either **convex** or **concave**. The term concave refers to a cave, or the polygon is “caving in”. All stars are concave polygons.



A convex polygon does not do this. Convex polygons look like:



Diagonals: Line segments that connect the vertices of a convex polygon that are not sides.



The red lines are all diagonals.

This pentagon has 5 diagonals.

Example 8: Determine if the shapes below are convex or concave.



Solution: To see if a polygon is concave, look at the polygons and see if any angle “caves in” to the interior of the polygon. The first polygon does not do this, so it is convex. The other two do, so they are concave.

Example 9: How many diagonals does a 7-sided polygon have?



Solution: Draw a 7-sided polygon, also called a heptagon.

Drawing in all the diagonals and counting them, we see there are 14.

Classifying Polygons

Whether a polygon is convex or concave, it is always named by the number of sides.

Table 1.11:


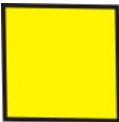

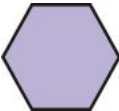

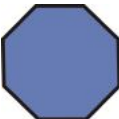

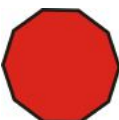

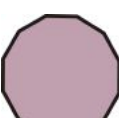
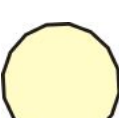
Polygon Name	Number of Sides	Number of Diagonals	Convex Example
Triangle	3	0	

Table 1.11: (continued)

Polygon Name	Number of Sides	Number of Diagonals	Convex Example
Quadrilateral	4	2	
Pentagon	5	5	
Hexagon	6	9	
Heptagon	7	14	
Octagon	8	?	
Nonagon	9	?	
Decagon	10	?	
Undecagon or hendecagon	11	?	
Dodecagon	12	?	
n-gon	n (where $n > 12$)	?	

Example 10: Name the three polygons below by their number of sides and if it is convex or concave.



Solution: The pink polygon is a concave hexagon (6 sides).

The green polygon convex pentagon (5 sides).

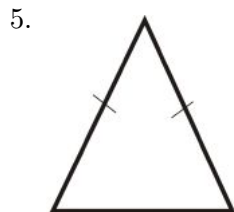
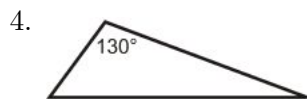
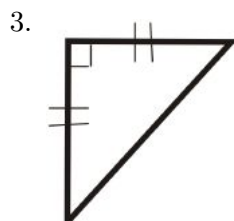
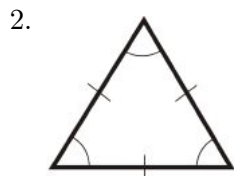
The yellow polygon is a convex decagon (10 sides).

Know What? Revisited The pentagon is divided up into 10 sections, all quadrilaterals. None of these dividing lines are diagonals because they are not drawn from vertices.

Review Questions

- Questions 1-8 are similar to Examples 3, 4 and 5.
- Questions 9-14 are similar to Examples 8 and 10
- Question 15 is similar to Example 6.
- Questions 16-19 are similar to Example 9 and the table.
- Questions 20-25 use the definitions, postulates and theorems in this section.

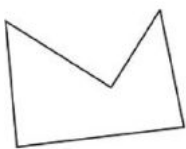
For questions 1-6, classify each triangle by its sides and by its angles.



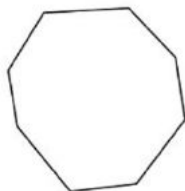
7. Can you draw a triangle with a right angle and an obtuse angle? Why or why not?
8. In an isosceles triangle, can the angles opposite the congruent sides be obtuse?

In problems 9-14, name each polygon in as much detail as possible.

9.



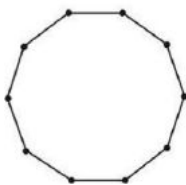
10.



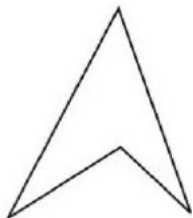
11.



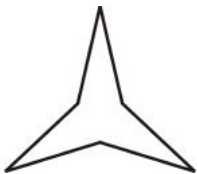
12.



13.



14.

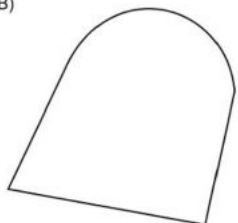


15. Explain why the following figures are NOT polygons:

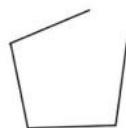
A)



B)



C)



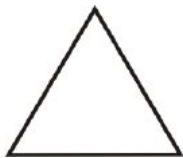
16. How many diagonals can you draw from **one vertex** of a pentagon? Draw a sketch of your answer.
17. How many diagonals can you draw from **one vertex** of an octagon? Draw a sketch of your answer.
18. How many diagonals can you draw from **one vertex** of a dodecagon?
19. Determine the number of total diagonals for an octagon, nonagon, decagon, undecagon, and dodecagon.

For 20-25, determine if the statement is true or false.

20. Obtuse triangles can be isosceles.
21. A polygon must be enclosed.
22. A star is a convex polygon.
23. A right triangle is acute.
24. An equilateral triangle is equiangular.
25. A quadrilateral is always a square.
26. A 5-point star is a decagon

Review Queue Answers

1.



2. Examples include: stop sign (8), table top (4), the Pentagon (5), snow crystals (6), bee hive combs (6), soccer ball pieces (5 and 6)
3. (a) congruent or equal
(b) supplementary
(c) sides, vertex

1.7 Chapter 1 Review

Symbol Toolbox

\overleftrightarrow{AB} , \overrightarrow{AB} , \overline{AB} - Line, ray, line segment

$\angle ABC$ - Angle with vertex B

$m\overline{AB}$ or AB - Distance between A and B

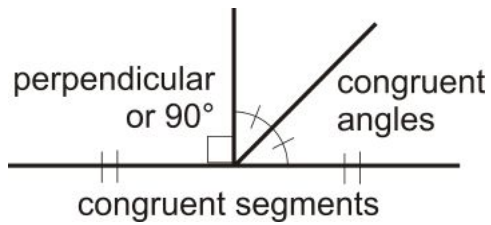
$m\angle ABC$ - Measure of $\angle ABC$

\perp - Perpendicular

$=$ - Equal

\cong - Congruent

Markings



Keywords, Postulates, and Theorems

Points, Lines, and Planes

- Geometry
- Point
- Line
- Plane
- Space
- Collinear
- Coplanar
- Endpoint
- Line Segment
- Ray
- Intersection
- Postulates
- Theorem
- Postulate 1-1
- Postulate 1-2
- Postulate 1-3
- Postulate 1-4
- Postulate 1-5

Segments and Distance

- Distance
- Measure
- Ruler Postulate
- Segment Addition Postulate

Angles and Measurement

- Angle
- Vertex
- Sides
- Protractor Postulate
- Straight Angle
- Right Angle
- Acute Angles
- Obtuse Angles
- Convex

- Concave
- Polygon
- Perpendicular
- Construction
- Compass
- Angle Addition Postulate

Midpoints and Bisectors

- Congruent
- Midpoint.
- Midpoint Postulate
- Segment Bisector
- Perpendicular Bisector
- Perpendicular Bisector Postulate
- Angle Bisector
- Angle Bisector Postulate

Angle Pairs

- Complementary
- Supplementary
- Adjacent Angles
- Linear Pair
- Linear Pair Postulate
- Vertical Angles
- Vertical Angles Theorem

Classifying Polygons

- Triangle
- Right Triangle
- Obtuse Triangle
- Acute Triangle
- Equiangular Triangle
- Scalene Triangle
- Isosceles Triangle
- Equilateral Triangle
- Vertices
- Diagonals

Review

Match the definition or description with the correct word.

1. When three points lie on the same line. — A. Measure
2. All vertical angles are _____. — B. Congruent
3. Linear pairs add up to _____. — C. Angle Bisector

4. The m in from of $m\angle ABC$. — D. Ray
5. What you use to measure an angle. — E. Collinear
6. When two sides of a triangle are congruent. — F. Perpendicular
7. \perp — G. Line
8. A line that passes through the midpoint of another line. — H. Protractor
9. An angle that is greater than 90° . — I. Segment Addition Postulate
10. The intersection of two planes is a _____. — J. Obtuse
11. $AB + BC = AC$ — K. Point
12. An exact location in space. — L. 180°
13. A sunbeam, for example. — M. Isosceles
14. Every angle has exactly one. — N. Pentagon
15. A closed figure with 5 sides. — O. Hexagon

P. Bisector

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9686>.

1.8 Study Guide

Keywords: Define, write theorems, and/or draw a diagram for each word below.

1st Section: Points, Lines, and Planes

Geometry

Point

Line

Plane

Space

Collinear

Coplanar

Endpoint

Line Segment

Ray

Intersection

Postulates

Theorem

Postulate 1-1

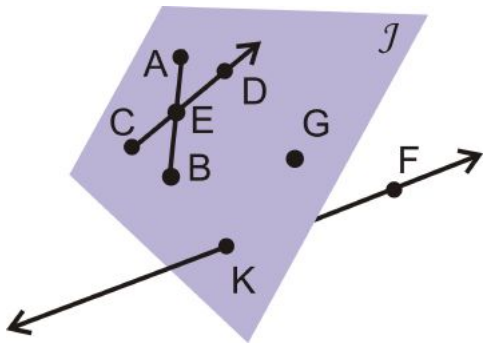
Postulate 1-2

Postulate 1-3

Postulate 1-4

Postulate 1-5

Use this picture to identify the geometric terms in this section.



Homework:

2nd Section: Segments and Distance

Distance

Measure

Ruler Postulate

Segment Addition Postulate



Homework:

3rd Section: Angles and Measurement

Angle

Vertex

Sides

Protractor Postulate

Straight Angle

Right Angle

Acute Angles

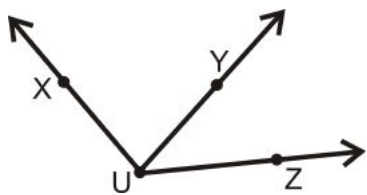
Obtuse Angles

Perpendicular

Construction

Compass

Angle Addition Postulate



Homework:

4th Section: Midpoints and Bisectors

Congruent

Midpoint.

Midpoint Postulate

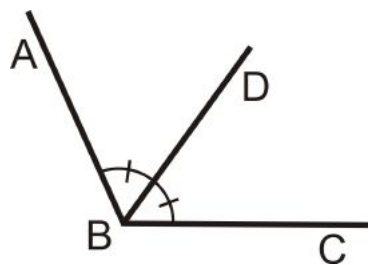
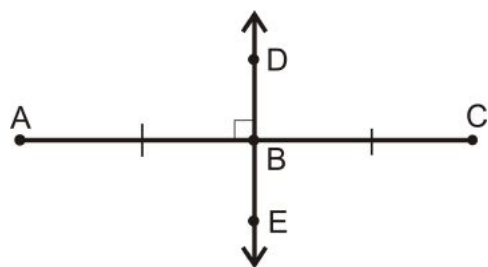
Segment Bisector

Perpendicular Bisector

Perpendicular Bisector Postulate

Angle Bisector

Angle Bisector Postulate



Homework:

5th Section: Angle Pairs

Complementary

Supplementary

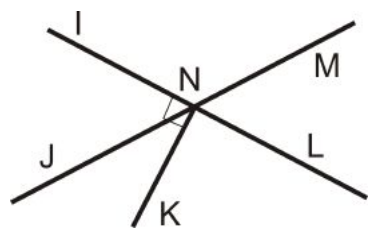
Adjacent Angles

Linear Pair

Linear Pair Postulate

Vertical Angles

Vertical Angles Theorem



Homework:

6th Section: Classifying Polygons

Draw your own pictures for this section

Triangle

Right Triangle

Obtuse Triangle

Acute Triangle

Equiangular Triangle

Scalene Triangle

Isosceles Triangle

Equilateral Triangle

Vertices

Sides

Polygon

Convex Polygon

Concave Polygon

Quadrilateral, Pentagon, Hexagon, Heptagon, Octagon, Nonagon, Decagon...

Diagonals

Homework:

Chapter 2

Coordinate Geometry

2.1 Vocabulary Self-Rating

Table 2.1: Rating Guide: DK: I am sure I don't know it K: I am sure I know it ?: I'm not sure

Word	Before Lesson/Unit	After Lesson/Unit
Distance		
Overbar		
Estimation		
Estimate		
Ruler		
Absolute value		
$x - y$ coordinate plane		
x -coordinate		
y -coordinate		
Right triangle		
Pythagorean Theorem		
Distance Formula		
Midpoint		
Equidistant		
Slope		
Undefined		
Parallel		
Perpendicular		
Slope-intercept form		
y -intercept		
Circle		
Center		
Radius		
Leg		
Hypotenuse		
Concentric		
Translation		

Table 2.1: (continued)

Word	Before Lesson/Unit	After Lesson/Unit
Prime		
Preimage		
Image		
Isometry		
Reflection		
Perpendicular bisector		
Rotation		
Angle of rotation		
Arc		
Central angle		
Diameter		
Acute		
Complementary		

2.2 Distance and Midpoint

Learning Objectives

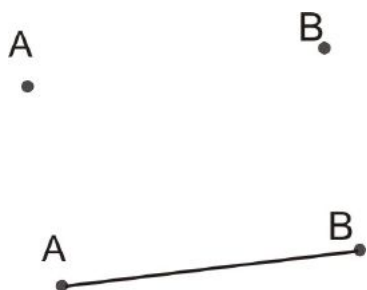
- Derive the *Distance Formula* using the Pythagorean Theorem.
- Use the *Distance Formula* to find the length of a line segment with known endpoints.
- Use the *Midpoint Formula* to calculate the coordinates of the midpoint of a line segment given both endpoints, or to determine the coordinates of one endpoint given the midpoint and the other endpoint.

Measuring Distances

There are many different ways to identify measurements. This lesson will present some that may be familiar, and probably a few that are new to you.

Before we begin to examine distances, however, it is important to identify the meaning of **distance** in the context of geometry. The **distance** between two points is defined by the *length of the line segment that connects them*.

- The **distance** between two points is the _____ of the line segment that connects them.



The most common way to *measure* **distance** is with a ruler. Also, distance can be estimated using scale on a map.

Notation Notes: When we name a segment we use the endpoints and an **overbar** (a bar or line above the letters) with no arrows. For example, "segment AB " is written \overline{AB} . The *length* of a segment is named by giving the endpoints *without* using an **overbar**. For example, the length of \overline{AB} is written AB . In some books you may also see $m\overline{AB}$, or *measure* of \overline{AB} , which means the same as AB , that is, it is the *length* of the segment with endpoints A and B .

Example 1

Use the scale to estimate the distance between Aaron's house and Bijal's house. Assume that the first third of the scale in black represents one inch.



You need to find the **distance** between the two houses in the map. The scale shows a sample distance. Use the scale to estimate the distance. You will find that *approximately* 3 segments of the length of the scale fit between the two points. Be careful — 3 is not the answer to this problem! As the scale shows 1 inch equal to 2 miles, you must multiply 3 units by 2 miles:

$$3 \text{ inches} \cdot \frac{2 \text{ miles}}{1 \text{ inch}} = 6 \text{ miles}$$

The **distance** between the houses is *about* six miles.

You can also use **estimation** to identify measurements in other geometric figures. Remember to include words like *approximately*, *about*, or *estimation* whenever you are finding an estimated answer.

The word "estimation" means using a non-exact guess of what a number is. Another similar word is "approximation."

Both of these words are nouns. The verb forms are: "to estimate" or "to approximate."

We use these words when we are not sure of the exact measurement of a distance, length, or other number, but when we can make an educated guess.

- To estimate (or to _____) a number means to give a non-exact but educated guess of what it is.

Rulers

You have probably been using **rulers** to measure distances for a long time and you know that a **ruler** is a *tool with measurement markings*.

- A **ruler** is a tool with _____ markings.

Using a ruler: If you use a ruler to find the distance between two points, the distance will be the **absolute value** of the *difference between the numbers* shown on the ruler.

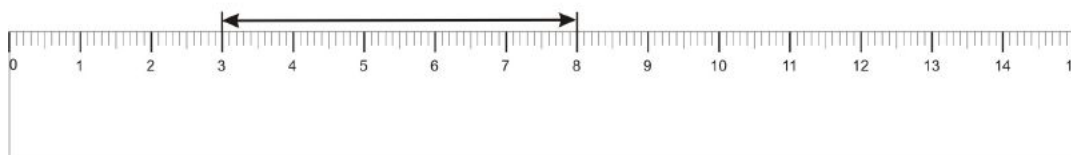
This means that you do not need to start measuring at the zero mark, as long as you use *subtraction* to find the distance.

Note: We say **absolute value** here since distances in geometry must always be *positive*, and subtraction can give a negative result.

- You do not need to measure from zero on a ruler; just _____ the start number from the end number to find the distance!
- The *distance* on a ruler is the _____ value of the *difference* between the numbers.
- Absolute value is always a _____ number.

Example 2

What distance is marked on the ruler in the diagram below? Assume that the scale is marked in centimeters.



The way to use the ruler is to find the **absolute value** of the *difference* between the numbers shown. This means you *subtract* the numbers and then make sure your answer is *positive*. The line segment spans from 3 cm to 8 cm:

$$|3 - 8| = |-5| = 5$$

The **absolute value** of the *difference* between the two numbers shown on the ruler above is 5 cm. So the line segment is 5 cm long.

Remember, we use vertical bars around an expression to show absolute value: $|x|$

Distances on a Grid

In algebra you most likely worked with graphing lines in the x – y **coordinate plane**. Sometimes you can find the distance between points on a **coordinate plane** using the values of the coordinates:

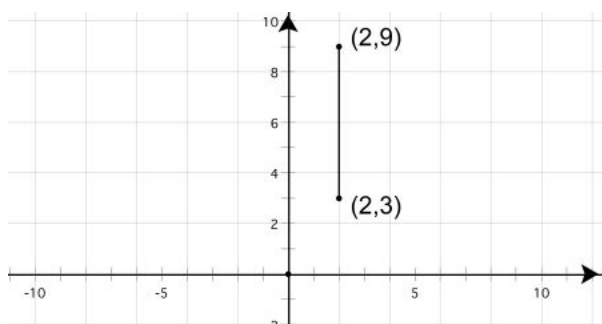
- If the two points line up *horizontally*, look at the change of value in the **x -coordinates**.
- If the two points line up *vertically*, look at the change of value in the **y -coordinates**.

The *change in value* will show the *distance* between the points. Remember to use **absolute value**, just like you did with the **ruler**. Later you will learn how to calculate distance between points that do not line up horizontally or vertically.

- When points line up *horizontally*, they have the *same* _____-coordinate. This means their _____-coordinates are *different* so we take their *difference* to find the distance between the points.
- When points line up *vertically*, they have the *same* _____-coordinate. This means their _____-coordinates are *different* so we take their *difference* to find the distance between the points.

Example 3

What is the distance between the two points shown below?



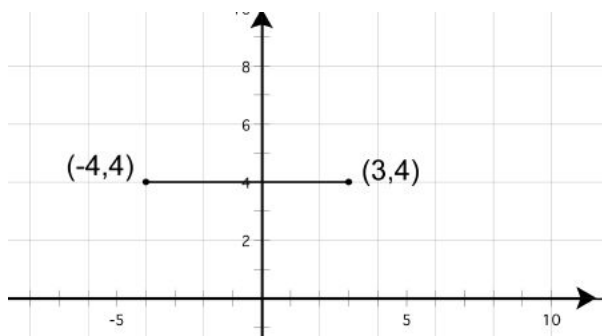
The two points shown on the grid are at (2, 9) and (2, 3). These points line up *vertically* (meaning they have the same **x-coordinate** of 2), so we can look at the *difference* in their **y-coordinates**:

$$|9 - 3| = |6| = 6$$

So, the distance between the two points is 6 units.

Example 4

What is the distance between the two points shown below?



The two points shown on the grid are at (-4, 4) and (3, 4). These points line up *horizontally* (meaning they have the same **y-coordinate** of 4), so we can look at the *difference* in their **x-coordinates**. Remember to take the **absolute value** of the *difference* between the values to find the distance:

$$|-4 - 3| = |-7| = 7$$

The distance between the two points is 7 units.

Reading Check:

1. What is absolute value? Explain in your own words.
2. When 2 points line up vertically, what value do they have in common?

3. When 2 points line up horizontally, what value do they have in common?

The Distance Formula

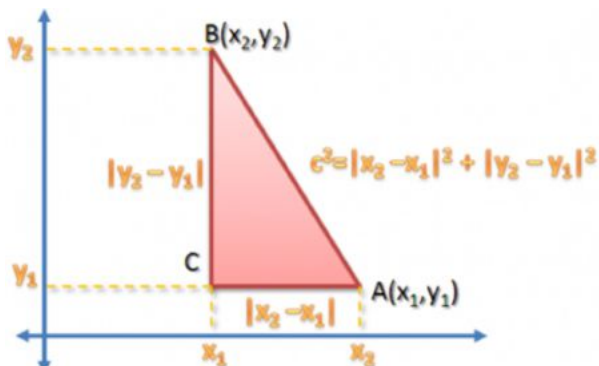
We have learned that a **right triangle** with sides of lengths a and b and hypotenuse of length c has a special relationship called the **Pythagorean Theorem**. The sum of the squares of a and b is equal to the square of c . Placing this in equation form we have:

$$a^2 + b^2 = c^2$$



If we put this triangle in a **coordinate plane** so A has coordinates of (x_1, y_1) and B has coordinates of (x_2, y_2) , we can find the lengths of the legs of the triangle using what we just learned about points that line up *horizontally* or *vertically*:

the length of AC is $|x_2 - x_1|$ and the length of BC is $|y_2 - y_1|$



We are finding the *length*, which means that we want a *positive* value; the **absolute value** bars guarantee

that our answer is always *positive*. But in the final equation,

$$c^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

the **absolute value** bars are *not* needed since we squared all three terms, and squared numbers are always *positive*.

Getting the square root of both sides we have,

$$c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We say that c is the distance between the points A and B , and we call the formula above the ***Distance Formula***.

Reading Check:

1. On which famous theorem is the Distance Formula based?
2. How can you find the distance of one of the *legs* of a right triangle like the one in the diagram on the previous page? Pick one leg and explain in your own words.

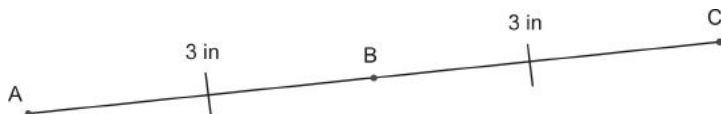
Segment Midpoints

Now that you understand congruent segments, there are a number of new terms and types of figures you can explore.

A **segment midpoint** is a point on a line segment that divides the segment into *two congruent segments*. So, each segment between the midpoint and an endpoint will have the same length.

- A **midpoint** divides a segment into two _____ parts.

In the diagram below, point B is the **midpoint** of \overline{AC} since \overline{AB} is *congruent* to \overline{BC} :



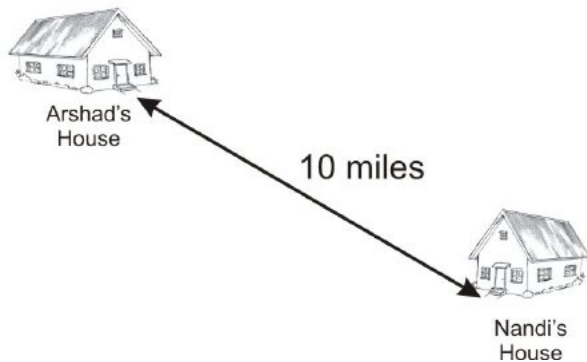
There is even a special postulate dedicated to midpoints:

Segment Midpoint Postulate

Any line segment will have exactly one midpoint—no more, and no less.

Example 5

Nandi and Arshad measure and find that their houses are 10 miles apart. If they agree to meet at the midpoint between their two houses, how far will each of them travel?



The easiest way to find the distance to the **midpoint** of the imagined segment connecting their houses is to divide the *length* (which is 10 miles) by 2:

$$10 \div 2 = 5$$

Each person will travel five miles to meet at the **midpoint** between their houses.

The Midpoint Formula

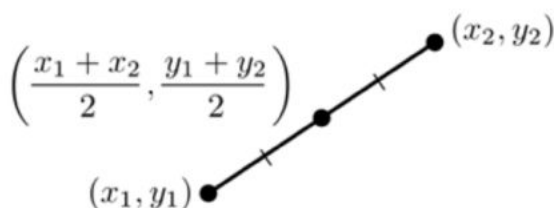
The **midpoint** is the *middle* point of a line segment. It is **equidistant** (*equal distances*) from both *endpoints*.

The formula for determining the **midpoint** of a segment in a **coordinate plane** is the *average* of the **x-coordinates** and the **y-coordinates**. Remember, to find the *average* of 2 numbers, you take the *sum* of the numbers and then *divide* by 2.

If a segment has endpoints (x_1, y_1) and (x_2, y_2) :

- the *average* of the **x-coordinates** is: $\frac{x_1 + x_2}{2}$
- and the *average* of the **y-coordinates** is: $\frac{y_1 + y_2}{2}$

Therefore, the **midpoint** is at:



Reading Check:

1. What is an average? Explain in your own words.

2. *Where is a midpoint located on a line segment? Describe.*
3. *What does the word equidistant mean?*
4. *How many midpoints can a line segment have?*
5. *In the space below, draw a line segment. Then draw and label its midpoint.*

2.3 Parallel and Perpendicular

Learning Objectives

- Identify and compute *slope* in the coordinate plane.
- Use the relationship between slopes of *parallel* lines.
- Use the relationship between slopes of *perpendicular* lines.
- Identify equations of *parallel* lines.
- Identify equations of *perpendicular* lines.

Slope in the Coordinate Plane

If you look at a graph of a line, you can think of the **slope** as the *steepness* of the line.

- **Slope** is the measure of the _____ of a line.

Mathematically, you can calculate the **slope** using two different points on a line. Given two points (x_1, y_1) and (x_2, y_2) the **slope** is:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

You may have also learned that **slope** equals “rise over run.”

This means that:

- The numerator (top) of the fraction is the “rise,” or how many units the slope goes *up* (positive) or *down* (negative).

→ *Up* or *down* is how the **slope** moves along the *y*-axis.

- The denominator (bottom) of the fraction is the “run,” or how many units the slope goes to the *right* (positive) or *left* (negative).

→ *Right* or *left* is how the **slope** moves along the *x*-axis.

You can remember “rise” as up or down because an elevator “rises” up or down.

*“Rise” (up/down) is in the *y* direction.*

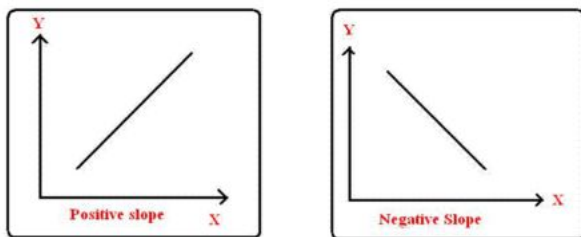
You can remember “run” as moving right or left because a person “runs” with her right and left feet.

*“Run” (right/left) is in the *x* direction.*

- The numerator of the **slope** represents the change in the _____ direction.
- The **slope’s** _____ represents the change in the *x* direction.

In other words, first calculate the distance that the line travels *up* (or *down*), and then divide that value by the distance the line travels *left to right*.

A line that goes *up* from *left to right* has *positive slope*, and a line that goes *down* from *left to right* has *negative slope*:



images from <http://www.tutorvista.com/math/positive-and-negative-slope>

- A line that goes *up* from left to right has a _____ slope.
- A line that goes *down* from left to right has a _____ slope.

Example 1

What is the slope of a line that goes through the points (2, 2) and (4, 6)?

You can use the **slope** formula on the previous page to find the **slope** of this line. When substituting values, (*x*₁, *y*₁) is (2, 2) and (*x*₂, *y*₂) is (4, 6):

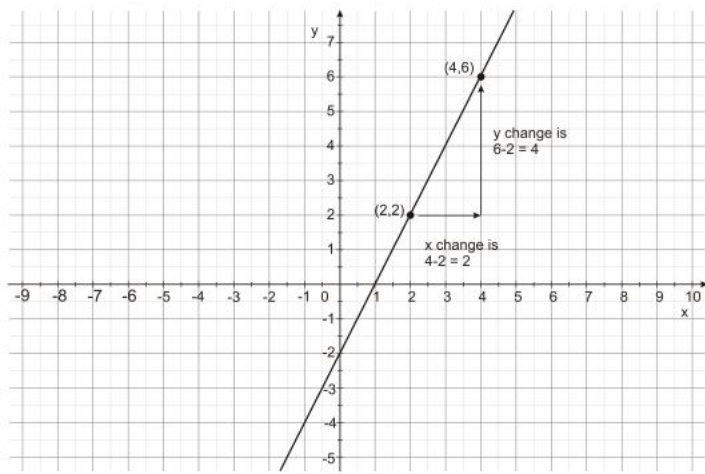
$x_1 =$ _____, $y_1 =$ _____, and $x_2 =$ _____, $y_2 =$ _____

The **slope** of this line is 2.

→ What does that mean graphically?

Look at the graph on the next page to see what the line looks like.

Notice: If the **slope** is *positive*, the line should go *up* from left to right. Does it?



You can see that the line “rises” 4 units as it “runs” 2 units to the right. So, the “rise” (numerator) is 4 units and the “run” (denominator) is 2 units. Since $4 \div 2 = 2$, the **slope** of this line is 2.

As you read on the previous page, the **slope** of this line is 2, a *positive* number.

- Any line with a *positive slope* will go _____ from *left* to *right*.
- Any line with a *negative slope* will go _____ from *left* to *right*.

Example 2

What is the slope of the line that goes through the points $(1, 9)$ and $(3, 3)$?

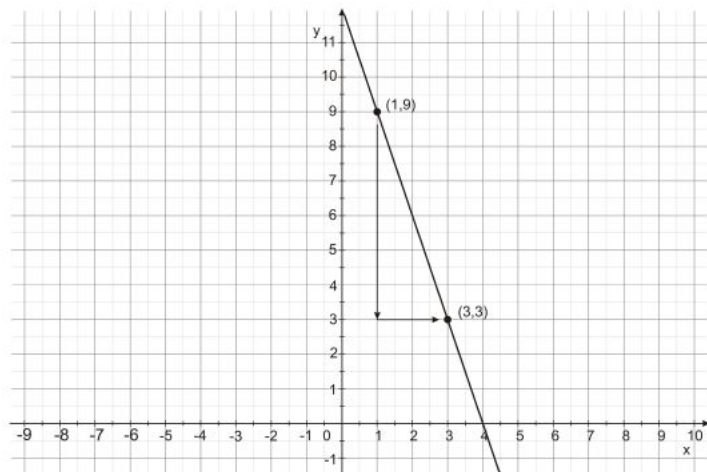
Again, use the formula to find the **slope** of this line:

$x_1 = \underline{\hspace{2cm}}, y_1 = \underline{\hspace{2cm}},$ and $x_2 = \underline{\hspace{2cm}}, y_2 = \underline{\hspace{2cm}}$

$$\begin{aligned} slope &= \frac{y_2 - y_1}{x_2 - x_1} \\ slope &= \frac{3 - 9}{3 - 1} = \frac{-6}{2} = -3 \end{aligned}$$

The **slope** of this line is -3 .

Because the **slope** of the line in example 2 is *negative*, it will go *down* to the *right*. The points and the line that connects them are shown below:



Some types of lines have special **slopes**. Check out following examples to see what happens with *horizontal* and *vertical* lines.

Example 3

What is the slope of a line that goes through the points (4, 4) and (8, 4)?

Use the formula to find the **slope** of this line:

$x_1 = \underline{\hspace{1cm}}, y_1 = \underline{\hspace{1cm}},$ and $x_2 = \underline{\hspace{1cm}}, y_2 = \underline{\hspace{1cm}}$

$$\begin{aligned} \text{slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ \text{slope} &= \frac{4 - 4}{8 - 4} = \frac{0}{8} = 0 \end{aligned}$$

The **slope** of this line is 0.

Every line with a **slope** of 0 is *horizontal*.

- A _____ line has a **slope** equal to zero.

Example 4

What is the slope of a line that goes through the points (3, 2) and (3, 6)?

Use the formula to find the **slope** of this line:

$x_1 = \underline{\hspace{1cm}}, y_1 = \underline{\hspace{1cm}},$ and $x_2 = \underline{\hspace{1cm}}, y_2 = \underline{\hspace{1cm}}$

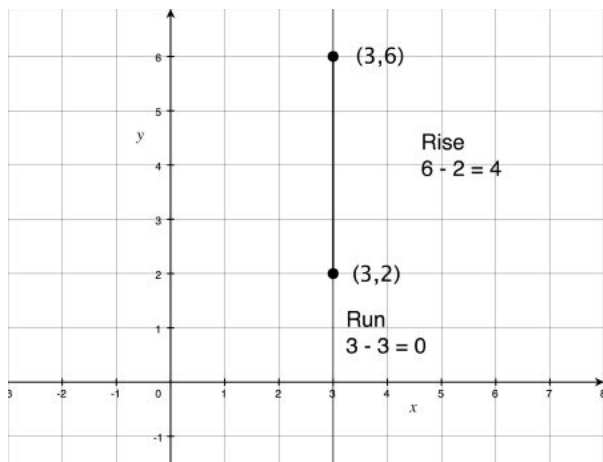
$$\begin{aligned} \text{slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ \text{slope} &= \frac{6 - 2}{3 - 3} = \frac{4}{0} \end{aligned}$$

Zero is *not* allowed to be in the denominator of a fraction! Therefore, the **slope** of this line is **undefined**.

Every line with an **undefined slope** is *vertical*.

- All *vertical* lines have **slopes** that are _____.

The line in example 4 is *vertical* and its slope is undefined:

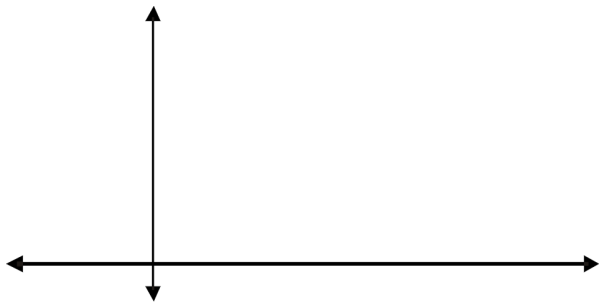


In review, if you look at a graph of a line from *left to right*, then:

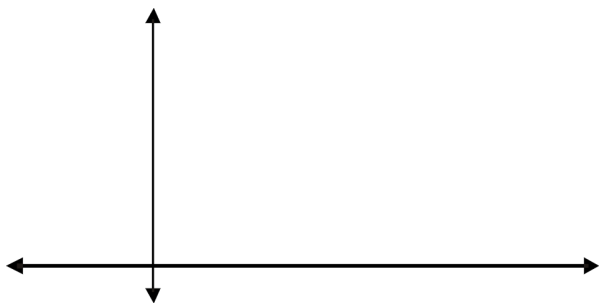
- Lines with *positive* slopes point *up* to the *right*.
- Lines with *negative* slopes point *down* to the *right*.
- *Horizontal* lines have a slope of *zero*.
- *Vertical* lines have *undefined* slope.

Reading Check:

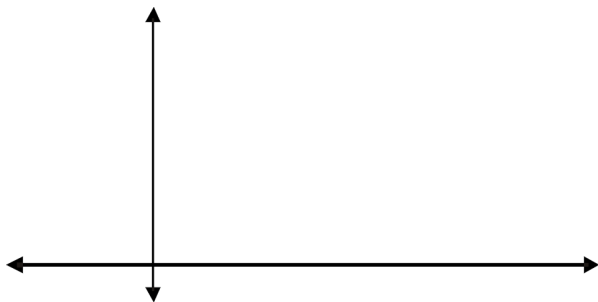
1. On the coordinate plane below, draw a line with a positive slope.



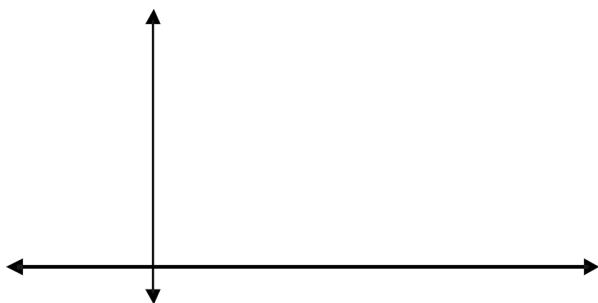
2. On the coordinate plane below, draw a line with a negative slope.



3. On the coordinate plane below, draw a line with a slope of zero.



4. On the coordinate plane below, draw a line with an undefined slope.



Slopes of Parallel Lines

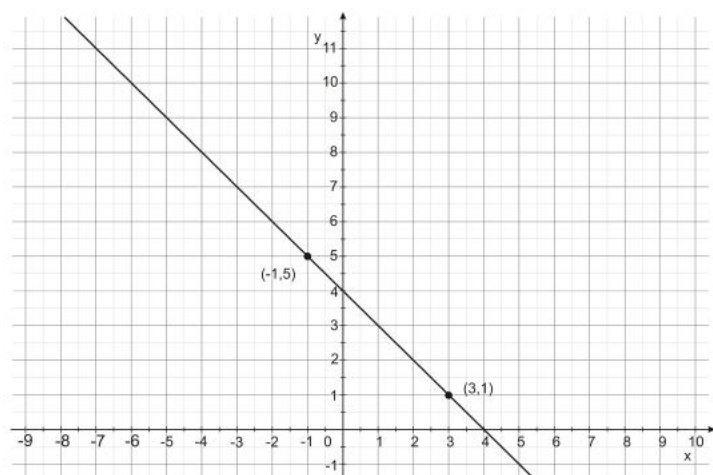
Now that you know how to find the **slope** of lines using x - and y -coordinates, you can think about how lines are related to their **slopes**.

If two lines in the coordinate plane are **parallel**, then they will have the *same* slope. Conversely, if two lines in the coordinate plane have the *same* slope, then those lines are **parallel**.

- **Parallel** lines have the _____ **slope**.

Example 5

Which of the following answers below could represent the slope of a line parallel to the one shown on the graph?



A. -4

B. -1

C. $\frac{1}{4}$

D. 1

Since you are looking for the **slope** of a **parallel** line, it will have the *same slope* as the line on the graph. First find the **slope** of the given line, and then choose the answer with that *same slope*. To do this, pick any two points on the line and use the **slope** formula.

For example, for the points $(-1, 5)$ and $(3, 1)$:

$x_1 = \underline{\hspace{2cm}}, y_1 = \underline{\hspace{2cm}},$ and $x_2 = \underline{\hspace{2cm}}, y_2 = \underline{\hspace{2cm}}$

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{slope} = \frac{1 - 5}{3 - (-1)} = \frac{-4}{3 + 1} = \frac{-4}{4} = -1$$

The **slope** of the line on the graph is -1 . The answer is **B**.

Slopes of Perpendicular Lines

Parallel lines have the *same slope*. There is also a mathematical relationship for the **slopes** of **perpendicular** lines.

The **slopes** of **perpendicular** lines will be the *opposite reciprocal* of each other.

Opposite here means the *opposite sign*.

If a **slope** is *positive*, then its *opposite* is *negative*.

If a **slope** is *negative*, then its *opposite* is *positive*.

A *reciprocal* is a fraction with its numerator and denominator *flipped*.

The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$. The reciprocal of $\frac{1}{2}$ is 2 . The reciprocal of 4 is $\frac{1}{4}$.

- The *opposite* of 5 is $\underline{\hspace{2cm}}$.
- The *reciprocal* of 5 is $\underline{\hspace{2cm}}$.

The *opposite reciprocal* can be found in two steps:

1. First, find the *reciprocal* of the given **slope**. If the **slope** is a fraction, you can simply *switch* the numbers in the numerator and the denominator to find the *reciprocal*. If the **slope** is not a fraction, you can make it into a fraction by putting a 1 in the denominator. Then find the *reciprocal* by flipping the numerator and denominator.
2. The second step is to find the *opposite* of the given number. If the value is *positive*, make it *negative*. If the value is *negative*, make it *positive*.

The *opposite reciprocal* of $\frac{5}{4}$ is $-\frac{4}{5}$ and the *opposite reciprocal* of -3 is $\frac{1}{3}$.

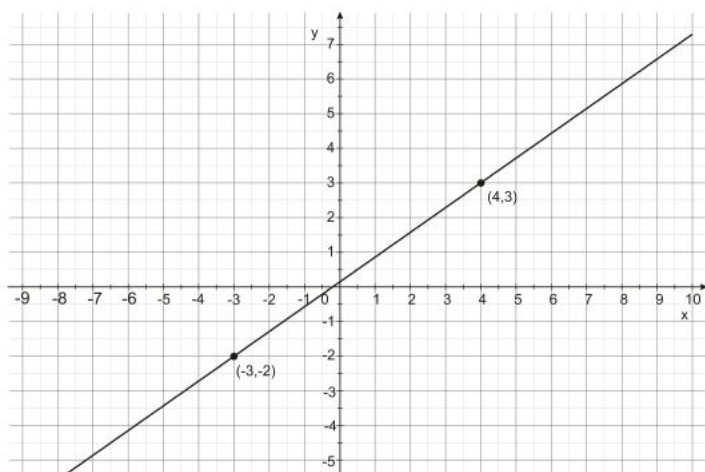
- The *opposite reciprocal* of 5 is $\underline{\hspace{2cm}}$.

Another way to check if lines are **perpendicular** is to *multiply* their **slopes**: if the **slopes** of two lines *multiply* to be -1 , then the two lines are **perpendicular**.

- The slopes of $\underline{\hspace{2cm}}$ lines multiply to be -1 .

Example 6

Which of the following numbers could represent the slope of a line perpendicular to the one shown below?



- A. $-\frac{7}{5}$
- B. $\frac{7}{5}$
- C. $-\frac{5}{7}$
- D. $\frac{5}{7}$

Since you are looking for the **slope** of a **perpendicular** line, it will be the *opposite reciprocal* of the **slope** of the line on the graph. First find the **slope** of the given line, then find its *opposite reciprocal*. You can use the slope formula to find the original line's slope. Pick two points on the line.

For example, for the points $(-3, -2)$ and $(4, 3)$:

$x_1 =$ _____, $y_1 =$ _____, and $x_2 =$ _____, $y_2 =$ _____

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{slope} = \frac{3 - (-2)}{4 - (-3)} = \frac{3 + 2}{4 + 3} = \frac{5}{7}$$

The **slope** of the line on the graph is $\frac{5}{7}$. Now find the *opposite reciprocal* of that value. First *switch* the numerator and denominator in the fraction, then find the *opposite sign*. The *opposite reciprocal* of $\frac{5}{7}$ is $-\frac{7}{5}$. The answer is A.

Slope-Intercept Equations

The most common type of linear equation to study is called **slope-intercept form**, which uses both the **slope** of the line and its **y-intercept**. A **y-intercept** is the point where the line crosses the vertical y-axis. This is the value of y when x is equal to 0.

- **Slope-intercept form** is an equation that uses the _____ and the _____ of a line.
- The **y-intercept** is the point where the line intersects the _____.
- At the **y-intercept**, x equals _____.

The formula for an equation in **slope-intercept form** is:

$$y = mx + b$$

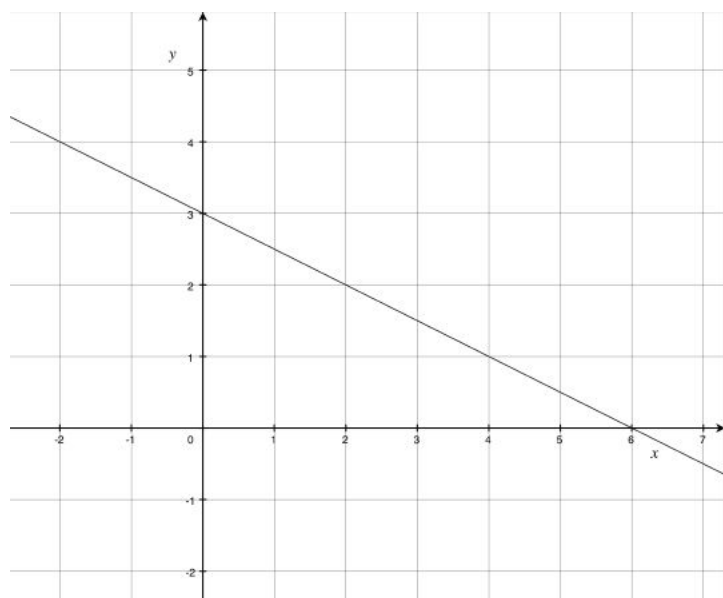
In this equation, y and x remain as variables, m is the **slope** of the line, and b is the **y-intercept** of the line. For example, if you know that a line has a **slope** of 4 and it crosses the y -axis at $(0, 8)$, then its equation in **slope-intercept form** is: $y = 4x + 8$.

- In **slope-intercept form**, m represents the _____.
- In **slope-intercept form**, b represents the _____.

This form is especially useful for finding the equation of a line given its graph. You already know how to calculate the **slope** by finding two points and using the slope formula. You can find the **y-intercept** by seeing where the line crosses the y -axis on the graph. The value of b is the y -coordinate of this point.

Example 7

Write an equation in slope-intercept form that represents the following line:



First find the **slope** of the line. You already know how to do this using the slope formula. There are no points given on the line, so you have to pick your own points. See where the line goes right through an intersection (corner point) on the graph paper. You can use the two points $(0, 3)$ and $(2, 2)$:

$x_1 = \underline{\hspace{2cm}}$, $y_1 = \underline{\hspace{2cm}}$, and $x_2 = \underline{\hspace{2cm}}$, $y_2 = \underline{\hspace{2cm}}$

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{slope} = \frac{2 - 3}{2 - 0} = \frac{-1}{2} = -\frac{1}{2}$$

The **slope** of the line is $-\frac{1}{2}$. This will replace m in the **slope-intercept** equation.

Now you need to find the **y-intercept**. On the graph, find where the line intersects the y -axis. It crosses the y -axis at $(0, 3)$ so the **y-intercept** is 3. This will replace b in the **slope-intercept** equation, so now you have all the information you need.

The equation for the line shown in the graph is: $y = -\frac{1}{2}x + 3$.

- In the slope-intercept equation, the slope is represented by the letter _____.
- In the slope-intercept equation, the y-intercept is the letter _____.

Equations of Parallel Lines

You studied **parallel** lines and their graphical relationships, so now you will learn how to easily identify equations of **parallel** lines. When looking for **parallel** lines, look for equations that have the *same slope*.

As long as the y-intercepts are *not* the same and the **slopes** are *equal*, the lines are **parallel**. If the y-intercept and the **slope** are *both* the *same*, then the two equations are for the *same* exact line, and a line cannot be parallel to itself.

- **Parallel** lines have the _____ **slope**.

Reading Check:

1. *True or false:*

The **reciprocal** of a fraction is when you flip the numerator and the denominator.

2. *Make up an example that supports the statement in #1 above.*

3. *What is the slope-intercept form of an equation?*

4. *What do the letters m and b stand for in the slope-intercept equation?*

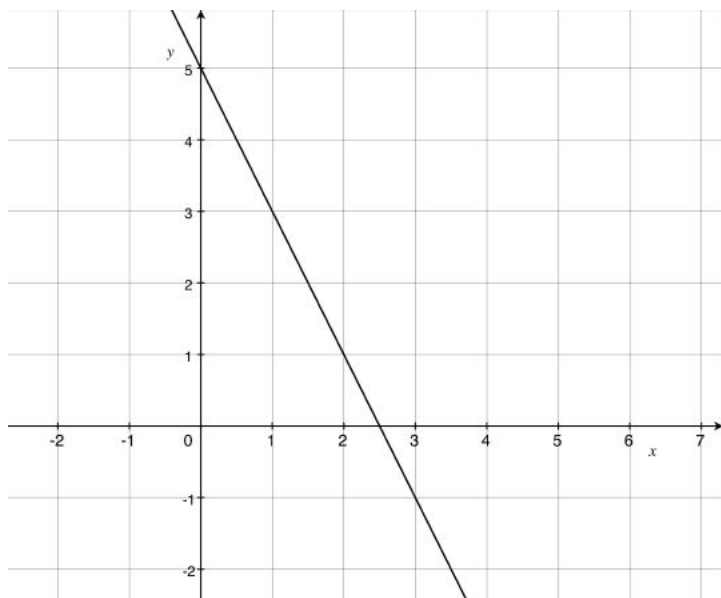
m :

b :

5. *How are the slopes of parallel lines related?*

Example 8

Juan drew the line below:



Which of the following equations could represent a line parallel to the one Juan drew?

- A. $y = -\frac{1}{2}x - 6$
- B. $y = \frac{1}{2}x + 9$
- C. $y = -2x - 18$
- D. $y = 2x + 1$

If you find the **slope** of the line in Juan's graph, you can find the **slope** of a **parallel** line because it will be the *same*. Pick two points on the graph and find the **slope** using the slope formula. Use the points (0, 5) and (1, 3) :

$x_1 = \underline{\hspace{2cm}}$, $y_1 = \underline{\hspace{2cm}}$, and $x_2 = \underline{\hspace{2cm}}$, $y_2 = \underline{\hspace{2cm}}$

$$\begin{aligned} \text{slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ \text{slope} &= \frac{3 - 5}{1 - 0} = \frac{-2}{1} = -2 \end{aligned}$$

The **slope** of Juan's line is -2 . Look at your four answer choices: which equation has a slope of -2 ? All other parts of the equation do not matter. The only equation that has a slope of -2 is choice C, so that is the correct answer.

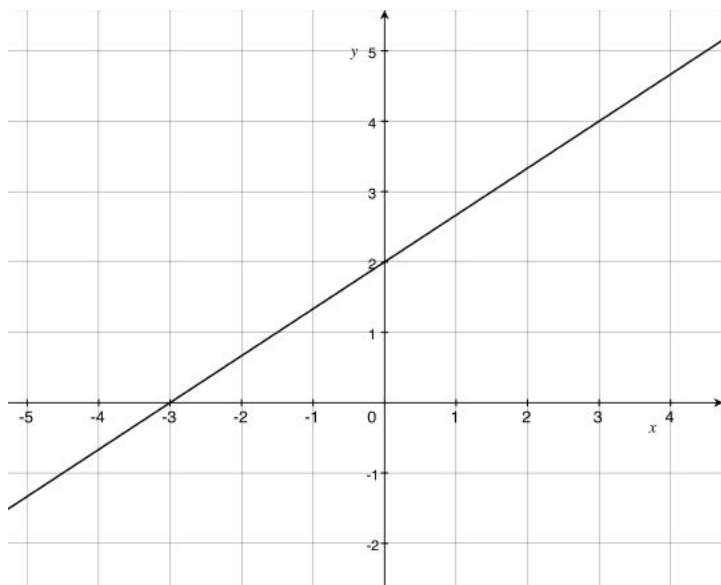
Equations of Perpendicular Lines

You also studied **perpendicular** lines and their graphical relationships: remember that the **slopes** of **perpendicular** lines are *opposite reciprocals*. To easily identify equations of **perpendicular** lines, look for equations that have **slopes** that are *opposite reciprocals* of each other.

Here, it does not matter what the **y-intercept** is; as long as the **slopes** are *opposite reciprocals*, the lines are **perpendicular**.

Example 9

Kara drew the line in this graph:



Which of the following equations could represent a line perpendicular to the one Kara drew above?

A. $y = \frac{3}{2}x + 10$

B. $y = -\frac{3}{2}x + 6$

C. $y = \frac{2}{3}x - 4$

D. $y = -\frac{2}{3}x - 1$

First find the **slope** of the line in Kara's graph. Then find the *opposite reciprocal* of this **slope**. To begin, pick two points on the graph and calculate the **slope** using the slope formula. Use the points (0, 2) and (3, 4) :

$x_1 = \underline{\hspace{2cm}}$, $y_1 = \underline{\hspace{2cm}}$, and $x_2 = \underline{\hspace{2cm}}$, $y_2 = \underline{\hspace{2cm}}$

$$\begin{aligned} \text{slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ \text{slope} &= \frac{4 - 2}{3 - 0} = \frac{2}{3} \end{aligned}$$

The **slope** of Kara's line on the graph is $\frac{2}{3}$.

Find the *opposite reciprocal*: the *reciprocal* of $\frac{2}{3}$ is $\frac{3}{2}$, and the *opposite* of $\frac{3}{2}$ is $-\frac{3}{2}$.

So, $-\frac{3}{2}$ is the *opposite reciprocal* of (or **perpendicular slope** to) $\frac{2}{3}$.

Now look in your answer choices for the equation that has a slope of $-\frac{3}{2}$.

The only equation that has a slope of $-\frac{3}{2}$ is choice B, so that is the correct answer.

Reading Check:

1. How are the slopes of perpendicular lines related to each other?

2. In the context of perpendicular slope values, what does *opposite* mean?

3. True or false: On a graph, perpendicular lines intersect at an angle of 45° .
4. Correct the statement in #3 above to make it true.

2.4 Equation of a Circle

Learning Objectives

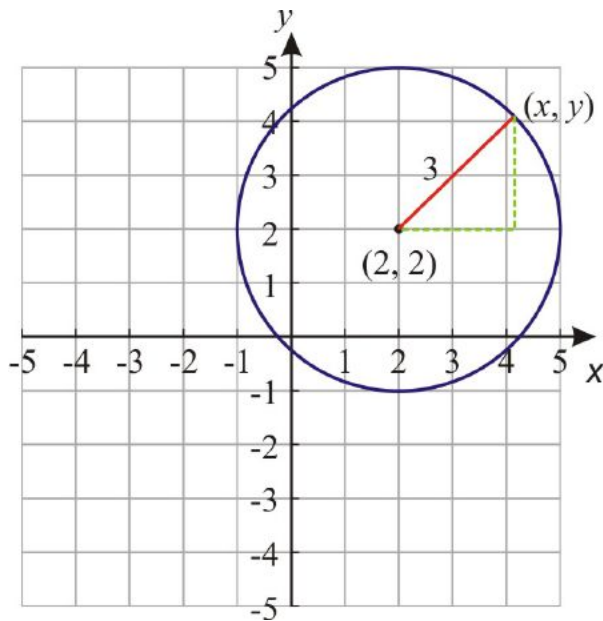
- Write the equation of a *circle*.

Equations and Graphs of Circles

A **circle** is defined as the set of all points that are the same distance from a single point called the **center**. This definition can be used to find an equation of a circle in the coordinate plane.

- A **circle** is the set of all points *equidistant* from the _____.

Look at the **circle** shown below. As you can see, this circle has its **center** at the point $(2, 2)$ and it has a **radius** of 3.



All of the points (x, y) on the **circle** are a *distance* of 3 units away from the **center** of the **circle**.

We can express this information as an equation with the help of the **Pythagorean Theorem**. The *right triangle* shown above has **legs** of lengths $(x - 2)$ and $(y - 2)$, and **hypotenuse** of length 3. We can write:

$$(x - 2)^2 + (y - 2)^2 = 3^2 \quad \text{or}$$

$$(x - 2)^2 + (y - 2)^2 = 9$$

We can *generalize* this equation for a **circle** with **center** at point (x_0, y_0) and **radius** r :

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

Example 1

Find the center and radius of the following circles:

A. $(x - 4)^2 + (y - 1)^2 = 25$

B. $(x + 1)^2 + (y - 2)^2 = 4$

A. We rewrite the equation as: $(x - 4)^2 + (y - 1)^2 = 5^2$. Compare this to the standard equation. The **center** of the **circle** is at the point $(4, 1)$ and the **radius** is 5.

B. We rewrite the equation as: $(x - (-1))^2 + (y - 2)^2 = 2^2$. The **center** of the **circle** is at the point $(-1, 2)$ and the **radius** is 2.

Example 2

Graph the following circles:

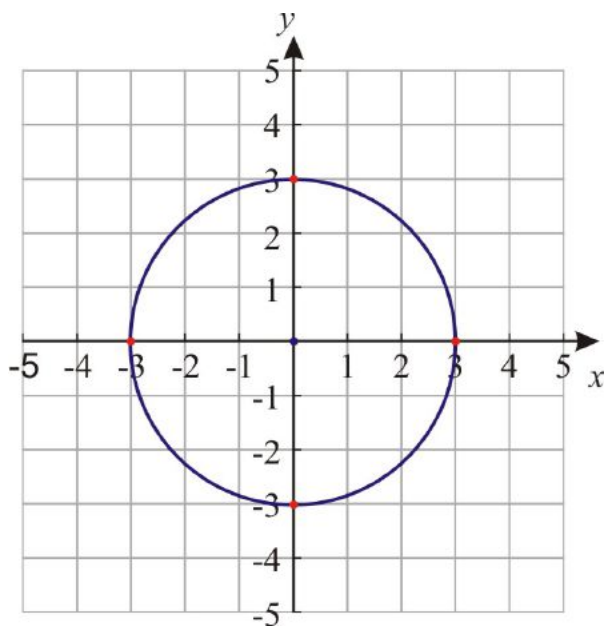
A. $x^2 + y^2 = 9$

B. $(x + 2)^2 + y^2 = 1$

In order to graph a **circle**, we first graph the **center** point and then draw points that are the *length* of the **radius** *away* from the **center** in the directions up, down, right, and left. Then connect the outer points in a smooth circle!

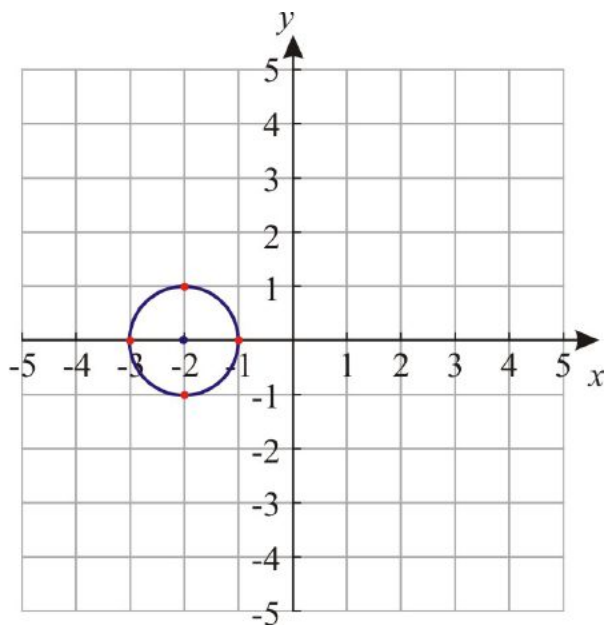
A. We rewrite the equation as: $(x - 0)^2 + (y - 0)^2 = 3^2$. The center of the circle is at the point $(0, 0)$ and the radius is 3.

Plot the center point and a point 3 units up at $(0, 3)$, 3 units down at $(0, -3)$, 3 units right at $(3, 0)$ and 3 units left at $(-3, 0)$:



B. We rewrite the equation as: $(x - (-2))^2 + (y - 0)^2 = 1^2$. The center of the circle is at the point $(-2, 0)$ and the radius is 1.

Plot the center point and a point 1 unit up at $(-2, 1)$, 1 unit down at $(-2, -1)$, 1 unit right at $(-1, 0)$ and 1 unit left at $(-3, 0)$:

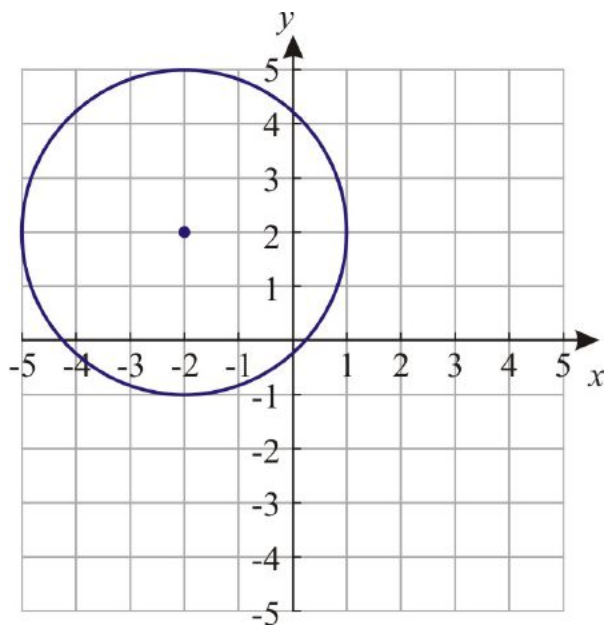


Reading Check:

1. *In your own words, describe the radius of a circle.*
2. *In the general equation of a circle $(x - x_0)^2 + (y - y_0)^2 = r^2$, the variables x_0 and y_0 stand for a special point. What point is this?*
3. *How can you find the radius of a circle from its equation? What do you need to do to the right side of the equation?*

Example 3

Write the equation of the circle in the graph below:



From the graph, we can see that the **center** of the **circle** is at the point $(-2, 2)$ and the **radius** is 3 units long, so we use these numbers in the standard circle equation:

$$\begin{aligned}(x + 2)^2 + (y - 2)^2 &= 3^2 \\(x + 2)^2 + (y - 2)^2 &= 9\end{aligned}$$

Example 4

Determine if the point $(1, 3)$ is on the circle given by the equation:

$$(x - 1)^2 + (y + 1)^2 = 16$$

In order to find the answer, we simply plug the point $(1, 3)$ into the equation of the **circle** given.

Substitute the number _____ for x and the number _____ for y :

$$\begin{aligned}(1 - 1)^2 + (3 + 1)^2 &= 16 \\(0)^2 + (4)^2 &= 16 \\16 &= 16\end{aligned}$$

Since we end up with a *true* statement, the point $(1, 3)$ satisfies the equation. Therefore, the point is on the circle.

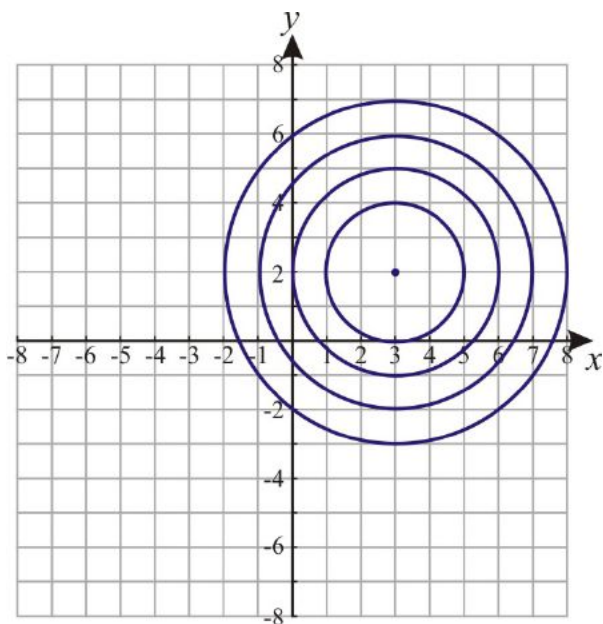
Concentric Circles

Concentric circles are circles of *different radii* that share the *same center* point.

- **Circles** with the *same* _____ but different _____ are called **concentric circles**.

Example 5

Write the equations of the concentric circles shown in the graph:



All 4 circles have the *same* **center** point at (3, 2) so we know the equations will all be:

$$(x - 3)^2 + (y - 2)^2$$

Since the circles have *different* **radius** lengths, the right side of the equations will all be different numbers.

The *smallest* **circle** has a radius of 2:

$$(x - 3)^2 + (y - 2)^2 = 2^2 \quad \text{or} \\ (x - 3)^2 + (y - 2)^2 = 4$$

The *next larger* **circle** has a radius of 3: $(x - 3)^2 + (y - 2)^2 = 9$

The *next larger* **circle** has a radius of 4: $(x - 3)^2 + (y - 2)^2 = 16$

The *largest* **circle** has a radius of 5: $(x - 3)^2 + (y - 2)^2 = 25$

Look at the word concentric:

In Spanish, the word “con” means “with.”

The second part of the word, “-centric” looks very similar to the word “center.”

When we put these two parts together, “concentric” means “with” the same “center.”

Reading Check:

1. If you are given a point and an equation of a circle, how can you tell if the given point is on the circle? Describe what you would do.

2. What are concentric circles?

3. If you are given two equations of two different circles, how can you tell if the circles are concentric? Describe what the two equations would have to have in common.

2.5 Translating and Reflecting

Learning Objectives

- Graph a *translation* in a coordinate plane.
- Recognize that a *translation* is an *isometry*.
- Find the *reflection* of a point in a line on a coordinate plane.
- Verify that a *reflection* is an *isometry*.

Translations

A **translation** moves *every* point a given *horizontal* distance and/or a given *vertical* distance.

- When a point is moved a certain distance horizontally and/or vertically, the move is called a ____-_____.

For example, if a **translation** moves point $A(3, 7)$ 2 units to the *right* and 4 units *up* to $A'(5, 11)$, then this **translation** moves *every* point in a larger figure the *same way*.

The symbol next to the letter A' above is called the **prime** symbol.

The **prime** symbol looks like an apostrophe like you may use to show possessive, such as, “that is my brother’s book.”

(The apostrophe is before the *s* in brother’s)

In math, we use the **prime** symbol to show that two things are related.

In the **translation** above, the original point is related to the translated point, so instead of renaming the translated point, we use the **prime** symbol to show this.

The *original* point (or figure) is called the **preimage** and the *translated* point (or figure) is called the **image**. In the example given above, the **preimage** is point $A(3, 7)$ and the **image** is point $A'(5, 11)$. The **image** is *designated* (or *shown*) with the **prime** symbol.

- Another name for the *original* point is the _____.
- Another name for the *translated* point is the _____.
- The *translated* point uses the _____ symbol next to its naming letter.

Example 1

The point $A(3, 7)$ in a **translation** becomes the point $A'(2, 4)$. What is the **image** of $B(-6, 1)$ in the same **translation**?

Point A moved 1 unit to the *left* and 3 units *down* to get to A' . Point B will also move 1 unit to the *left* and 3 units *down*.

We *subtract* 1 from the x -coordinate and 3 from the y -coordinate of point B :

$$B' = (-6 - 1, 1 - 3) = (-7, -2)$$

$B'(-7, -2)$ is the **image** of $B(-6, 1)$.

Using the **Distance Formula**, you can notice the following:

$$\begin{aligned} AB &= \sqrt{(-6 - 3)^2 + (1 - 7)^2} = \sqrt{(-9)^2 + (-6)^2} = \sqrt{117} \\ A'B' &= \sqrt{(-7 - 2)^2 + (-2 - 4)^2} = \sqrt{(-9)^2 + (-6)^2} = \sqrt{117} \end{aligned}$$

Since the endpoints of \overline{AB} and $\overline{A'B'}$ moved the *same* distance horizontally and vertically, both segments have the *same length*.

Translation is an Isometry

An **isometry** is a transformation in which *distance* is “preserved.” This means that the *distance* between any two points in the **preimage** (*before* the **translation**) is the *same* as the *distance* between the points in the **image** (*after* the **translation**).

- An **isometry** is when _____ is *preserved* from the **preimage** to the **image**.

As you saw in Example 1 above:

The **preimage** AB = the **image** $A'B'$ (since they are both equal to $\sqrt{117}$)

Would we get the same result for any other point in this translation? The answer is yes. It is clear that for any point X , the distance from X to X' will be $\sqrt{117}$. Every point moves $\sqrt{117}$ units to its **image**.

This is true in general:

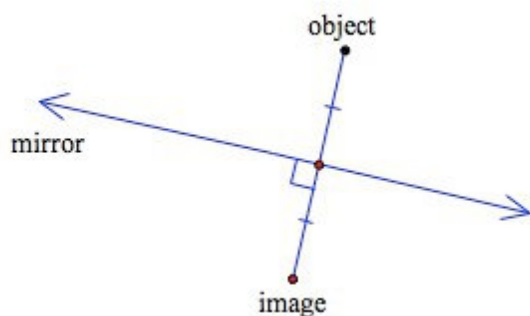
Translation Isometry Theorem

Every **translation** in the coordinate plane is an **isometry**.

- Every translation in an $x - y$ coordinate plane is an _____.

Reflection in a Line

A **reflection** in a line is as if the line were a *mirror*:



- When an object is **reflected** in a line, the line is like a _____.

An object **reflects** in the mirror, and we see the **image** of the object.

- The **image** is the *same* distance behind the mirror line as the object is in front of the mirror line.
- The “line of sight” from the *object* to the *mirror* is **perpendicular** to the mirror line itself.
- The “line of sight” from the *image* to the *mirror* is also **perpendicular** to the mirror line.

Reflection of a Point in a Line

Point P' is the **reflection** of point P in line k if and only if line k is the **perpendicular bisector** of $\overline{PP'}$.

- The mirror line is a perpendicular _____ of the line that connects the *object* to its reflected image.

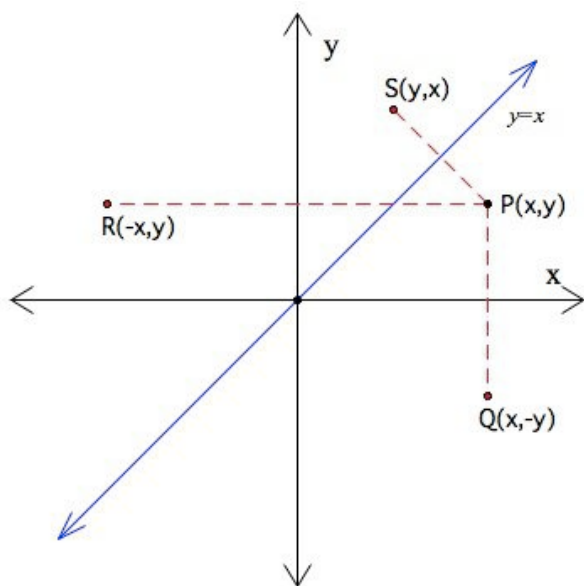
Reflections in Special Lines

In a coordinate plane there are some “special” lines for which it is relatively easy to create **reflections**:

- the x -axis
- the y -axis
- the line $y = x$ (this line makes a 45° angle between the x -axis and the y -axis)
- The _____-axis, the _____-axis, and the line _____ = _____ are “special” lines to use as *mirrors* when finding **reflections** of figures.

We can develop simple formulas for reflections in these lines.

Let $P(x, y)$ be a point in the coordinate plane:



We now have the following reflections of $P(x, y)$:

- Reflection of P in the x -axis is $Q(x, -y)$

[the x -coordinate stays the *same*, and the y -coordinate is *opposite*]

- Reflection of P in the y -axis is $R(-x, y)$

[the x -coordinate is *opposite*, and the y -coordinate stays the *same*]

- Reflection of P in the line $y = x$ is $S(y, x)$

[switch the x -coordinate and the y -coordinate]

Look at the graph above and you will be convinced of the first two **reflections** in the axes. We will prove the third **reflection** in the line $y = x$ on the next page.

- Reflections in the x -axis have the same _____-coordinate, but the y -coordinate has the _____ value.
- Reflections in the y -axis have an _____ x -coordinate, and the y -coordinate stays the _____.
- For reflections in the $y = x$ line, _____ the x - and y -coordinates.

Example 2

Prove that the reflection of point $P(h, k)$ in the line $y = x$ is the point $S(k, h)$.

Here is an “outline” proof:

First, we know the **slope** of the line $y = x$ is 1 because $y = 1x + 0$.

Next, we will investigate the **slope** of the line that connects our two points, \overline{PS} . Use the slope formula and the values of the points’ coordinates given above:

Slope of \overline{PS} is $\frac{k-h}{h-k} = \frac{-1(h-k)}{h-k} = -1$

Therefore, we have just shown that \overline{PS} and $y = x$ are **perpendicular** because the *product* of their **slopes** is -1 .

Finally, we can show that $y = x$ is the **perpendicular bisector** of \overline{PS} by finding the **midpoint** of \overline{PS} :
Midpoint of \overline{PS} is $\left(\frac{h+k}{2}, \frac{h+k}{2}\right)$

We know the **midpoint** of \overline{PS} is on the line $y = x$ because the x -coordinate and the y -coordinate of the **midpoint** are the same.

Therefore, the line $y = x$ is the **perpendicular bisector** of \overline{PS} .

Conclusion: The points P and S are **reflections** in the line $y = x$.

Example 3

Point $P(5, 2)$ is reflected in the line $y = x$. The image is P' . P' is then reflected in the y -axis. The image is P'' . What are the coordinates of P'' ?

We find one reflection at a time:

- Reflect P in the line $y = x$ to find P' :

For reflections in the line $y = x$ we _____ coordinates.

Therefore, P' is $(2, 5)$.

- Reflect P' in the y -axis:

For reflections in the y -axis, the x -coordinate is _____ and the y -coordinate stays the _____.

Therefore, P'' is $(-2, 5)$.

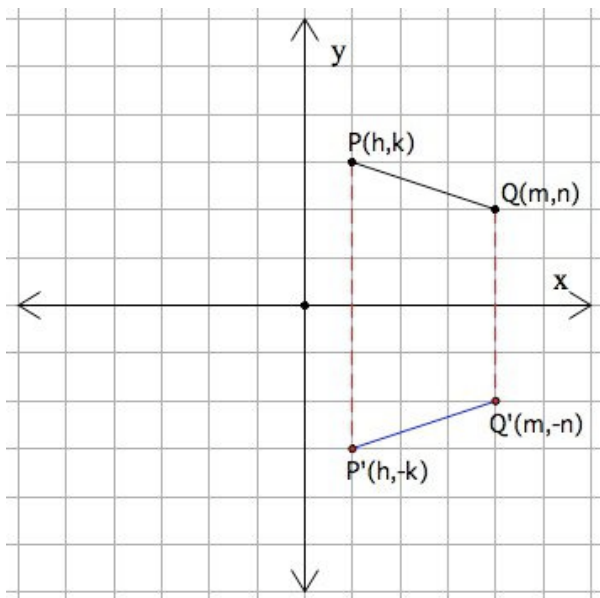
Reflections Are Isometries

Like a **translation**, a **reflection** in a line is also an **isometry**. Distance between points is “preserved” (stays the same).

- A reflection in a line is an _____, which means that *distance* is *preserved*.

We will verify the **isometry** for **reflection** in the x -axis. The proof is very similar for **reflection** in the y -axis.

The diagram below shows \overline{PQ} and its reflection in the x -axis, $\overline{P'Q'}$:



Use the Distance Formula:

$$\begin{aligned}
 PQ &= \sqrt{(m-h)^2 + (n-k)^2} \\
 P'Q' &= \sqrt{(m-h)^2 + (-n-(-k))^2} = \sqrt{(m-h)^2 + (k-n)^2} \\
 &= \sqrt{(m-h)^2 + (n-k)^2}
 \end{aligned}$$

So $PQ = P'Q'$

Conclusion: When a segment is **reflected** in the x -axis, the image segment has the *same length* as the *original preimage* segment. This is the meaning of **isometry**. You can see that a similar argument would apply to **reflection** in *any* line.

Reading Check:

1. *True or false:* Both translations and reflections are isometries.
2. *What is the meaning of the statement in #1 above?*

3. *If a translation rule is $(x + 3, y - 1)$, in which directions is a point moved?*

4. *When a point or figure is reflected in a line, that line acts as a mirror.*
 - a. *How does the x -axis change a point that is reflected? What do you do to the coordinates of the point in this type of reflection?*

b. How does the y -axis change a point that is reflected? What do you do to the coordinates of the point in this type of reflection?

c. How does the line $y = x$ change a point that is reflected? What do you do to the coordinates of the point in this type of reflection?

2.6 Rotating

Learning Objectives

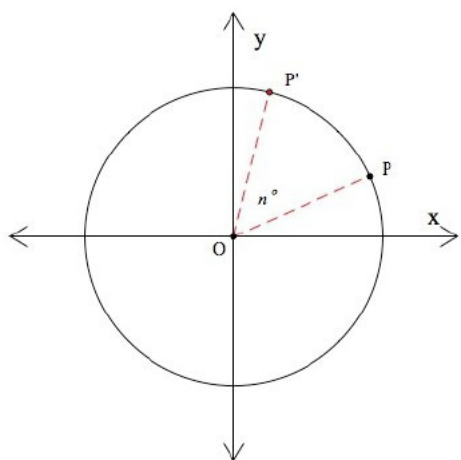
- Find the image of a point in a *rotation* in a coordinate plane.
- Recognize that a *rotation* is an *isometry*.

Sample Rotations

In this lesson we will study **rotations** *centered* at the *origin* of a coordinate plane. We begin with some specific examples of **rotations**. Later we will see how these **rotations** fit into a general formula.

We define a **rotation** as follows: In a **rotation** centered at the *origin* with an **angle of rotation** of n° , a point moves *counterclockwise* along an **arc** of a circle. The **central angle** of the circle measures n° .

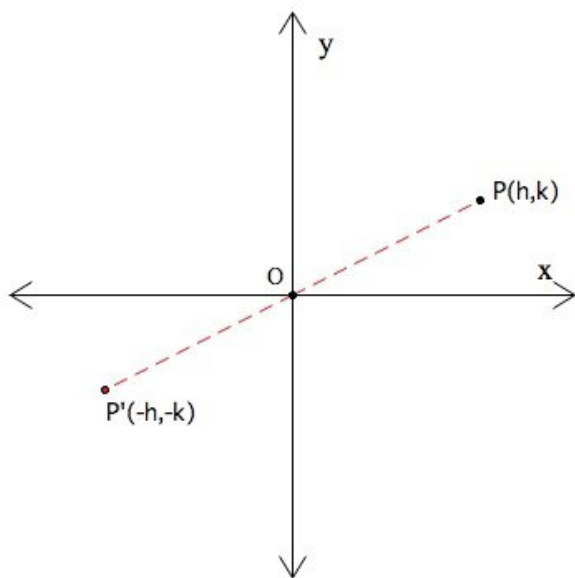
The *original* **preimage** point is one *endpoint* of the **arc**, and the **image** of the *original* point is the *other endpoint* of the **arc**:



- **Rotations** centered at the origin move points _____ along an **arc** of a circle.
- For a **rotation** of n° , the **central angle** of the circle measures _____.
- The **preimage** point is one endpoint of the _____ and the **image** is the other endpoint.

180° Rotation

Our first example is **rotation** through an angle of 180°:



In a 180° **rotation**, the **image** of $P(h, k)$ is the point $P'(-h, -k)$.

Notice:

- P and P' are the *endpoints* of a **diameter** of a circle.

→ This means that the distance from the point P to the *origin* (or the distance from the point P' to the *origin*) is a **radius** of the circle.

The distance from P to the origin *equals* the distance from _____ to the origin.

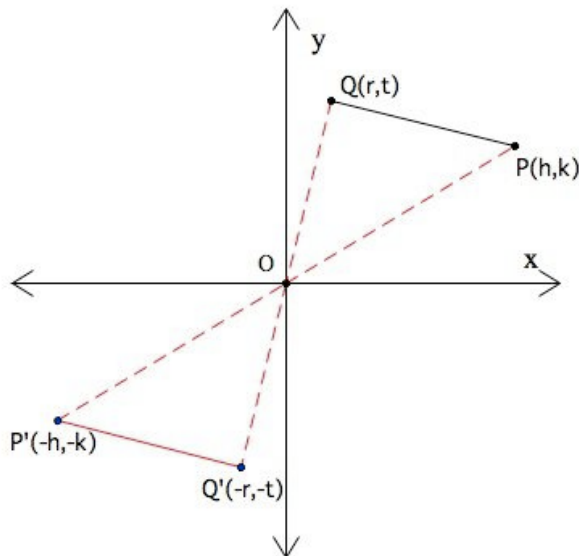
- The **rotation** is the *same* as a “**reflection** in the *origin*.”

→ This means that if we use the *origin* as a mirror, the point P is directly across from the point P' .

A 180° _____ is also a **reflection** in the *origin*.

In a **rotation** of 180° , the x -coordinate and the y -coordinate of the _____ become the *negative* versions of the values in the **image**.

A 180° **rotation** is an **isometry**. The **image** of a segment is a *congruent* segment:



Use the Distance Formula:

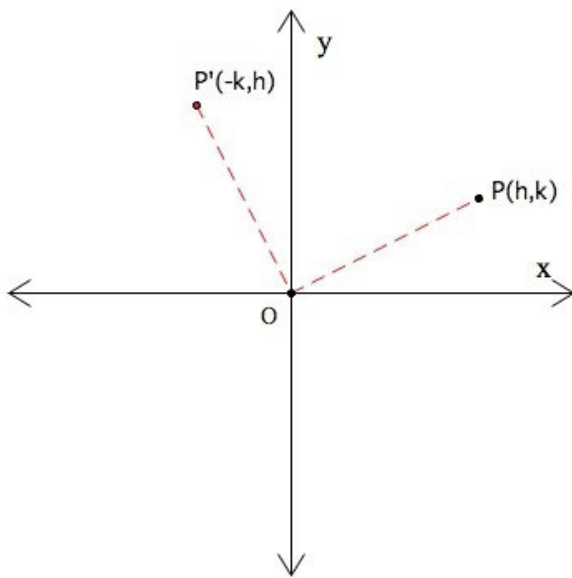
$$\begin{aligned}
 PQ &= \sqrt{(k-t)^2 + (h-r)^2} \\
 P'Q' &= \sqrt{(-k-(-t))^2 + (-h-(-r))^2} = \sqrt{(-k+t)^2 + (-h+r)^2} \\
 &= \sqrt{(t-k)^2 + (r-h)^2} \\
 &= \sqrt{(k-t)^2 + (h-r)^2}
 \end{aligned}$$

So $PQ = P'Q'$

- A 180° **rotation** is an _____, so distance is preserved.
- When a segment is **rotated** 180° (or **reflected** in the *origin*), its **image** is a _____-
_____ segment.

90° Rotation

The next example is a **rotation** through an angle of 90° . The **rotation** is in the *counterclockwise* direction:



In a 90° **rotation**, the **image** of $P(h, k)$ is the point $P'(-k, h)$.

Notice:

- \overline{PO} and $\overline{P'O}$ are both **radii** of the same circle, so $PO = P'O$.

If PO and $P'O$ are both **radii**, then they are the *same* _____.

- $\angle POP'$ is a *right* angle.
- The **acute** angle formed by \overline{PO} and the x -axis and the **acute** angle formed by $\overline{P'O}$ and the x -axis are **complementary** angles.

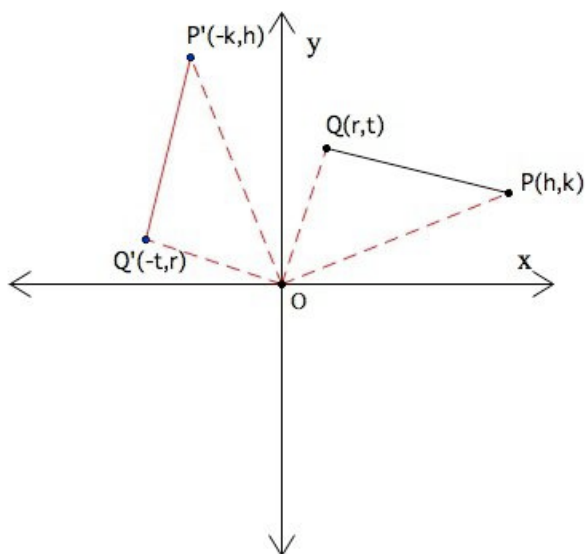
Remember, **complementary** angles add up to _____ $^\circ$.

You can see by the coordinates of the **preimage** and **image** points, in a 90° **rotation**:

- the x - and y -coordinates are *switched* AND
- the x -coordinate is *negative*.

In a 90° **rotation**, *switch* the _____ - and _____-coordinates and make the new x -coordinate _____.

A 90° **rotation** is an **isometry**. The **image** of a segment is a *congruent* segment.



Use the Distance Formula:

$$PQ = \sqrt{(k-t)^2 + (h-r)^2}$$

$$P'Q' = \sqrt{(h-r)^2 + (-k-(-t))^2} = \sqrt{(h-r)^2 + (t-k)^2}$$

$$= \sqrt{(k-t)^2 + (h-r)^2}$$

So $PQ = P'Q'$

Reading Check:

Which of the following are isometries? Circle all that apply:

30° rotation

45° rotation

60° rotation

90° rotation

150° rotation

180° rotation

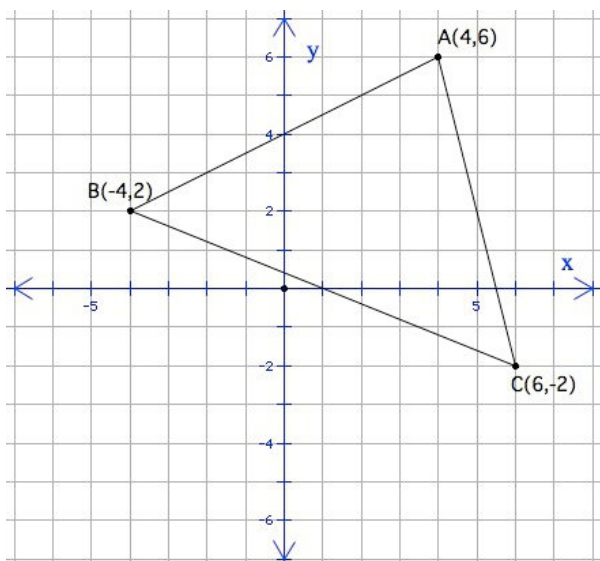
Reflection

Translation

Bisection

Example 1

What are the coordinates of the vertices of $\triangle ABC$ in a rotation of 90° ?



Point A is $(4, 6)$, B is $(-4, 2)$, and C is $(6, -2)$.

In a 90° **rotation**, the x -coordinate and the y -coordinate are *switched* AND the new x -coordinate is made *negative*:

- A becomes A' : switch x and y to $(6, 4)$ and make x negative $(-6, 4)$
- B becomes B' : switch x and y to $(2, -4)$ and make x negative $(-2, -4)$
- C becomes C' : switch x and y to $(-2, 6)$ and make x negative $(-(-2), 6) = (2, 6)$

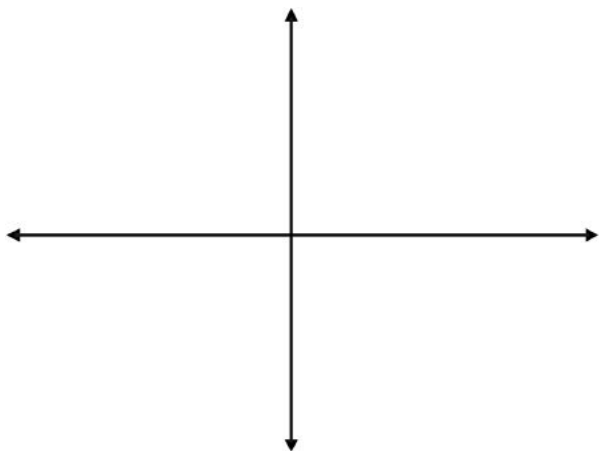
So the vertices of $\triangle A'B'C'$ are $(-6, 4)$, $(-2, -4)$, and $(2, 6)$.

Plot each of these points on the coordinate plane above and draw in each side of the new **rotated** triangle. Can you see how $\triangle ABC$ is **rotated** 90° to $\triangle A'B'C'$?

Reading Check:

1. *True or false:* A rotation is always in the counterclockwise direction.
2. *On the coordinate plane below, create a point anywhere you like, and label it P .*

Then draw a second point W that is the image of point P rotated 180° .



3. *On the coordinate plane above, draw a third point R that is the image of your original point P rotated 90° .*

4. *Is a 90° rotation an isometry? Explain.*
5. *Is a 180° rotation an isometry? Explain.*
6. *What type of rotation is the same as a reflection in the origin?*

Chapter 3

Triangles and Congruence

In this chapter, you will learn all about triangles. First, we will find out how many degrees are in a triangle and other properties of the angles within a triangle. Second, we will use that information to determine if two different triangles are congruent. Finally, we will investigate the properties of isosceles and equilateral triangles.

3.1 Triangle Sums

Learning Objectives

- Understand the Triangle Sum Theorem.
- Identify interior and exterior angles in a triangle.
- Use the Exterior Angle Theorem.

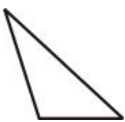
Review Queue

Classify the triangles below by their angles and sides.

1.



2.



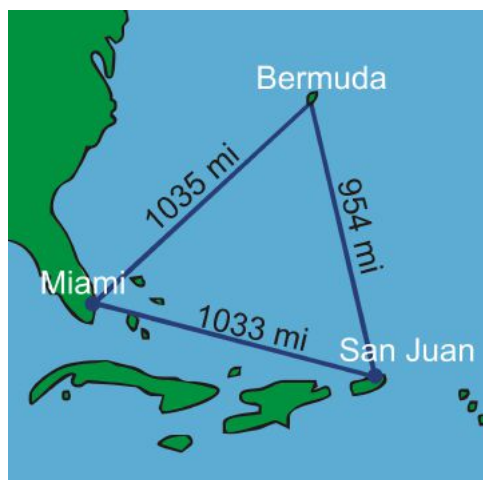
3.



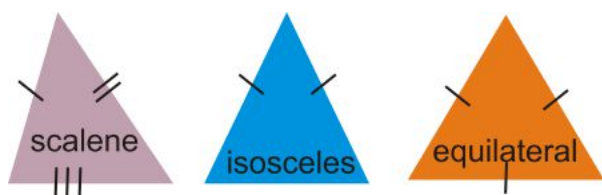
4. Draw and label a straight angle, $\angle ABC$. Which point is the vertex? How many degrees does a straight angle have?

Know What? To the right is the Bermuda Triangle. The myth of this triangle is that ships and planes have passed through and mysteriously disappeared.

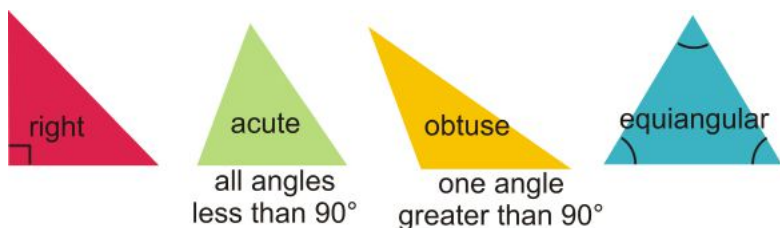
The measurements of the sides of the triangle are in the picture. Classify the Bermuda triangle by its sides and angles. Then, using a protractor, find the measure of each angle. What do they add up to?



Recall that a triangle can be classified by its sides...

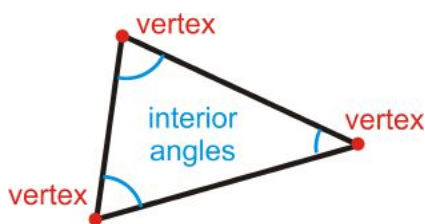


and its angles...



Interior Angles: The angles inside of a polygon.

Vertex: The point where the sides of a polygon meet.



Triangles have three interior angles, three vertices, and three sides.

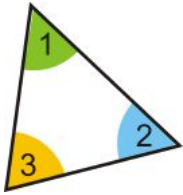
A triangle is labeled by its vertices with a Δ . This triangle can be labeled ΔABC , ΔACB , ΔBCA , ΔBAC , ΔCBA or ΔCAB .

Triangle Sum Theorem The interior angles in a polygon are measured in degrees. How many degrees are there in a triangle?

Investigation 4-1: Triangle Tear-Up

Tools Needed: paper, ruler, pencil, colored pencils

1. Draw a triangle on a piece of paper. Make all three angles different sizes. Color the three interior angles three different colors and label each one, $\angle 1$, $\angle 2$, and $\angle 3$.



2. Tear off the three colored angles, so you have three separate angles.

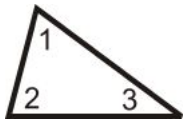


3. Line up the angles so the vertices points all match up. What happens? What measure do the three angles add up to?



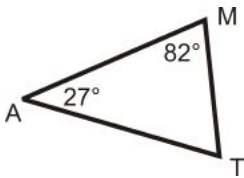
This investigation shows us that the sum of the angles in a triangle is 180° because the three angles fit together to form a straight angle where all the vertices meet.

Triangle Sum Theorem: The interior angles of a triangle add up to 180° .



$$m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$$

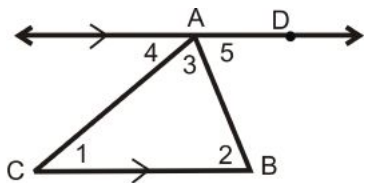
Example 1: What $m\angle T$?



Solution: Set up an equation.

$$\begin{aligned} m\angle M + m\angle A + m\angle T &= 180^\circ \\ 82^\circ + 27^\circ + m\angle T &= 180^\circ \\ 109^\circ + m\angle T &= 180^\circ \\ m\angle T &= 71^\circ \end{aligned}$$

Even though Investigation 4-1 is a way to show that the angles in a triangle add up to 180° , it is not a proof. Here is the proof of the Triangle Sum Theorem.



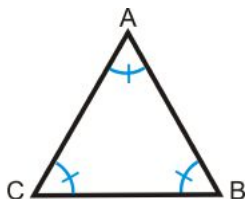
Given: $\triangle ABC$ with $\overleftrightarrow{AD} \parallel \overline{BC}$

Prove: $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$

Table 3.1:

<i>Statement</i>	<i>Reason</i>
1. $\triangle ABC$ above with $\overleftrightarrow{AD} \parallel \overline{BC}$	Given
2. $\angle 1 \cong \angle 4$, $\angle 2 \cong \angle 5$	Alternate Interior Angles Theorem
3. $m\angle 1 = m\angle 4$, $m\angle 2 = m\angle 5$	\cong angles have $=$ measures
4. $m\angle 4 + m\angle CAD = 180^\circ$	Linear Pair Postulate
5. $m\angle 3 + m\angle 5 = m\angle CAD$	Angle Addition Postulate
6. $m\angle 4 + m\angle 3 + m\angle 5 = 180^\circ$	Substitution PoE
7. $m\angle 1 + m\angle 3 + m\angle 2 = 180^\circ$	Substitution PoE

Example 2: What is the measure of each angle in an equiangular triangle?



Solution: $\triangle ABC$ is an equiangular triangle, where all three angles are equal. Write an equation.

$$m\angle A + m\angle B + m\angle C = 180^\circ$$

$$m\angle A + m\angle A + m\angle A = 180^\circ$$

Substitute, all angles are equal.

$$3m\angle A = 180^\circ$$

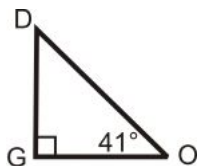
Combine like terms.

$$m\angle A = 60^\circ$$

If $m\angle A = 60^\circ$, then $m\angle B = 60^\circ$ and $m\angle C = 60^\circ$.

Each angle in an equiangular triangle is 60° .

Example 3: Find the measure of the missing angle.



Solution: $m\angle O = 41^\circ$ and $m\angle G = 90^\circ$ because it is a right angle.

$$\begin{aligned}
 m\angle D + m\angle O + m\angle G &= 180^\circ \\
 m\angle D + 41^\circ + 90^\circ &= 180^\circ \\
 m\angle D + 41^\circ &= 90^\circ \\
 m\angle D &= 49^\circ
 \end{aligned}$$

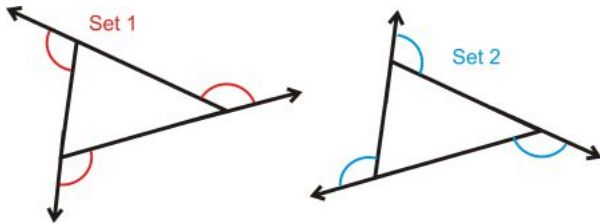
Notice that $m\angle D + m\angle O = 90^\circ$.

The acute angles in a right triangle are always complementary.

Exterior Angles

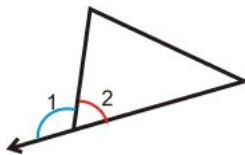
Exterior Angle: The angle formed by one side of a polygon and the extension of the adjacent side.

In all polygons, there are two sets of exterior angles, one that goes around clockwise and the other goes around counterclockwise.

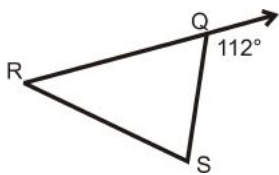


Notice that the interior angle and its adjacent exterior angle form a linear pair and add up to 180° .

$$m\angle 1 + m\angle 2 = 180^\circ$$



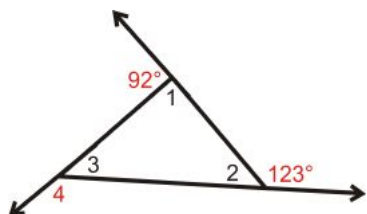
Example 4: Find the measure of $\angle RQS$.



Solution: 112° is an exterior angle of $\triangle RQS$ and is supplementary to $\angle RQS$.

$$\begin{aligned}
 112^\circ + m\angle RQS &= 180^\circ \\
 m\angle RQS &= 68^\circ
 \end{aligned}$$

Example 5: Find the measure of the numbered interior and exterior angles in the triangle.



Solution: $m\angle 1 + 92^\circ = 180^\circ$ by the Linear Pair Postulate. $m\angle 1 = 88^\circ$

$m\angle 2 + 123^\circ = 180^\circ$ by the Linear Pair Postulate. $m\angle 2 = 57^\circ$

$$m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ \quad \text{by the Triangle Sum Theorem.}$$

$$88^\circ + 57^\circ + m\angle 3 = 180$$

$$m\angle 3 = 35^\circ$$

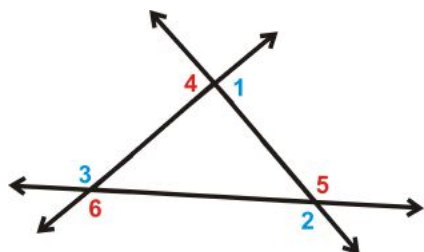
Lastly, $m\angle 3 + m\angle 4 = 180^\circ$ by the Linear Pair Postulate.

$$35^\circ + m\angle 4 = 180^\circ$$

$$m\angle 4 = 145^\circ$$

In Example 5, the exterior angles are 92° , 123° , and 145° . Adding these angles together, we get $92^\circ + 123^\circ + 145^\circ = 360^\circ$. This is true for any set of exterior angles for any polygon.

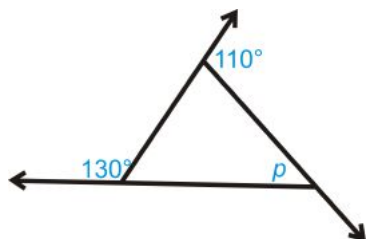
Exterior Angle Sum Theorem: The exterior angles of a polygon add up to 360° .



$$m\angle 1 + m\angle 2 + m\angle 3 = 360^\circ$$

$$m\angle 4 + m\angle 5 + m\angle 6 = 360^\circ$$

Example 6: What is the value of p in the triangle below?



Solution: First, we need to find the missing exterior angle, let's call it x . Set up an equation using the Exterior Angle Sum Theorem.

$$130^\circ + 110^\circ + x = 360^\circ$$

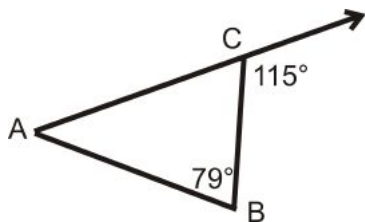
$$x = 360^\circ - 130^\circ - 110^\circ$$

$$x = 120^\circ$$

x and p add up to 180° because they are a linear pair.

$$\begin{aligned}x + p &= 180^\circ \\120^\circ + p &= 180^\circ \\p &= 60^\circ\end{aligned}$$

Example 7: Find $m\angle A$.



Solution:

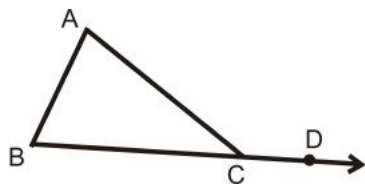
$$\begin{aligned}m\angle ACB + 115^\circ &= 180^\circ && \text{because they are a linear pair} \\m\angle ACB &= 65^\circ \\m\angle A + 65^\circ + 79^\circ &= 180^\circ && \text{by the Triangle Sum Theorem} \\m\angle A &= 36^\circ\end{aligned}$$

Remote Interior Angles: The two angles in a triangle that are not adjacent to the indicated exterior angle.

In Example 7 above, $\angle A$ and 79° are the remote interior angles relative to 115° .

Exterior Angle Theorem From Example 7, we can find the sum of $m\angle A$ and $m\angle B$, which is $36^\circ + 79^\circ = 115^\circ$. This is equal to the exterior angle at C .

Exterior Angle Theorem: The sum of the remote interior angles is equal to the non-adjacent exterior angle.



$$m\angle A + m\angle B = m\angle ACD$$

Proof of the Exterior Angle Theorem

Given: Triangle with exterior $\angle 4$

Prove: $m\angle 1 + m\angle 2 = m\angle 4$

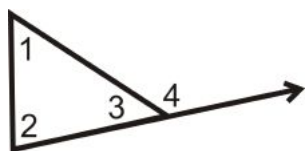
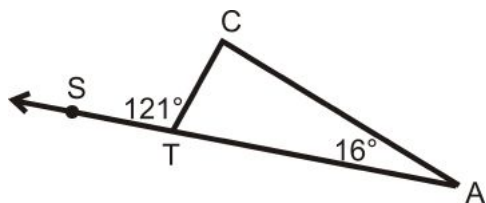


Table 3.2:

<i>Statement</i>	<i>Reason</i>
1. Triangle with exterior $\angle 4$	Given
2. $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$	Triangle Sum Theorem
3. $m\angle 3 + m\angle 4 = 180^\circ$	Linear Pair Postulate
4. $m\angle 1 + m\angle 2 + m\angle 3 = m\angle 3 + m\angle 4$	Transitive PoE
5. $m\angle 1 + m\angle 2 = m\angle 4$	Subtraction PoE

Example 8: Find $m\angle C$.



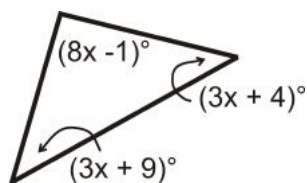
Solution: Using the Exterior Angle Theorem

$$m\angle C + 16^\circ = 121^\circ$$

$$m\angle C = 105^\circ$$

If you forget the Exterior Angle Theorem, you can do this problem just like Example 7.

Example 9: Algebra Connection Find the value of x and the measure of each angle.



Solution: All the angles add up to 180° .

$$(8x - 1)^\circ + (3x + 9)^\circ + (3x + 4)^\circ = 180^\circ$$

$$(14x + 12)^\circ = 180^\circ$$

$$14x = 168^\circ$$

$$x = 12^\circ$$

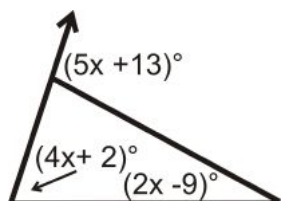
Substitute in 12° for x to find each angle.

$$3(12^\circ) + 9^\circ = 45^\circ$$

$$3(12^\circ) + 4^\circ = 40^\circ$$

$$8(12^\circ) - 1^\circ = 95^\circ$$

Example 10: Algebra Connection Find the value of x and the measure of each angle.



Solution: Set up an equation using the Exterior Angle Theorem.

$$\begin{array}{ccc}
 (4x + 2)^\circ + (2x - 9)^\circ & = & (5x + 13)^\circ \\
 \uparrow \quad \nearrow & & \uparrow \\
 \text{interior angles} & & \text{exterior angle} \\
 (6x - 7)^\circ & = & (5x + 13)^\circ \\
 x & = & 20^\circ
 \end{array}$$

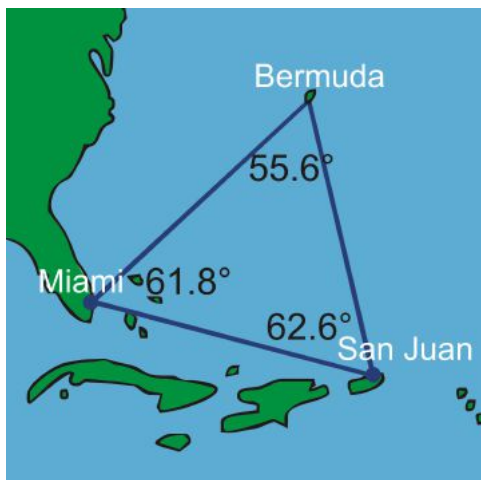
Substitute in 20° for x to find each angle.

$$4(20^\circ) + 2^\circ = 82^\circ$$

$$2(20^\circ) - 9^\circ = 31^\circ$$

$$\text{Exterior angle: } 5(20^\circ) + 13^\circ = 113^\circ$$

Know What? Revisited The Bermuda Triangle is an acute scalene triangle. The actual angle measures are in the picture to the right. Your measured angles should be within a degree or two of these measures and should add up to 180° . However, because your measures are estimates using a protractor, they might not exactly add up.

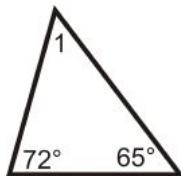


Review Questions

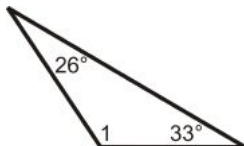
- Questions 1-16 are similar to Examples 1-8.
- Questions 17 and 18 use the definition of an Exterior Angle and the Exterior Angle Sum Theorem.
- Question 19 is similar to Example 3.
- Questions 20-27 are similar to Examples 9 and 10.

Determine $m\angle 1$.

1.



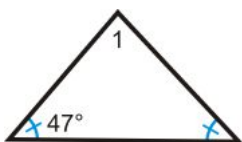
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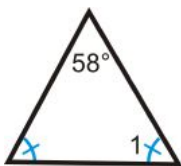
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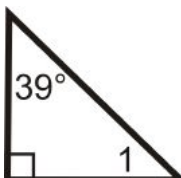
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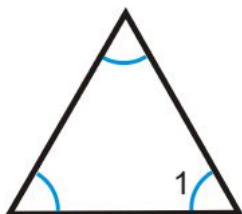
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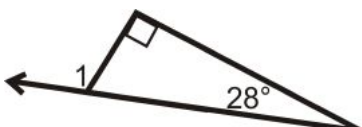
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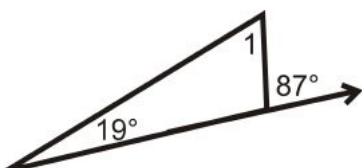
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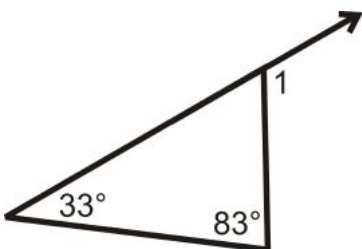
10.



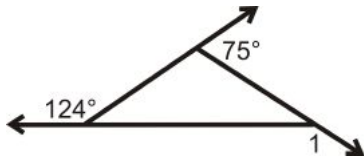
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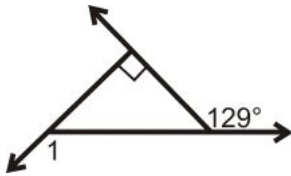
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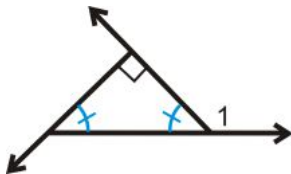
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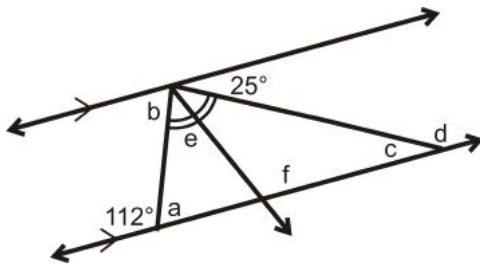
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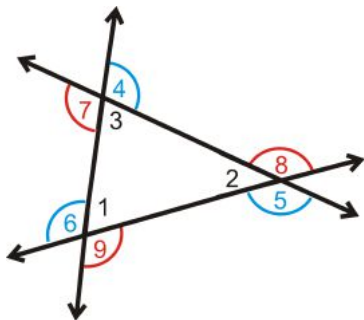
15.



16. Find the lettered angles, $a - f$, in the picture to the right. Note that the two lines are parallel.



17. Draw both sets of exterior angles on the same triangle.



- What is $m\angle 1 + m\angle 2 + m\angle 3$?
- What is $m\angle 4 + m\angle 5 + m\angle 6$?
- What is $m\angle 7 + m\angle 8 + m\angle 9$?
- List all pairs of congruent angles.

18. Fill in the blanks in the proof below.

Given: The triangle to the right with interior angles and exterior angles.

Prove: $m\angle 4 + m\angle 5 + m\angle 6 = 360^\circ$

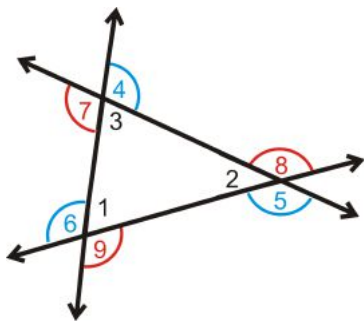


Table 3.3:

<i>Statement</i>	<i>Reason</i>
1. Triangle with interior and exterior angles.	Given
2. $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$	
3. $\angle 3$ and $\angle 4$ are a linear pair, $\angle 2$ and $\angle 5$ are a linear pair, and $\angle 1$ and $\angle 6$ are a linear pair	
4.	Linear Pair Postulate (do all 3)
5. $m\angle 1 + m\angle 6 = 180^\circ$	
$m\angle 2 + m\angle 5 = 180^\circ$	
$m\angle 3 + m\angle 4 = 180^\circ$	
6. $m\angle 1 + m\angle 6 + m\angle 2 + m\angle 5 + m\angle 3 + m\angle 4 = 540^\circ$	
7. $m\angle 4 + m\angle 5 + m\angle 6 = 360^\circ$	

19. Fill in the blanks in the proof below.

Given: $\triangle ABC$ with right angle B .

Prove: $\angle A$ and $\angle C$ are complementary.

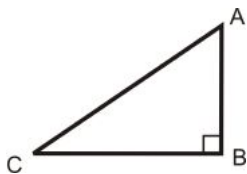
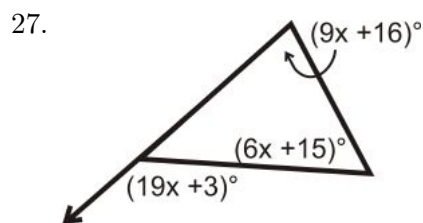
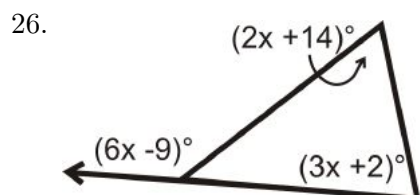
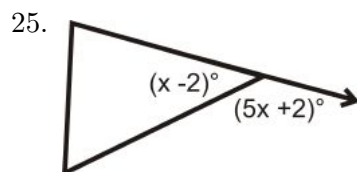
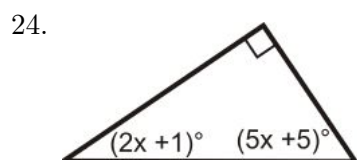
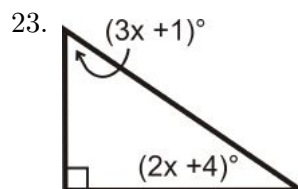
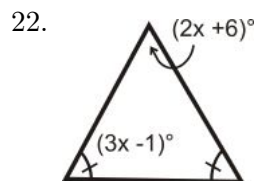
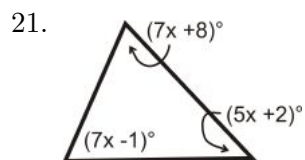
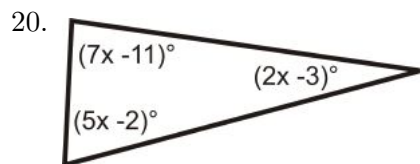


Table 3.4:

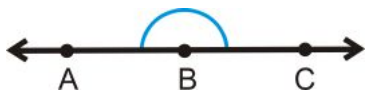
<i>Statement</i>	<i>Reason</i>
1. $\triangle ABC$ with right angle B .	Given
2.	Definition of a right angle
3. $m\angle A + m\angle B + m\angle C = 180^\circ$	
4. $m\angle A + 90^\circ + m\angle C = 180^\circ$	
5.	
6. $\angle A$ and $\angle C$ are complementary	

Algebra Connection Solve for x .



Review Queue Answers

1. acute isosceles
2. obtuse scalene
3. right scalene
4. B is the vertex, 180° ,



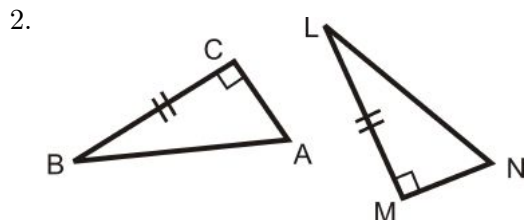
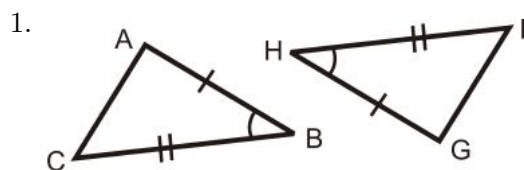
3.2 Congruent Figures

Learning Objectives

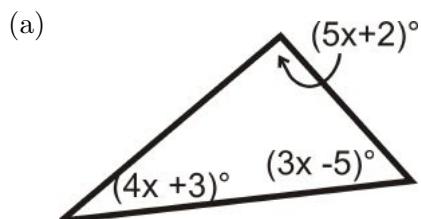
- Define congruent triangles and use congruence statements.
- Understand the Third Angle Theorem.

Review Queue

What part of each pair of triangles are congruent? Write out each congruence statement for the marked congruent sides and angles.



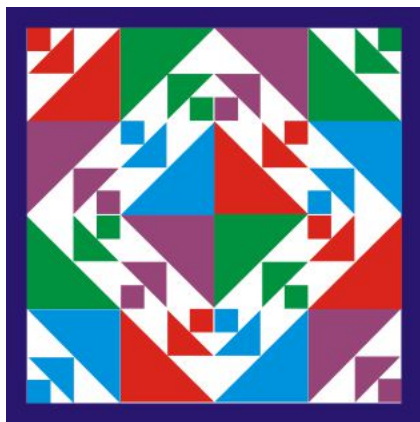
3. Determine the measure of x .



- (b) What is the measure of each angle?
 (c) What type of triangle is this?

Know What? Quilt patterns are very geometrical. The pattern to the right is made up of several congruent figures. In order for these patterns to come together, the quilter rotates and flips each block (in this case, a large triangle, smaller triangle, and a smaller square) to get new patterns and arrangements.

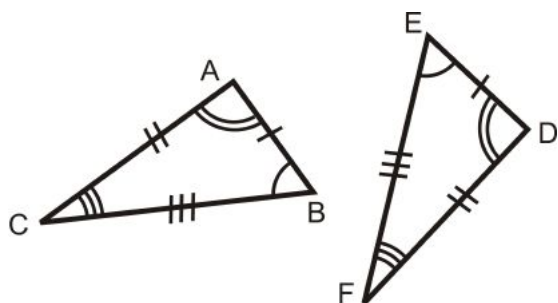
How many different sets of colored congruent triangles are there? How many triangles are in each set? How do you know these triangles are congruent?



Congruent Triangles

Two figures are congruent if they have exactly the same size and shape.

Congruent Triangles: Two triangles are congruent if the three corresponding angles and sides are congruent.



$\triangle ABC$ and $\triangle DEF$ are congruent because

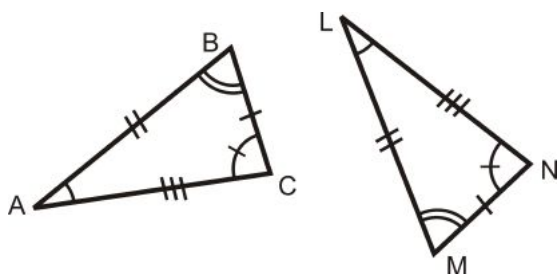
$$\overline{AB} \cong \overline{DE} \quad \angle A \cong \angle D$$

$$\overline{BC} \cong \overline{EF} \text{ and } \angle B \cong \angle E$$

$$\overline{AC} \cong \overline{DF} \quad \angle C \cong \angle F$$

When referring to corresponding congruent parts of congruent triangles it is called **Corresponding Parts of Congruent Triangles are Congruent**, or **CPCTC**.

Example 1: Are the two triangles below congruent?



Solution: To determine if the triangles are congruent, match up sides with the same number of tic marks:
 $\overline{BC} \cong \overline{MN}$, $\overline{AB} \cong \overline{LM}$, $\overline{AC} \cong \overline{LN}$.

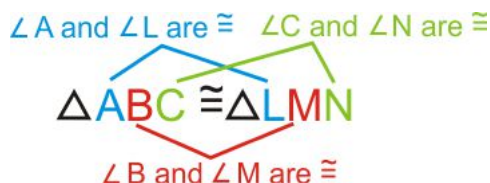
Next match up the angles with the same markings:

$\angle A \cong \angle L$, $\angle B \cong \angle M$, and $\angle C \cong \angle N$.

Lastly, we need to make sure these are *corresponding* parts. To do this, check to see if the congruent angles are opposite congruent sides. Here, $\angle A$ is opposite \overline{BC} and $\angle L$ is opposite \overline{MN} . Because $\angle A \cong \angle L$ and $\overline{BC} \cong \overline{MN}$, they are corresponding. Doing this check for the other sides and angles, we see that everything matches up and the two triangles are congruent.

Creating Congruence Statements

In Example 1, we determined that $\triangle ABC$ and $\triangle LMN$ are congruent. When stating that two triangles are congruent, the corresponding parts must be written in the same order. Using Example 1, we would have:



Notice that the congruent sides also line up within the congruence statement.

$\overline{AB} \cong \overline{LM}$, $\overline{BC} \cong \overline{MN}$, $\overline{AC} \cong \overline{LN}$

We can also write this congruence statement five other ways, as long as the congruent angles match up. For example, we can also write $\triangle ABC \cong \triangle LMN$ as:

$$\triangle ACB \cong \triangle LNM$$

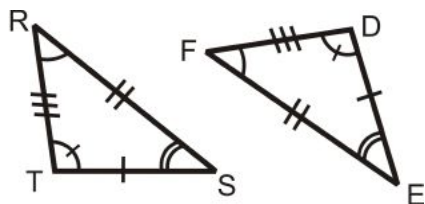
$$\triangle BCA \cong \triangle MNL$$

$$\triangle BAC \cong \triangle MLN$$

$$\triangle CBA \cong \triangle NML$$

$$\triangle CAB \cong \triangle NLM$$

Example 2: Write a congruence statement for the two triangles below.



Solution: Line up the corresponding angles in the triangles:

$\angle R \cong \angle F$, $\angle S \cong \angle E$, and $\angle T \cong \angle D$.

$\triangle RST \cong \triangle FED$

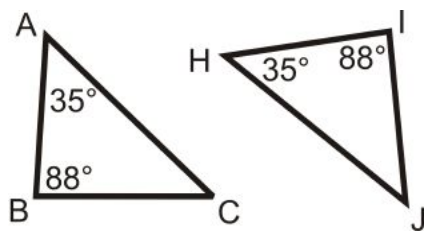
Example 3: If $\triangle CAT \cong \triangle DOG$, what else do you know?

Solution: From this congruence statement, we know three pairs of angles and three pairs of sides are congruent.



Third Angle Theorem

Example 4: Find $m\angle C$ and $m\angle J$.



Solution: The sum of the angles in a triangle is 180° .

$$\triangle ABC : 35^\circ + 88^\circ + m\angle C = 180^\circ$$

$$m\angle C = 57^\circ$$

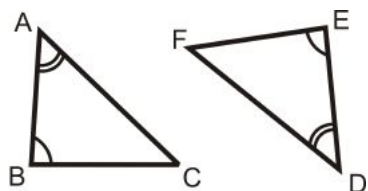
$$\triangle HIJ : 35^\circ + 88^\circ + m\angle J = 180^\circ$$

$$m\angle J = 57^\circ$$

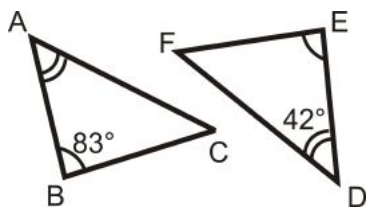
Notice we were given $m\angle A = m\angle H$ and $m\angle B = m\angle I$ and we found out $m\angle C = m\angle J$. This can be generalized into the Third Angle Theorem.

Third Angle Theorem: If two angles in one triangle are congruent to two angles in another triangle, then the third pair of angles must also be congruent.

If $\angle A \cong \angle D$ and $\angle B \cong \angle E$, then $\angle C \cong \angle F$.



Example 5: Determine the measure of the missing angles.



Solution: From the Third Angle Theorem, we know $\angle C \cong \angle F$.

$$m\angle A + m\angle B + m\angle C = 180^\circ$$

$$m\angle D + m\angle E + m\angle F = 180^\circ$$

$$42^\circ + 83^\circ + m\angle C = 180^\circ$$

$$m\angle C = 55^\circ = m\angle F$$

Congruence Properties Recall the Properties of Congruence from Chapter 2. They will be very useful in the upcoming sections.

Reflexive Property of Congruence:

$$\overline{AB} \cong \overline{AB} \text{ or } \triangle ABC \cong \triangle ABC$$

Symmetric Property of Congruence:

$$\angle EFG \cong \angle XYZ \text{ and } \angle XYZ \cong \angle EFG$$

$$\triangle ABC \cong \triangle DEF \text{ and } \triangle DEF \cong \triangle ABC$$

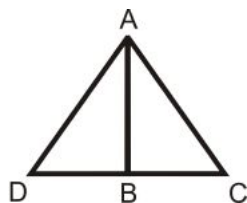
Transitive Property of Congruence:

$$\triangle ABC \cong \triangle DEF \text{ and } \triangle DEF \cong \triangle GHI \text{ then}$$

$$\triangle ABC \cong \triangle GHI$$

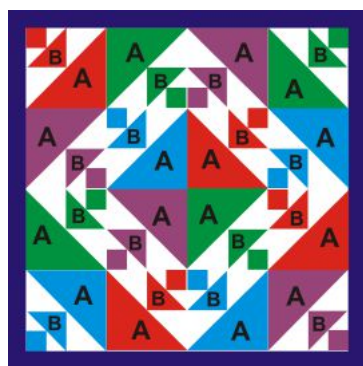
These three properties will be very important when you begin to prove that two triangles are congruent.

Example 6: In order to say that $\triangle ABD \cong \triangle ABC$, you must show the three corresponding angles and sides are congruent. Which pair of sides is congruent by the Reflexive Property?



Solution: The side \overline{AB} is shared by both triangles. In a geometric proof, $\overline{AB} \cong \overline{AB}$ by the Reflexive Property.

Know What? Revisited The 16 “A” triangles are congruent. The 16 “B” triangles are also congruent. The quilt pattern is made from dividing up the entire square into smaller squares. Both the “A” and “B” triangles are right triangles.

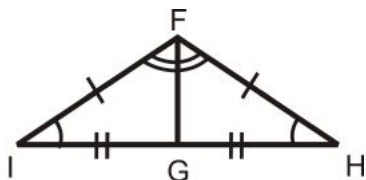


Review Questions

- Questions 1 and 2 are similar to Example 3.
- Questions 3-12 are a review and use the definitions and theorems explained in this section.
- Questions 13-17 are similar to Example 1 and 2.
- Question 18 the definitions and theorems explained in this section.
- Questions 19-22 are similar to Examples 4 and 5.
- Question 23 is a proof of the Third Angle Theorem.
- Questions 24-28 are similar to Example 6.
- Questions 29 and 30 are investigations using congruent triangles, a ruler and a protractor.

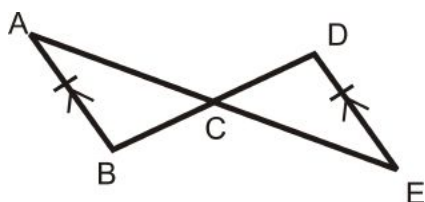
1. If $\triangle RAT \cong \triangle UGH$, what is also congruent?
2. If $\triangle BIG \cong \triangle TOP$, what is also congruent?

For questions 3-7, use the picture to the right.



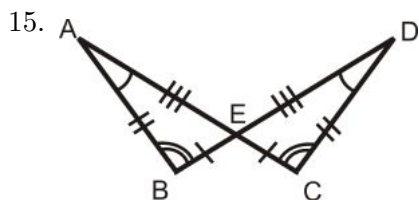
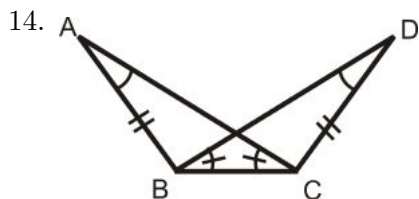
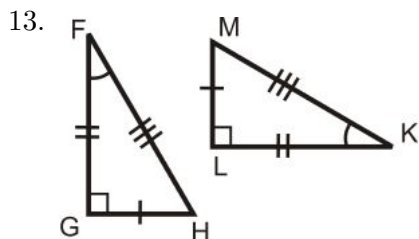
3. What theorem tells us that $\angle FGH \cong \angle FGI$?
4. What is $m\angle FGI$ and $m\angle FGH$? How do you know?
5. What property tells us that the third side of each triangle is congruent?
6. How does \overline{FG} relate to $\angle IFH$?
7. Write the congruence statement for these two triangles.

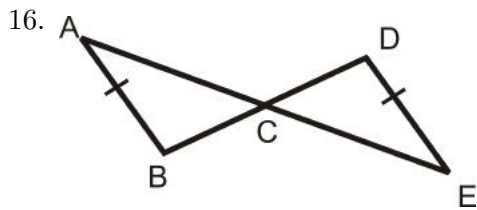
For questions 8-12, use the picture to the right.



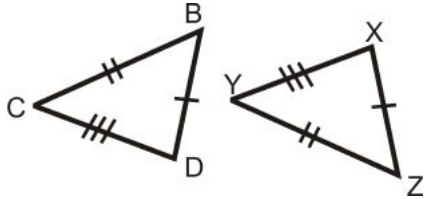
8. $\overline{AB} \parallel \overline{DE}$, what angles are congruent? How do you know?
9. Why is $\angle ACB \cong \angle ECD$? It is not the same reason as #8.
10. Are the two triangles congruent with the information you currently have? Why or why not?
11. If you are told that C is the midpoint of \overline{AE} and \overline{BD} , what segments are congruent?
12. Write a congruence statement.

For questions 13-16, determine if the triangles are congruent. If they are, write the congruence statement.



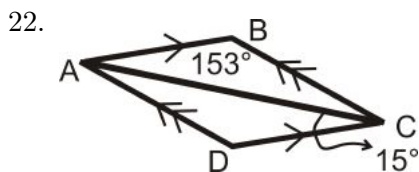
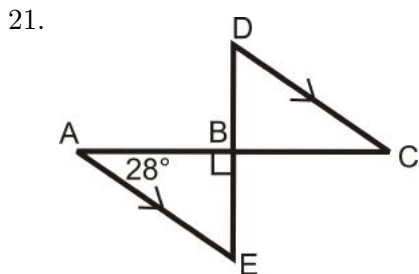
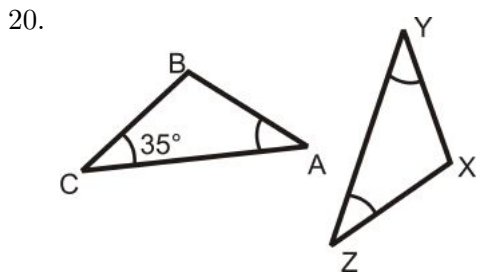
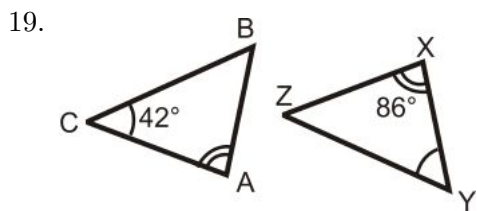


17. Suppose the two triangles to the right are congruent. Write a congruence statement for these triangles.



18. Explain how we know that if the two triangles are congruent, then $\angle B \cong \angle Z$.

For questions 19-22, determine the measure of all the angles in the each triangle.



23. Fill in the blanks in the Third Angle Theorem proof below.

Given: $\angle A \cong \angle D$, $\angle B \cong \angle E$

Prove: $\angle C \cong \angle F$

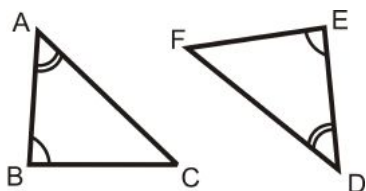


Table 3.5:

<i>Statement</i>	<i>Reason</i>
1. $\angle A \cong \angle D$, $\angle B \cong \angle E$	
2.	\cong angles have = measures
3. $m\angle A + m\angle B + m\angle C = 180^\circ$ $m\angle D + m\angle E + m\angle F = 180^\circ$	
4.	Substitution PoE
5.	Substitution PoE
6. $m\angle C = m\angle F$	
7. $\angle C \cong \angle F$	

For each of the following questions, determine if the Reflexive, Symmetric or Transitive Properties of Congruence is used.

24. $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$
25. $\overline{AB} \cong \overline{AB}$
26. $\triangle XYZ \cong \triangle LMN$ and $\triangle LMN \cong \triangle XYZ$
27. $\triangle ABC \cong \triangle BAC$
28. What type of triangle is $\triangle ABC$ in #27? How do you know?

Review Queue Answers

1. $\angle B \cong \angle H$, $\overline{AB} \cong \overline{GH}$, $\overline{BC} \cong \overline{HI}$
2. $\angle C \cong \angle M$, $\overline{BC} \cong \overline{LM}$
3. The angles add up to 180°
 - (a) $(5x + 2)^\circ + (4x + 3)^\circ + (3x - 5)^\circ = 180^\circ$
 $12x = 180^\circ$
 $x = 15^\circ$
 - (b) $77^\circ, 63^\circ, 40^\circ$
 - (c) acute scalene

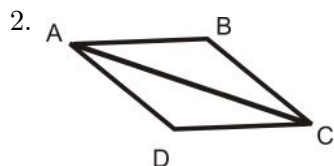
3.3 Triangle Congruence using SSS and SAS

Learning Objectives

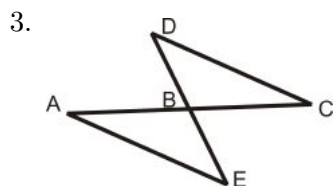
- Use the distance formula to analyze triangles on the $x - y$ plane.
- Apply the SSS and SAS Postulate to show two triangles are congruent.

Review Queue

- Use the distance formula, $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ to find the distance between the two points.
 - $(-1, 5)$ and $(4, 12)$
 - $(-6, -15)$ and $(-3, 8)$



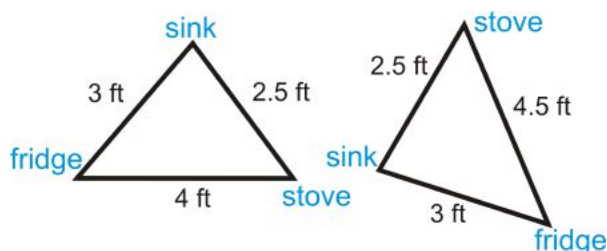
- If we know that $\overline{AB} \parallel \overline{CD}$, $\overline{AD} \parallel \overline{BC}$, what angles are congruent? By which theorem?
- Which side is congruent by the Reflexive Property?
- Is this enough to say $\triangle ADC \cong \triangle CBA$?



- If we know that B is the midpoint of \overline{AC} and \overline{DE} , what segments are congruent?
- Are there any angles that are congruent by looking at the picture? Which ones and why?
- Is this enough to say $\triangle ABE \cong \triangle CBD$?

Know What?

The “ideal” measurements in a kitchen from the sink, refrigerator and oven are as close to an equilateral triangle as possible. Your parents are remodeling theirs to be as close to this as possible and the measurements are in the picture at the left, below. Your neighbor’s kitchen has the measurements on the right. Are the two triangles congruent? Why or why not?



SSS Postulate of Triangle Congruence

Consider the question: If I have three lengths: 3 in, 4 in, and 5 in, can I construct more than one triangle?

Investigation 4-2: Constructing a Triangle Given Three Sides

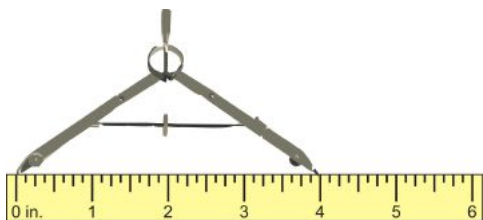
Tools Needed: compass, pencil, ruler, and paper

- Draw the longest side (5 in) horizontally, halfway down the page.

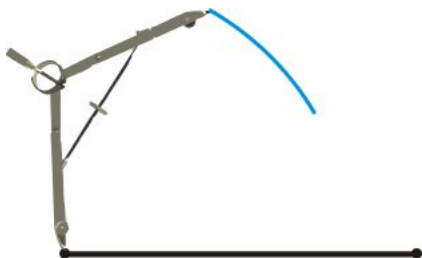


The drawings in this investigation are to scale.

2. Take the compass and, using the ruler, widen the compass to measure 4 in, the second side.



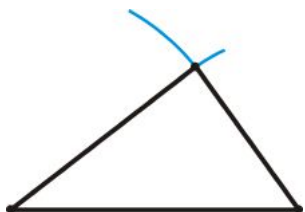
3. Using the measurement from Step 2, place the pointer of the compass on the left endpoint of the side drawn in Step 1. Draw an arc mark above the line segment.



4. Repeat Step 2 with the third measurement, 3 in. Then, like Step 3, place the pointer of the compass on the **right** endpoint of the side drawn in Step 1. Draw an arc mark above the line segment. Make sure it intersects the arc mark drawn in Step 3.



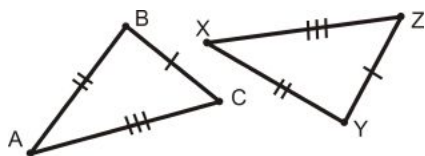
5. Draw lines from each endpoint to the arc intersections. These segments are the other two sides of the triangle.



An animation of this construction can be found at: <http://www.mathsisfun.com/geometry/construct-ruler-compass.html>

Can another triangle be drawn with these measurements that look different? NO. ***Only one triangle can be created from any given three lengths. You can rotate, flip, or move this triangle but it will still be the same size.***

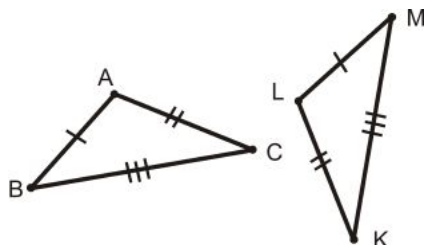
Side-Side-Side (SSS) Triangle Congruence Postulate: If 3 sides in one triangle are congruent to 3 sides in another triangle, then the triangles are congruent.



$\overline{BC} \cong \overline{YZ}$, $\overline{AB} \cong \overline{XY}$, and $\overline{AC} \cong \overline{XZ}$ then $\triangle ABC \cong \triangle XYZ$.

The SSS Postulate is a shortcut. Before, you had to show **3 sides and 3 angles** in one triangle were congruent to **3 sides and 3 angles** in another triangle. Now you only have to show **3 sides** in one triangle are congruent to **3 sides** in another.

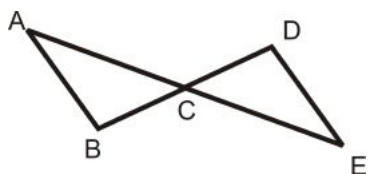
Example 1: Write a triangle congruence statement based on the picture below:



Solution: From the tic marks, we know $\overline{AB} \cong \overline{LM}$, $\overline{AC} \cong \overline{LK}$, $\overline{BC} \cong \overline{MK}$. From the SSS Postulate, the triangles are congruent. Lining up the corresponding sides, we have $\triangle ABC \cong \triangle LMK$.

Don't forget ORDER MATTERS when writing congruence statements. Line up the sides with the same number of tic marks.

Example 2: Write a two-column proof to show that the two triangles are congruent.



Given: $\overline{AB} \cong \overline{DE}$

C is the midpoint of \overline{AE} and \overline{DB} .

Prove: $\triangle ACB \cong \triangle ECD$

Solution:

Table 3.6:

<i>Statement</i>	<i>Reason</i>
1. $\overline{AB} \cong \overline{DE}$ C is the midpoint of \overline{AE} and \overline{DB}	Given
2. $\overline{AC} \cong \overline{CE}$, $\overline{BC} \cong \overline{CD}$	Definition of a midpoint
3. $\triangle ACB \cong \triangle ECD$	SSS Postulate

Prove Move: You must clearly state the three sets of sides are congruent BEFORE stating the triangles are congruent.

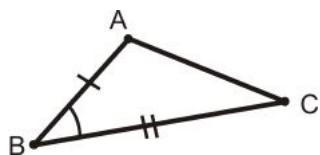
Prove Move: Mark the picture with the information you are given as well as information that you see in

the picture (vertical angles, information from parallel lines, midpoints, angle bisectors, right angles). This information may be used in a proof.

SAS Triangle Congruence Postulate

SAS refers to Side-Angle-Side. The placement of the word Angle is important because it indicates that the angle you are given is *between* the two sides.

Included Angle: When an angle is between two given sides of a polygon.



$\angle B$ would be the included angle for sides \overline{AB} and \overline{BC} .

Consider the question: If I have two sides of length 2 in and 5 in and the angle between them is 45° , can I construct one triangle?

Investigation 4-3: Constructing a Triangle Given Two Sides and Included Angle

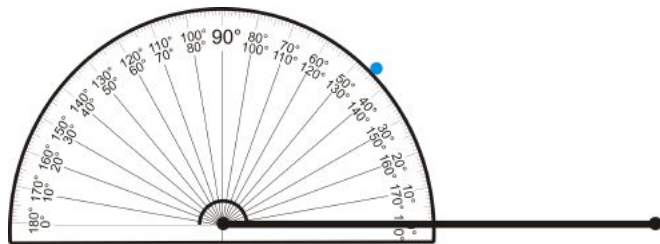
Tools Needed: protractor, pencil, ruler, and paper

1. Draw the longest side (5 in) horizontally, halfway down the page.



The drawings in this investigation are to scale.

2. At the left endpoint of your line segment, use the protractor to measure a 45° angle. Mark this measurement.



3. Connect your mark from Step 2 with the left endpoint. Make your line 2 in long, the length of the second side.

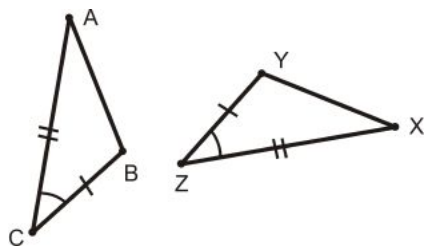


4. Connect the two endpoints to draw the third side.



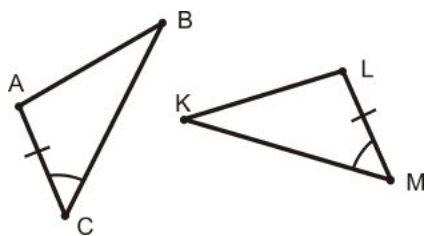
Can you draw another triangle, with these measurements that looks different? NO. *Only one triangle can be created from any two lengths and the INCLUDED angle.*

Side-Angle-Side (SAS) Triangle Congruence Postulate: If two sides and the included angle in one triangle are congruent to two sides and the included angle in another triangle, then the two triangles are congruent.



$\overline{AC} \cong \overline{XZ}$, $\overline{BC} \cong \overline{YZ}$, and $\angle C \cong \angle Z$, then $\triangle ABC \cong \triangle XYZ$.

Example 3: What additional piece of information do you need to show that these two triangles are congruent using the SAS Postulate?



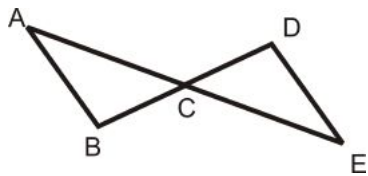
- a) $\angle ABC \cong \angle LKM$
- b) $\overline{AB} \cong \overline{LK}$
- c) $\overline{BC} \cong \overline{KM}$
- d) $\angle BAC \cong \angle KLM$

Solution: For the SAS Postulate, you need the side on the other side of the angle. In $\triangle ABC$, that is \overline{BC} and in $\triangle LKM$ that is \overline{KM} . The answer is c.

Example 4: Write a two-column proof to show that the two triangles are congruent.

Given: C is the midpoint of \overline{AE} and \overline{DB}

Prove: $\triangle ACB \cong \triangle ECD$



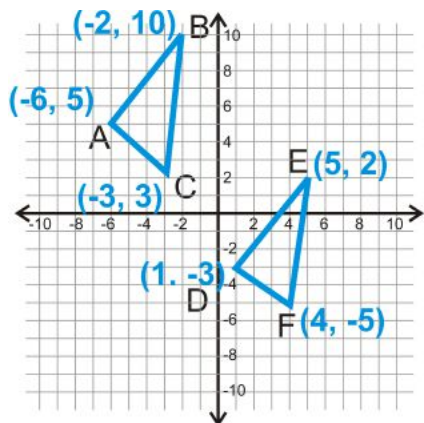
Solution:

Table 3.7:

<i>Statement</i>	<i>Reason</i>
1. C is the midpoint of \overline{AE} and \overline{DB}	Given
2. $\overline{AC} \cong \overline{CE}$, $\overline{BC} \cong \overline{CD}$	Definition of a midpoint
3. $\angle ACB \cong \angle DCE$	Vertical Angles Postulate
4. $\triangle ACB \cong \triangle ECD$	SAS Postulate

SSS in the Coordinate Plane

The only way we will show two triangles are congruent in an $x-y$ plane is using SSS. To do this, you need to use the distance formula.



Example 5: Find the distances of all the line segments from both triangles to see if the two triangles are congruent.

Solution: Begin with $\triangle ABC$ and its sides.

$$\begin{aligned} AB &= \sqrt{(-6 - (-2))^2 + (5 - 10)^2} \\ &= \sqrt{(-4)^2 + (-5)^2} \\ &= \sqrt{16 + 25} \\ &= \sqrt{41} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(-2 - (-3))^2 + (10 - 3)^2} \\ &= \sqrt{(1)^2 + (7)^2} \\ &= \sqrt{1 + 49} \\ &= \sqrt{50} = 5\sqrt{2} \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{(-6 - (-3))^2 + (5 - 3)^2} \\ &= \sqrt{(-3)^2 + (2)^2} \\ &= \sqrt{9 + 4} \\ &= \sqrt{13} \end{aligned}$$

Now, find the distances of all the sides in $\triangle DEF$.

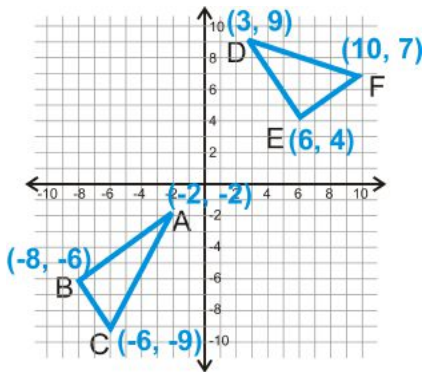
$$\begin{aligned} DE &= \sqrt{(1 - 5)^2 + (-3 - 2)^2} \\ &= \sqrt{(-4)^2 + (-5)^2} \\ &= \sqrt{16 + 25} \\ &= \sqrt{41} \end{aligned}$$

$$\begin{aligned}
 EF &= \sqrt{(5-4)^2 + (2-(-5))^2} \\
 &= \sqrt{(1)^2 + (7)^2} \\
 &= \sqrt{1+49} \\
 &= \sqrt{50} = 5\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 DF &= \sqrt{(1-4)^2 + (-3-(-5))^2} \\
 &= \sqrt{(-3)^2 + (2)^2} \\
 &= \sqrt{9+4} \\
 &= \sqrt{13}
 \end{aligned}$$

$AB = DE$, $BC = EF$, and $AC = DF$, so two triangles are congruent by SSS.

Example 6: Determine if the two triangles are congruent.



Solution: Start with $\triangle ABC$.

$$\begin{aligned}
 AB &= \sqrt{(-2-(-8))^2 + (-2-(-6))^2} \\
 &= \sqrt{(6)^2 + (4)^2} \\
 &= \sqrt{36+16} \\
 &= \sqrt{52} = 2\sqrt{13}
 \end{aligned}$$

$$\begin{aligned}
 BC &= \sqrt{(-8-(-6))^2 + (-6-(-9))^2} \\
 &= \sqrt{(-2)^2 + (3)^2} \\
 &= \sqrt{4+9} \\
 &= \sqrt{13}
 \end{aligned}$$

$$\begin{aligned}
 AC &= \sqrt{(-2-(-6))^2 + (-2-(-9))^2} \\
 &= \sqrt{(4)^2 + (7)^2} \\
 &= \sqrt{16+49} \\
 &= \sqrt{65}
 \end{aligned}$$

Now find the sides of $\triangle DEF$.

$$\begin{aligned} DE &= \sqrt{(3-6)^2 + (9-4)^2} \\ &= \sqrt{(-3)^2 + (5)^2} \\ &= \sqrt{9+25} \\ &= \sqrt{34} \end{aligned}$$

$$\begin{aligned} EF &= \sqrt{(6-10)^2 + (4-7)^2} \\ &= \sqrt{(-4)^2 + (-3)^2} \\ &= \sqrt{16+9} \\ &= \sqrt{25} = 5 \end{aligned}$$

$$\begin{aligned} DF &= \sqrt{(3-10)^2 + (9-7)^2} \\ &= \sqrt{(-7)^2 + (2)^2} \\ &= \sqrt{49+4} \\ &= \sqrt{53} \end{aligned}$$

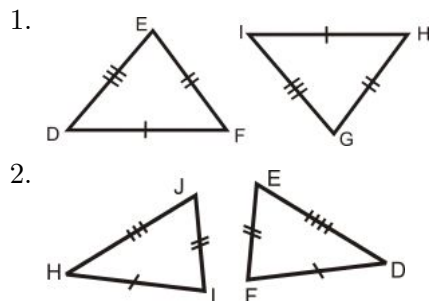
No sides have equal measures, so the triangles are not congruent.

Know What? Revisited From what we have learned in this section, the two triangles are not congruent because the distance from the fridge to the stove in your house is 4 feet and in your neighbor's it is 4.5 ft. The SSS Postulate tells us that all three sides have to be congruent in order for the triangles to be congruent.

Review Questions

- Questions 1-10 are similar to Example 1.
- Questions 11-16 are similar to Example 3.
- Questions 17-23 are similar to Examples 2 and 4.
- Questions 24-27 are similar to Examples 5 and 6.

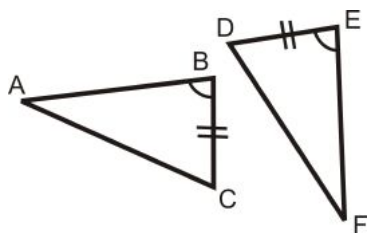
Are the pairs of triangles congruent? If so, write the congruence statement and why.



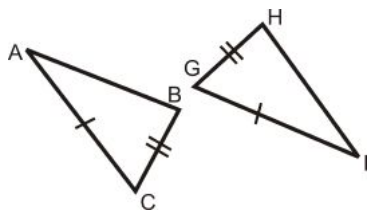
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.

State the additional piece of information needed to show that each pair of triangles is congruent.

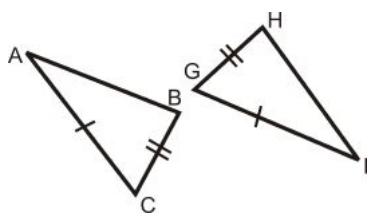
11. Use SAS



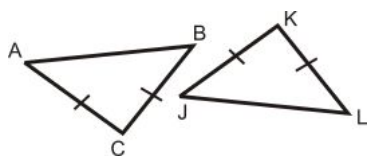
12. Use SSS



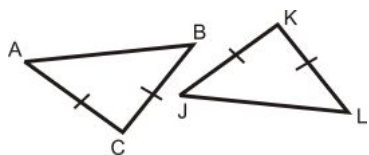
13. Use SAS



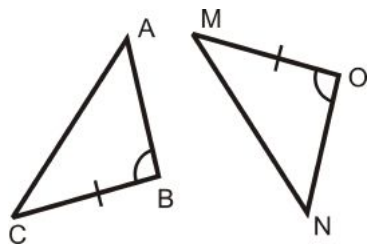
14. Use SAS



15. Use SSS



16. Use SAS



Fill in the blanks in the proofs below.

17. Given: $\overline{AB} \cong \overline{DC}$, $\overline{BE} \cong \overline{CE}$
Prove: $\triangle ABE \cong \triangle ACE$

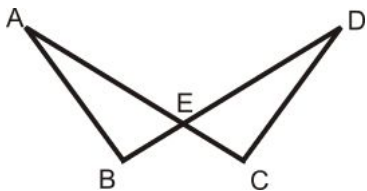


Table 3.8:

<i>Statement</i>	<i>Reason</i>
1.	1.
2. $\angle AEB \cong \angle DEC$	2.
3. $\triangle ABE \cong \triangle DEC$	3.

18. Given: $\overline{AB} \cong \overline{DC}$, $\overline{AC} \cong \overline{DB}$
Prove: $\triangle ABC \cong \triangle DCB$

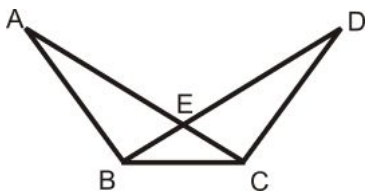


Table 3.9:

<i>Statement</i>	<i>Reason</i>
1.	1.
2.	2. Reflexive PoC
3. $\triangle ABC \cong \triangle DCB$	3.

19. Given: B is a midpoint of \overline{DC}
 $\overline{AB} \perp \overline{DC}$
Prove: $\triangle ABD \cong \triangle ABC$

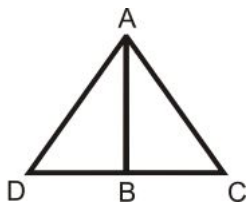


Table 3.10:

<i>Statement</i>	<i>Reason</i>
1. B is a midpoint of \overline{DC} , $\overline{AB} \perp \overline{DC}$	1.
2.	2. Definition of a midpoint
3. $\angle ABD$ and $\angle ABC$ are right angles	3.

Table 3.10: (continued)

<i>Statement</i>	<i>Reason</i>
4.	4. All right angles are \cong
5.	5.
6. $\triangle ABD \cong \triangle ABC$	6.

20. Given: \overline{AB} is an angle bisector of $\angle DAC$

$$\overline{AD} \cong \overline{AC}$$

Prove: $\triangle ABD \cong \triangle ABC$

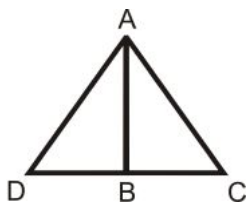


Table 3.11:

<i>Statement</i>	<i>Reason</i>
1.	
2. $\angle DAB \cong \angle BAC$	
3.	Reflexive PoC
4. $\triangle ABD \cong \triangle ABC$	

21. Given: B is the midpoint of \overline{DC}

$$\overline{AD} \cong \overline{AC}$$

Prove: $\triangle ABD \cong \triangle ABC$

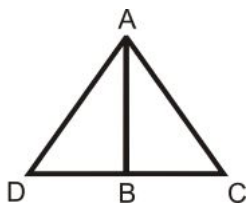


Table 3.12:

<i>Statement</i>	<i>Reason</i>
1.	
2.	Definition of a Midpoint
3.	Reflexive PoC
4. $\triangle ABD \cong \triangle ABC$	

22. Given: B is the midpoint of \overline{DE} and \overline{AC}

$\angle ABE$ is a right angle

Prove: $\triangle ABE \cong \triangle CBD$

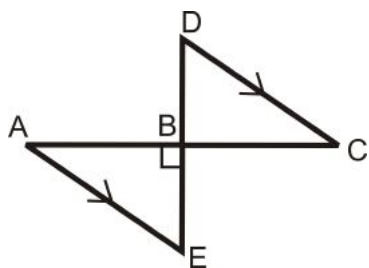


Table 3.13:

<i>Statement</i>	<i>Reason</i>
1.	Given
2. $\overline{DB} \cong \overline{BE}$, $\overline{AB} \cong \overline{BC}$	
3.	Definition of a Right Angle
4.	Vertical Angle Theorem
5. $\triangle ABE \cong \triangle CBD$	

23. Given: \overline{DB} is the angle bisector of $\angle ADC$

$$\overline{AD} \cong \overline{DC}$$

Prove: $\triangle ABD \cong \triangle CBD$

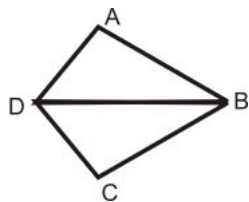
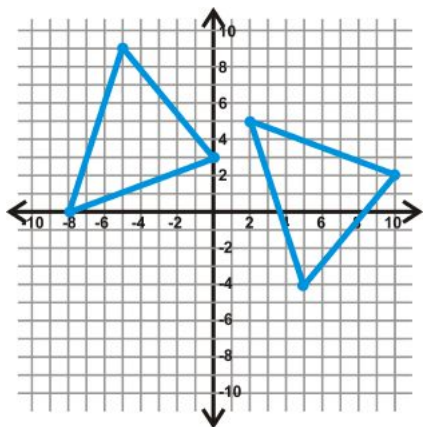


Table 3.14:

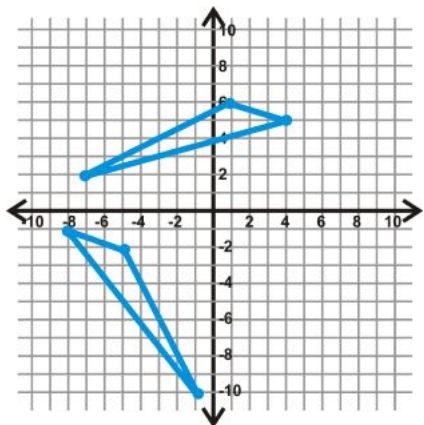
<i>Statement</i>	<i>Reason</i>
1.	
2. $\angle ADB \cong \angle BDC$	
3.	
4. $\triangle ABD \cong \triangle CBD$	

Find the lengths of the sides of each triangle to see if the two triangles are congruent. Leave your answers under the radical.

24.



25.



26. $\triangle ABC$: $A(-1, 5)$, $B(-4, 2)$, $C(2, -2)$ and $\triangle DEF$: $D(7, -5)$, $E(4, 2)$, $F(8, -9)$

27. $\triangle ABC$: $A(-8, -3)$, $B(-2, -4)$, $C(-5, -9)$ and $\triangle DEF$: $D(-7, 2)$, $E(-1, 3)$, $F(-4, 8)$

Review Queue Answers

- $\sqrt{74}$
 - $\sqrt{538}$
- $\angle BAC \cong \angle DCA$, $\angle DAC \cong \angle BCA$ by the Alternate Interior Angles Theorem.
 - $\overline{AC} \cong \overline{AC}$
 - Not yet, this would be ASA.
- $\overline{DB} \cong \overline{BE}$, $\overline{AB} \cong \overline{BC}$
 - $\angle DBC \cong \angle ABE$ by the Vertical Angles Theorem.
 - By the end of this section, yes, we will be able to show that these two triangles are congruent by SAS.

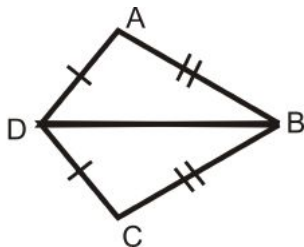
3.4 Triangle Congruence using ASA, AAS, and HL

Learning Objectives

- Use and understand the ASA, AAS, and HL Congruence Postulate.
- Complete two-column proofs using SSS, SAS, ASA and AAS.

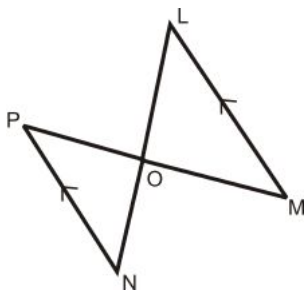
Review Queue

1.



- What sides are marked congruent?
- Is third side congruent? Why?
- Write the congruence statement for the two triangles. Why are they congruent?

2.

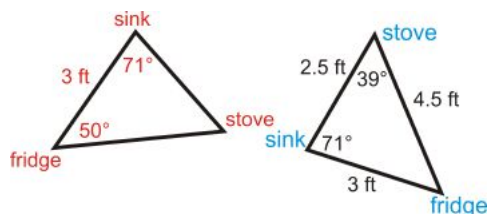


- From the parallel lines, what angles are congruent?
- How do you know the third angle is congruent?
- Are any sides congruent? How do you know?
- Are the two triangles congruent? Why or why not?

3. If $\triangle DEF \cong \triangle PQR$, can it be assumed that:

- $\angle F \cong \angle R$? Why or why not?
- $\overline{EF} \cong \overline{PR}$? Why or why not?

Know What? Your parents changed their minds at the last second about their kitchen layout. Now, the measurements are in the triangle on the left, below. Your neighbor's kitchen is in blue on the right. Are the kitchen triangles congruent now?



ASA Congruence

ASA refers to Angle-Side-Angle. The placement of the word Side is important because it indicates that the side that you are given is between the two angles.

Consider the question: If I have two angles that are 45° and 60° and the side between them is 5 in, can I construct only one triangle?

Investigation 4-4: Constructing a Triangle Given Two Angles and Included Side

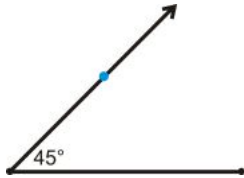
Tools Needed: protractor, pencil, ruler, and paper

1. Draw the side (5 in) horizontally, about halfway down the page.

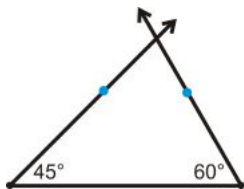


The drawings in this investigation are to scale.

2. At the left endpoint of your line segment, use the protractor to measure the 45° angle. Mark this measurement and draw a ray from the left endpoint through the 45° mark.



3. At the right endpoint of your line segment, use the protractor to measure the 60° angle. Mark this measurement and draw a ray from the right endpoint through the 60° mark. Extend this ray so that it crosses through the ray from Step 2.

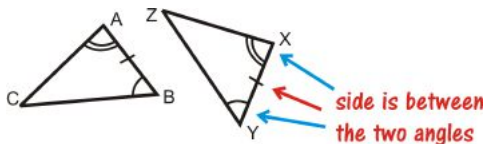


4. Erase the extra parts of the rays from Steps 2 and 3 to leave only the triangle.

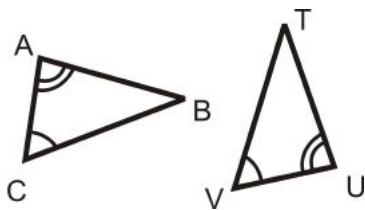
Can you draw another triangle, with these measurements that looks different? NO. *Only one triangle can be created from any given two angle measures and the INCLUDED side.*

Angle-Side-Angle (ASA) Congruence Postulate: If two angles and the included side in one triangle are congruent to two angles and the included side in another triangle, then the two triangles are congruent.

$\angle A \cong \angle X$, $\angle B \cong \angle Y$, and $\overline{AB} \cong \overline{XY}$, then $\triangle ABC \cong \triangle XYZ$.



Example 1: What information do you need to prove that these two triangles are congruent using the ASA Postulate?



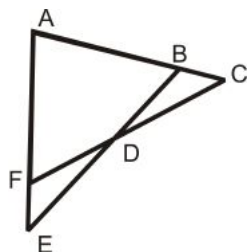
- a) $\overline{AB} \cong \overline{UT}$
- b) $\overline{AC} \cong \overline{UV}$
- c) $\overline{BC} \cong \overline{TV}$
- d) $\angle B \cong \angle T$

Solution: For ASA, we need the side between the two given angles, which is \overline{AC} and \overline{UV} . The answer is b.

Example 2: Write a 2-column proof.

Given: $\angle C \cong \angle E$, $\overline{AC} \cong \overline{AE}$

Prove: $\triangle ACF \cong \triangle AEB$



Solution:

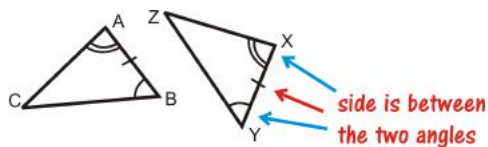
Table 3.15:

<i>Statement</i>	<i>Reason</i>
1. $\angle C \cong \angle E$, $\overline{AC} \cong \overline{AE}$	Given
2. $\angle A \cong \angle A$	Reflexive PoC
3. $\triangle ACF \cong \triangle AEB$	ASA

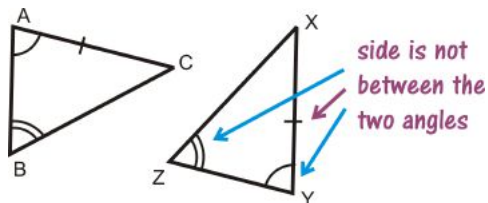
AAS Congruence

A variation on ASA is AAS, which is Angle-Angle-Side. For ASA you need two angles and the side **between** them. But, if you know two pairs of angles are congruent, the third pair will also be congruent by the 3rd Angle Theorem. This means you can prove two triangles are congruent when you have any two pairs of **corresponding** angles and a pair of sides.

ASA



AAS



Angle-Angle-Side (AAS) Congruence Theorem: If two angles and a non-included side in one triangle are congruent to two angles and a non-included side in another triangle, then the triangles are congruent.

Proof of AAS Theorem

Given: $\angle A \cong \angle Y$, $\angle B \cong \angle Z$, $\overline{AC} \cong \overline{XY}$

Prove: $\triangle ABC \cong \triangle YZX$

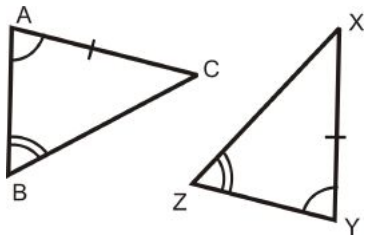


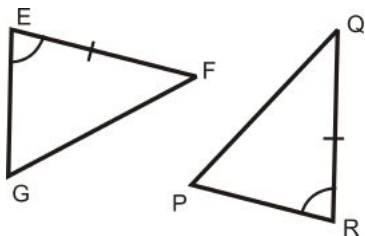
Table 3.16:

<i>Statement</i>	<i>Reason</i>
1. $\angle A \cong \angle Y$, $\angle B \cong \angle Z$, $\overline{AC} \cong \overline{XY}$	Given
2. $\angle C \cong \angle X$	3 rd Angle Theorem
3. $\triangle ABC \cong \triangle YZX$	ASA

By proving $\triangle ABC \cong \triangle YZX$ with ASA, we have also proved that the AAS Theorem is true.

Example 3: What information do you need to prove that these two triangles are congruent using:

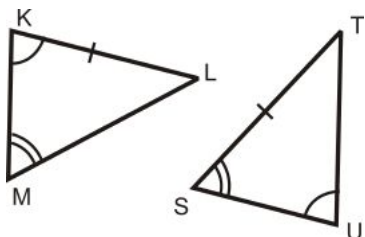
- a) ASA?
- b) AAS?
- c) SAS?



Solution:

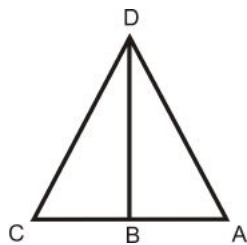
- a) For ASA, we need the angles on the other side of \overline{EF} and \overline{QR} . $\angle F \cong \angle Q$
- b) For AAS, we would need the other angle. $\angle G \cong \angle P$
- c) For SAS, we need the side on the other side of $\angle E$ and $\angle R$. $\overline{EG} \cong \overline{RP}$

Example 4: Can you prove that the following triangles are congruent? Why or why not?



Solution: We cannot show the triangles are congruent because \overline{KL} and \overline{ST} are *not corresponding*, even though they are congruent. To determine if \overline{KL} and \overline{ST} are corresponding, look at the angles around them, $\angle K$ and $\angle L$ and $\angle S$ and $\angle T$. $\angle K$ has **one** arc and $\angle L$ is unmarked. $\angle S$ has **two** arcs and $\angle T$ is unmarked. In order to use AAS, $\angle S$ needs to be congruent to $\angle K$.

Example 5: Write a 2-column proof.



Given: \overline{BD} is an angle bisector of $\angle CDA$, $\angle C \cong \angle A$

Prove: $\triangle CBD \cong \triangle ABD$

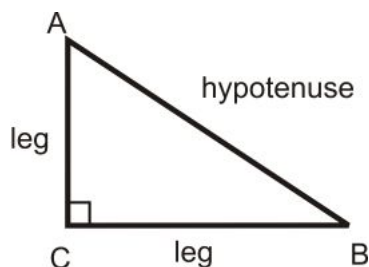
Solution:

Table 3.17:

<i>Statement</i>	<i>Reason</i>
1. \overline{BD} is an angle bisector of $\angle CDA$, $\angle C \cong \angle A$	Given
2. $\angle CDB \cong \angle ADB$	Definition of an Angle Bisector
3. $\overline{DB} \cong \overline{DB}$	Reflexive PoC
4. $\triangle CBD \cong \triangle ABD$	AAS

Hypotenuse-Leg

So far, the congruence postulates we have used will work for any triangle. The last congruence theorem can only be used on **right** triangles. A right triangle has exactly one right angle. The two sides adjacent to the right angle are called legs and the side opposite the right angle is called the hypotenuse.



You may or may not know the Pythagorean Theorem, which says, for any **right** triangle, this equation is true:

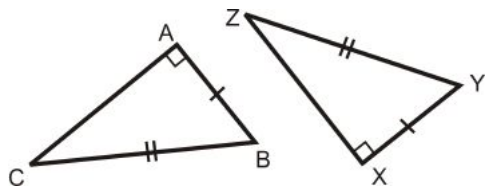
$$(\text{leg})^2 + (\text{leg})^2 = (\text{hypotenuse})^2$$

What this means is that if you are given two sides of a right triangle, you can always find the third. Therefore, if you have two sides of a right triangle are congruent to two sides of another right triangle; you can conclude that third sides are also congruent.

The Hypotenuse-Leg (HL) Congruence Theorem is a shortcut of this process.

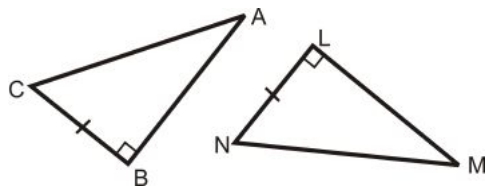
HL Congruence Theorem: If the hypotenuse and leg in one *right* triangle are congruent to the hypotenuse and leg in another *right* triangle, then the two triangles are congruent.

$\triangle ABC$ and $\triangle XYZ$ are both right triangles and $\overline{AB} \cong \overline{XY}$ and $\overline{BC} \cong \overline{YZ}$ then $\triangle ABC \cong \triangle XYZ$.



Example 6: What information would you need to prove that these two triangles were congruent using the:

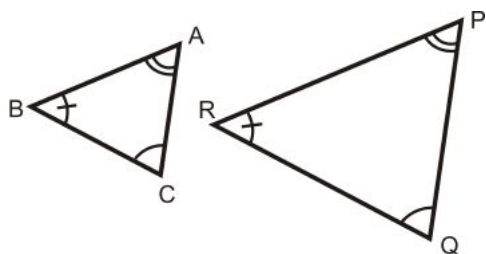
- HL Theorem?
- SAS Theorem?



Solution:

- For HL, you need the hypotenuses to be congruent. $\overline{AC} \cong \overline{MN}$.
- To use SAS, we would need the other legs to be congruent. $\overline{AB} \cong \overline{ML}$.

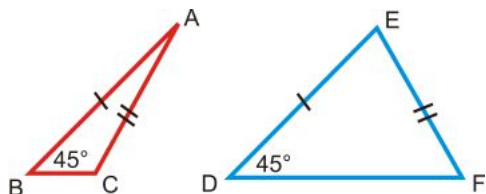
AAA and SSA Relationships There are two other side-angle relationships that we have not discussed: AAA and SSA.



AAA implies that all the angles are congruent.

As you can see, $\triangle ABC$ and $\triangle PRQ$ are not congruent, even though all the angles are.

SSA relationships do not prove congruence either. See $\triangle ABC$ and $\triangle DEF$ below.



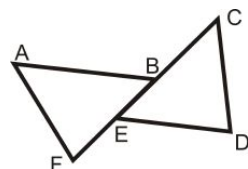
Because $\angle B$ and $\angle D$ are **not** the included angles between the congruent sides, we cannot prove that these two triangles are congruent.

Recap

Table 3.18:

Side-Angle Relationship	Picture	Determine Congruence?
SSS		Yes $\triangle ABC \cong \triangle XYZ$
SAS		Yes $\triangle ABC \cong \triangle XYZ$
ASA		Yes $\triangle ABC \cong \triangle XYZ$
AAS (or SAA)		Yes $\triangle ABC \cong \triangle YZX$
HL		Yes, Right Triangles Only $\triangle ABC \cong \triangle XYZ$
SSA		NO
AAA		NO

Example 7: Write a 2-column proof.



Given: $\overline{AB} \parallel \overline{ED}$, $\angle C \cong \angle F$, $\overline{AB} \cong \overline{ED}$

Prove: $\overline{AF} \cong \overline{CD}$

Solution:

Table 3.19:

<i>Statement</i>	<i>Reason</i>
1. $\overline{AB} \parallel \overline{ED}$, $\angle C \cong \angle F$, $\overline{AB} \cong \overline{ED}$	Given
2. $\angle ABE \cong \angle DDB$	Alternate Interior Angles Theorem
3. $\triangle ABF \cong \triangle DEC$	ASA
4. $\overline{AF} \cong \overline{CD}$	CPCTC

Prove Move: At the beginning of this chapter we introduced CPCTC. Now, it can be used in a proof once two triangles are proved congruent. It is used to prove the parts of congruent triangles are congruent.

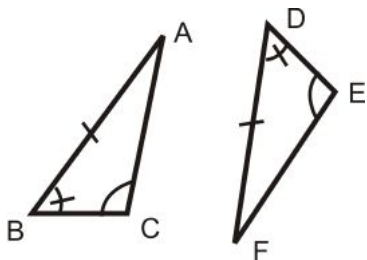
Know What? Revisited Even though we do not know all of the angle measures in the two triangles, we can find the missing angles by using the Third Angle Theorem. In your parents' kitchen, the missing angle is 39° . The missing angle in your neighbor's kitchen is 50° . From this, we can conclude that the two kitchens are now congruent, either by ASA or AAS.

Review Questions

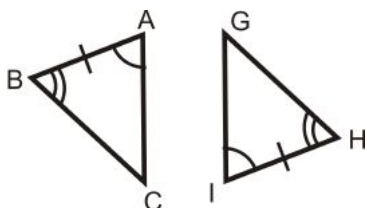
- Questions 1-10 are similar to Examples 1, 3, 4, and 6.
- Questions 11-20 are review and use the definitions and theorems explained in this section.
- Question 21-26 are similar to Examples 1, 3, 4 and 6.
- Questions 27 and 28 are similar to Examples 2 and 5.
- Questions 29-31 are similar to Example 4 and Investigation 4-4.

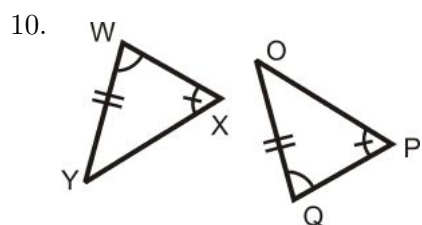
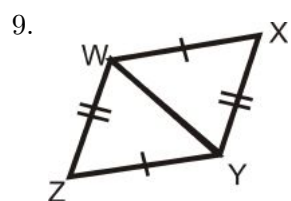
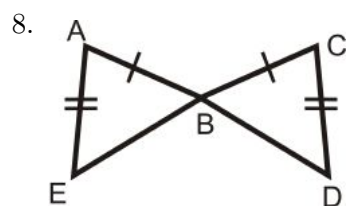
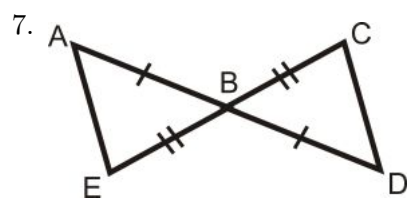
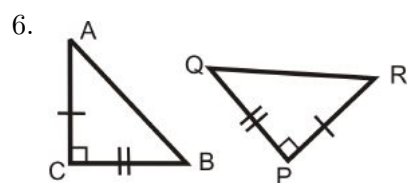
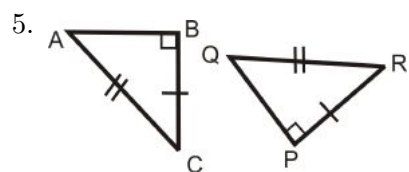
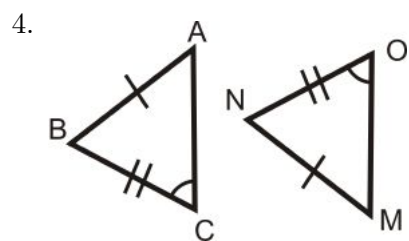
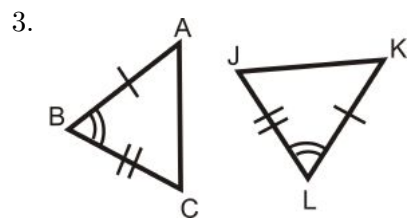
For questions 1-10, determine if the triangles are congruent. If they are, write the congruence statement and which congruence postulate or theorem you used.

1.

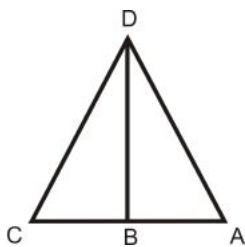


2.





For questions 11-15, use the picture to the right and the given information below.

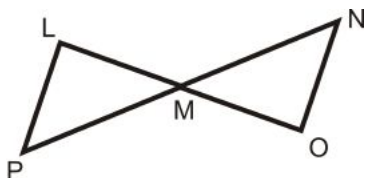


Given: $\overline{DB} \perp \overline{AC}$, \overline{DB} is the angle bisector of $\angle CDA$

11. From $\overline{DB} \perp \overline{AC}$, which angles are congruent and why?
12. Because \overline{DB} is the angle bisector of $\angle CDA$, what two angles are congruent?
13. From looking at the picture, what additional piece of information are you given? Is this enough to prove the two triangles are congruent?
14. Write a 2-column proof to prove $\triangle CDB \cong \triangle ADB$, using #11-13.
15. What would be your reason for $\angle C \cong \angle A$?

For questions 16-20, use the picture to the right and the given information.

Given: $\overline{LP} \parallel \overline{NO}$, $\overline{LP} \cong \overline{NO}$



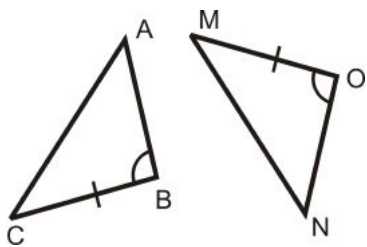
16. From $\overline{LP} \parallel \overline{NO}$, which angles are congruent and why?
17. From looking at the picture, what additional piece of information can you conclude?
18. Write a 2-column proof to prove $\triangle LMP \cong \triangle OMN$.
19. What would be your reason for $\overline{LM} \cong \overline{MO}$?
20. Fill in the blanks for the proof below. Use the given from above.
Prove: M is the midpoint of \overline{PN} .

Table 3.20:

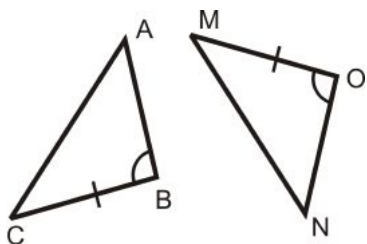
<i>Statement</i>	<i>Reason</i>
1. $\overline{LP} \parallel \overline{NO}$, $\overline{LP} \cong \overline{NO}$	Given
2.	Alternate Interior Angles
3.	ASA
4. $\overline{LM} \cong \overline{MO}$	
5. M is the midpoint of \overline{PN} .	

Determine the additional piece of information needed to show the two triangles are congruent by the given postulate.

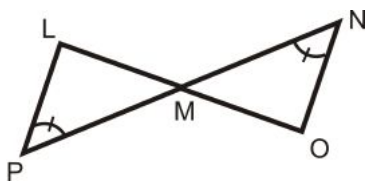
21. AAS



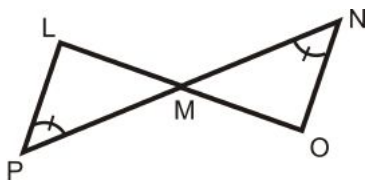
22. ASA



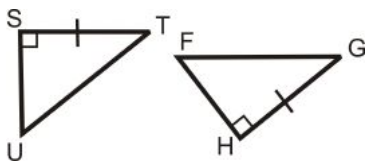
23. ASA



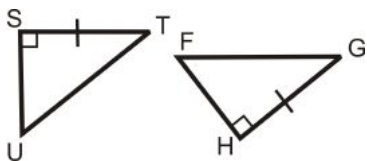
24. AAS



25. HL



26. SAS



Fill in the blanks in the proofs below.

27. Given: $\overline{SV} \perp \overline{WU}$
 T is the midpoint of \overline{SV} and \overline{WU}
 Prove: $\overline{WS} \cong \overline{UV}$

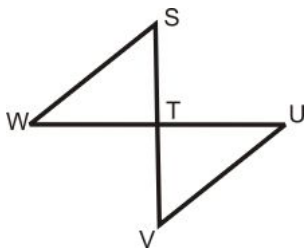


Table 3.21:

<i>Statement</i>	<i>Reason</i>
1.	
2. $\angle STW$ and $\angle UTV$ are right angles	
3.	
4. $\overline{ST} \cong \overline{TV}$, $\overline{WT} \cong \overline{TU}$	
5. $\triangle STW \cong \triangle UTV$	
6. $\overline{WS} \cong \overline{UV}$	

28. Given: $\angle K \cong \angle T$, \overline{EI} is the angle bisector of $\angle KET$
Prove: \overline{EI} is the angle bisector of $\angle KIT$

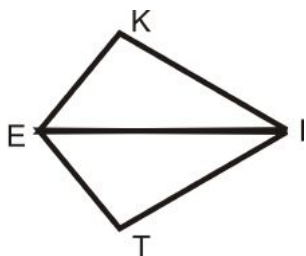
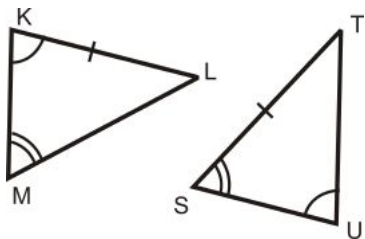


Table 3.22:

<i>Statement</i>	<i>Reason</i>
1.	
2.	Definition of an angle bisector
3. $\overline{EI} \cong \overline{EI}$	
4. $\triangle KEI \cong \triangle TEI$	
5. $\angle KIE \cong \angle TIE$	
6. \overline{EI} is the angle bisector of $\angle KIT$	

Construction Let's see if we can construct two different triangles like $\triangle KLM$ and $\triangle STU$ from Example 4.



29. Look at $\triangle KLM$.

- If $m\angle K = 70^\circ$ and $m\angle M = 60^\circ$, what is $m\angle L$?
- If $KL = 2 \text{ in}$, construct $\triangle KLM$ using $\angle L$, $\angle K$, \overline{KL} and Investigation 4-4 (ASA Triangle construction).

30. Look at $\triangle STU$.

- If $m\angle S = 60^\circ$ and $m\angle U = 70^\circ$, what is $m\angle T$?
- If $ST = 2 \text{ in}$, construct $\triangle STU$ using $\angle S$, $\angle T$, \overline{ST} and Investigation 4-4 (ASA Triangle construction).

31. Are the two triangles congruent?

Review Queue Answers

- $\overline{AD} \cong \overline{DC}, \overline{AB} \cong \overline{BC}$
 - Yes, by the Reflexive Property
 - $\triangle DAB \cong \triangle DCB$ by SSS
- $\angle L \cong \angle N$ and $\angle M \cong \angle P$ by the Alternate Interior Angles Theorem
 - $\angle PON \cong \angle LOM$ by Vertical Angles or the 3rd Angle Theorem
 - No, no markings or midpoints
 - No, no congruent sides.
- Yes, CPCTC
 - No, these sides do not line up in the congruence statement.

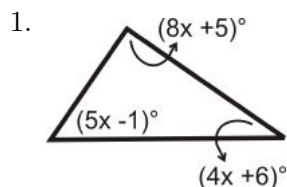
3.5 Isosceles and Equilateral Triangles

Learning Objectives

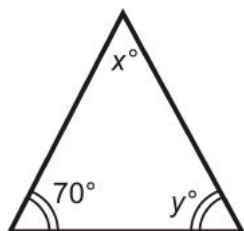
- Understand the properties of isosceles and equilateral triangles.
- Use the Base Angles Theorem and its converse.
- Understand that an equilateral triangle is also equiangular.

Review Queue

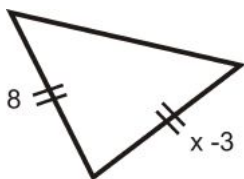
Find the value of x and/or y .



2.

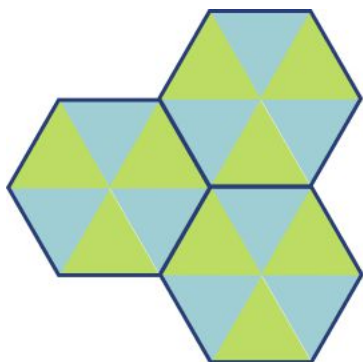


3.

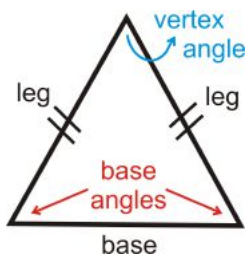


4. If a triangle is equiangular, what is the measure of each angle?

Know What? Your parents now want to redo the bathroom. To the right are 3 of the tiles they would like to place in the shower. Each blue and green triangle is an equilateral triangle. What shape is each dark blue polygon? Find the number of degrees in each of these figures?



Isosceles Triangle Properties

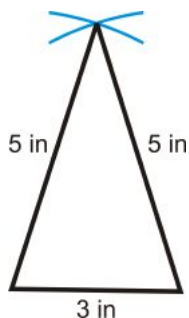


An isosceles triangle is a triangle that has **at least** two congruent sides. The congruent sides of the isosceles triangle are called the **legs**. The other side is called the **base**. The angles between the base and the legs are called **base angles**. The angle made by the two legs is called the **vertex angle**.

Investigation 4-5: Isosceles Triangle Construction

Tools Needed: pencil, paper, compass, ruler, protractor

1. Refer back to Investigation 4-2. Using your compass and ruler, draw an isosceles triangle with sides of 3 in, 5 in and 5 in. Draw the 3 in side (the base) horizontally at least 6 inches down the page.



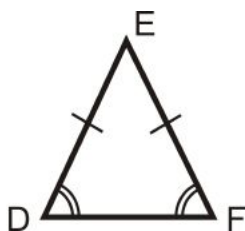
2. Now that you have an isosceles triangle, use your protractor to measure the base angles and the vertex angle.

The base angles should each be 72.5° and the vertex angle should be 35° .

We can generalize this investigation for all isosceles triangles.

Base Angles Theorem: The base angles of an isosceles triangle are congruent.

For $\triangle DEF$, if $\overline{DE} \cong \overline{EF}$, then $\angle D \cong \angle F$.

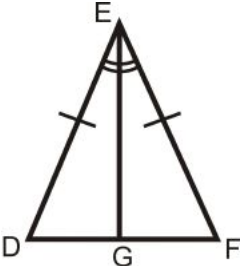


To prove the Base Angles Theorem, we need to draw the angle bisector (Investigation 1-5) of $\angle E$.

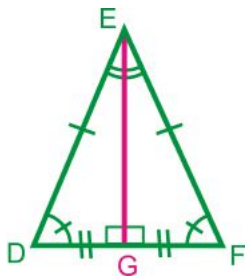
Given: Isosceles triangle $\triangle DEF$ above, with $\overline{DE} \cong \overline{EF}$.

Prove: $\angle D \cong \angle F$

Table 3.23:

<i>Statement</i>	<i>Reason</i>
1. Isosceles triangle $\triangle DEF$ with $\overline{DE} \cong \overline{EF}$	Given
2. Construct angle bisector \overline{EG} of $\angle E$	Every angle has one angle bisector
	
3. $\angle DEG \cong \angle FEG$	Definition of an angle bisector
4. $\overline{EG} \cong \overline{EG}$	Reflexive PoC
5. $\triangle DEG \cong \triangle FEG$	SAS
6. $\angle D \cong \angle F$	CPCTC

Let's take a further look at the picture from step 2 of our proof.



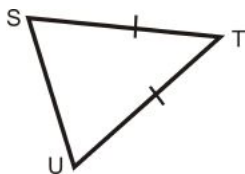
Because $\triangle DEG \cong \triangle FEG$, we know $\angle EGD \cong \angle EGF$ by CPCTC. These two angles are also a linear pair, so 90° each and $\overline{EG} \perp \overline{DF}$.

Additionally, $\overline{DG} \cong \overline{GF}$ by CPCTC, so G is the midpoint of \overline{DF} . This means that \overline{EG} is the **perpendicular bisector** of \overline{DF} .

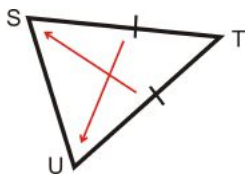
Isosceles Triangle Theorem: The angle bisector of the vertex angle in an isosceles triangle is also the perpendicular bisector of the base.

Note this is ONLY true of the vertex angle. We will prove this theorem in the review questions.

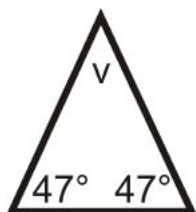
Example 1: Which two angles are congruent?



Solution: This is an isosceles triangle. The congruent angles, are opposite the congruent sides. From the arrows we see that $\angle S \cong \angle U$.



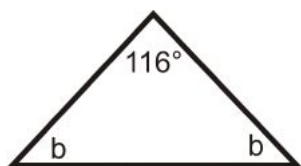
Example 2: If an isosceles triangle has base angles with measures of 47° , what is the measure of the vertex angle?



Solution: Draw a picture and set up an equation to solve for the vertex angle, v .

$$\begin{aligned} 47^\circ + 47^\circ + v &= 180^\circ \\ v &= 180^\circ - 47^\circ - 47^\circ \\ v &= 86^\circ \end{aligned}$$

Example 3: If an isosceles triangle has a vertex angle with a measure of 116° , what is the measure of each base angle?



Solution: Draw a picture and set up an equation to solve for the base angles, b .

$$116^\circ + b + b = 180^\circ$$

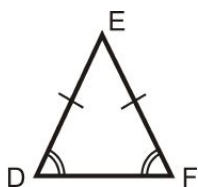
$$2b = 64^\circ$$

$$b = 32^\circ$$

The converses of the Base Angles Theorem and the Isosceles Triangle Theorem are both true.

Base Angles Theorem Converse: If two angles in a triangle are congruent, then the opposite sides are also congruent.

For $\triangle DEF$, if $\angle D \cong \angle F$, then $\overline{DE} \cong \overline{EF}$.



Isosceles Triangle Theorem Converse: The perpendicular bisector of the base of an isosceles triangle is also the angle bisector of the vertex angle.

For isosceles $\triangle DEF$, if $\overline{EG} \perp \overline{DF}$ and $\overline{DG} \cong \overline{GF}$, then $\angle DEG \cong \angle FEG$.

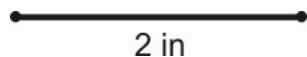


Equilateral Triangles By definition, all sides in an equilateral triangle have the same length.

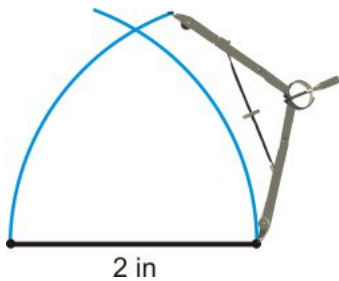
Investigation 4-6: Constructing an Equilateral Triangle

Tools Needed: pencil, paper, compass, ruler, protractor

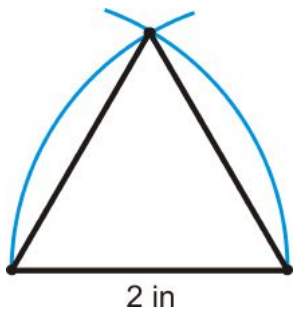
1. Because all the sides of an equilateral triangle are equal, pick one length to be all the sides of the triangle. Measure this length and draw it horizontally on your paper.



2. Put the pointer of your compass on the left endpoint of the line you drew in Step 1. Open the compass to be the same width as this line. Make an arc above the line. Repeat Step 2 on the right endpoint.



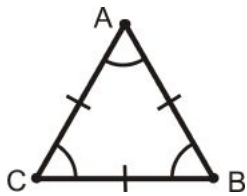
4. Connect each endpoint with the arc intersections to make the equilateral triangle.



Use the protractor to measure each angle of your constructed equilateral triangle. What do you notice?

From the Base Angles Theorem, the angles opposite congruent sides in an isosceles triangle are congruent. So, if all three sides of the triangle are congruent, then all of the angles are congruent, 60° each.

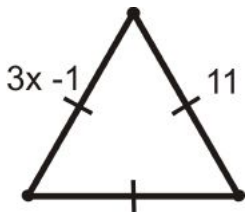
Equilateral Triangle Theorem: All equilateral triangles are also equiangular. Also, all equiangular triangles are also equilateral.



If $\overline{AB} \cong \overline{BC} \cong \overline{AC}$, then $\angle A \cong \angle B \cong \angle C$.

If $\angle A \cong \angle B \cong \angle C$, then $\overline{AB} \cong \overline{BC} \cong \overline{AC}$.

Example 4: Algebra Connection Find the value of x .



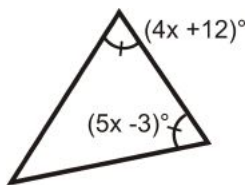
Solution: Because this is an equilateral triangle $3x - 1 = 11$. Solve for x .

$$3x - 1 = 11$$

$$3x = 12$$

$$x = 4$$

Example 5: Algebra Connection Find the value of x and the measure of each angle.

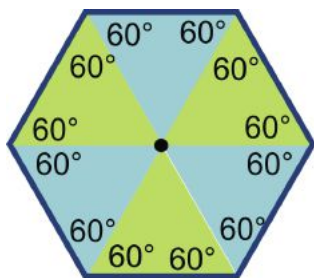


Solution: Similar to Example 4, the two angles are equal, so set them equal to each other and solve for x .

$$\begin{aligned}(4x + 12)^\circ &= (5x - 3)^\circ \\ 15^\circ &= x\end{aligned}$$

Substitute $x = 15^\circ$; the base angles are $4(15^\circ) + 12$, or 72° . The vertex angle is $180^\circ - 72^\circ - 72^\circ = 36^\circ$.

Know What? Revisited Let's focus on one tile. First, these triangles are all equilateral, so this is an equilateral hexagon (6 sides). Second, we now know that every equilateral triangle is also equiangular, so every triangle within this tile has 3 60° angles. This makes our equilateral hexagon also equiangular, with each angle measuring 120° . Because there are 6 angles, the sum of the angles in a hexagon are $6 \cdot 120^\circ$ or 720° .



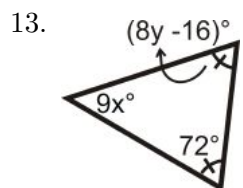
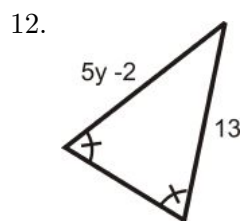
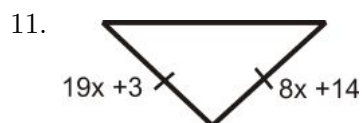
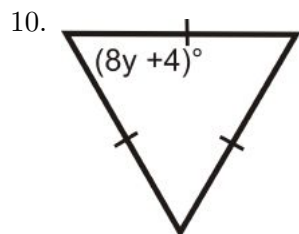
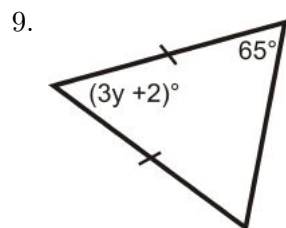
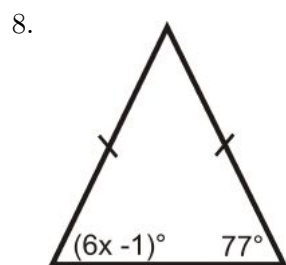
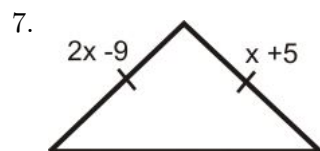
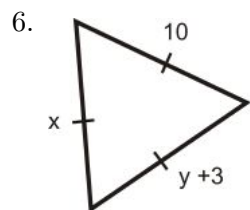
Review Questions

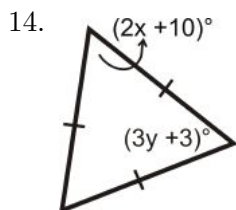
- Questions 1-5 are similar to Investigations 4-5 and 4-6.
- Questions 6-14 are similar to Examples 2-5.
- Question 15 uses the definition of an equilateral triangle.
- Questions 16-20 use the definition of an isosceles triangle.
- Question 21 is similar to Examples 2 and 3.
- Questions 22-25 are proofs and use definitions and theorems learned in this section.
- Questions 26-30 use the distance formula.

Constructions For questions 1-5, use your compass and ruler to:

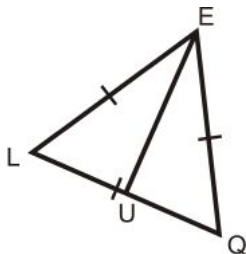
1. Draw an isosceles triangle with sides 3.5 in, 3.5 in, and 6 in.
2. Draw an isosceles triangle that has a vertex angle of 100° and legs with length of 4 cm. (you will also need your protractor for this one)
3. Draw an equilateral triangle with sides of length 7 cm.
4. Using what you know about constructing an equilateral triangle, construct (without a protractor) a 60° angle.
5. Draw an isosceles right triangle. What is the measure of the base angles?

For questions 6-14, find the measure of x and/or y .





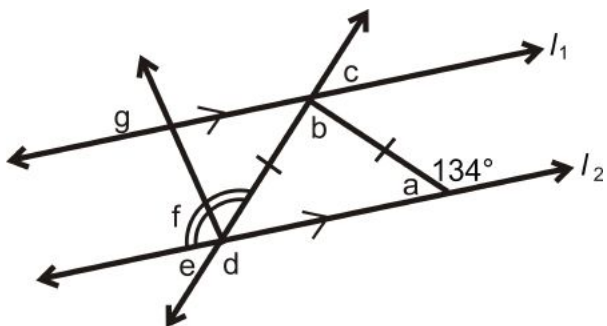
15. $\triangle EQG$ is an equilateral triangle. If \overline{EU} bisects $\angle LEQ$, find:



- (a) $m\angle EUL$
- (b) $m\angle UEL$
- (c) $m\angle ELQ$
- (d) If $EQ = 4$, find LU .

Determine if the following statements are true or false.

- 16. Base angles of an isosceles triangle are congruent.
- 17. Base angles of an isosceles triangle are complementary.
- 18. Base angles of an isosceles triangle can be equal to the vertex angle.
- 19. Base angles of an isosceles triangle can be right angles.
- 20. Base angles of an isosceles triangle are acute.
- 21. In the diagram below, $l_1 \parallel l_2$. Find all of the lettered angles.



Fill in the blanks in the proofs below.

22. Given: Isosceles $\triangle CIS$, with base angles $\angle C$ and $\angle S$
 \overline{IO} is the angle bisector of $\angle CIS$
Prove: \overline{IO} is the perpendicular bisector of \overline{CS}

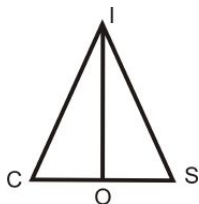


Table 3.24:

<i>Statement</i>	<i>Reason</i>
1.	Given
2.	Base Angles Theorem
3. $\angle CIO \cong \angle SIO$	
4.	Reflexive PoC
5. $\triangle CIO \cong \triangle SIO$	
6. $\overline{CO} \cong \overline{OS}$	
7.	CPCTC
8. $\angle IOC$ and $\angle IOS$ are supplementary	
9.	Congruent Supplements Theorem
10. \overline{IO} is the perpendicular bisector of \overline{CS}	

23. Given: Equilateral $\triangle RST$ with $\overline{RT} \cong \overline{ST} \cong \overline{RS}$
Prove: $\triangle RST$ is equiangular

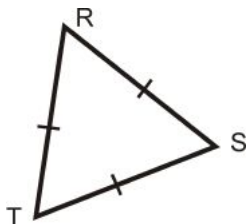


Table 3.25:

<i>Statement</i>	<i>Reason</i>
1.	Given
2.	Base Angles Theorem
3.	Base Angles Theorem
4.	Transitive PoC
5. $\triangle RST$ is equiangular	

24. Given: Isosceles $\triangle ICS$ with $\angle C$ and $\angle S$
 \overline{IO} is the perpendicular bisector of \overline{CS}
Prove: \overline{IO} is the angle bisector of $\angle CIS$

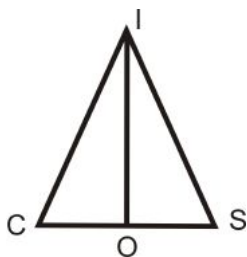


Table 3.26:

<i>Statement</i>	<i>Reason</i>
1.	
2. $\angle C \cong \angle S$	
3. $\overline{CO} \cong \overline{OS}$	
4. $m\angle IOC = m\angle IOS = 90^\circ$	
5.	
6.	CPCTC
7. \overline{IO} is the angle bisector of $\angle CIS$	

25. Given: Isosceles $\triangle ABC$ with base angles $\angle B$ and $\angle C$

Isosceles $\triangle XYZ$ with base angles $\angle Y$ and $\angle Z$

$\angle C \cong \angle Z$, $\overline{BC} \cong \overline{YZ}$

Prove: $\triangle ABC \cong \triangle XYZ$

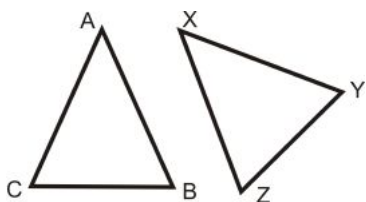


Table 3.27:

<i>Statement</i>	<i>Reason</i>
1.	
2. $\angle B \cong \angle C$, $\angle Y \cong \angle Z$	
3. $\angle B \cong \angle Y$	
4. $\triangle ABC \cong \triangle XYZ$	

Coordinate Plane Geometry On the $x - y$ plane, plot the coordinates and determine if the given three points make a scalene or isosceles triangle.

26. $(-2, 1)$, $(1, -2)$, $(-5, -2)$

27. $(-2, 5)$, $(2, 4)$, $(0, -1)$

28. $(6, 9)$, $(12, 3)$, $(3, -6)$

29. $(-10, -5)$, $(-8, 5)$, $(2, 3)$

30. $(-1, 2)$, $(7, 2)$, $(3, 9)$

Review Queue Answers

$$1. (5x - 1)^\circ + (8x + 5)^\circ + (4x + 6)^\circ = 180^\circ$$

$$17x + 10 = 180^\circ$$

$$17x = 170^\circ$$

$$x = 10^\circ$$

$$2. x = 40^\circ, y = 70^\circ$$

3. $x - 3 = 8$
 $x = 5$
4. Each angle is $\frac{180^\circ}{3}$, or 60°

3.6 Chapter 4 Review

Symbols Toolbox

Congruent Triangles and their corresponding parts



Definitions, Postulates, and Theorems

Triangle Sums

- Interior Angles
- Vertex
- Triangle Sum Theorem
- Exterior Angle
- Exterior Angle Sum Theorem
- Remote Interior Angles
- Exterior Angle Theorem

Congruent Figures

- Congruent Triangles
- Congruence Statements
- Third Angle Theorem
- Reflexive Property of Congruence
- Symmetric Property of Congruence
- Transitive Property of Congruence

Triangle Congruence using SSS and SAS

- Side-Side-Side (SSS) Triangle Congruence Postulate
- Included Angle
- Side-Angle-Side (SAS) Triangle Congruence Postulate
- Distance Formula

Triangle Congruence using ASA, AAS, and HL

- Angle-Side-Angle (ASA) Congruence Postulate

- Angle-Angle-Side (AAS) Congruence Theorem
- Hypotenuse
- Legs (of a right triangle)
- HL Congruence Theorem

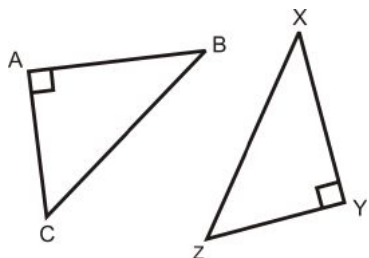
Isosceles and Equilateral Triangles

- Base
- Base Angles
- Vertex Angle
- Legs (of an isosceles triangle)
- Base Angles Theorem
- Isosceles Triangle Theorem
- Base Angles Theorem Converse
- Isosceles Triangle Theorem Converse
- Equilateral Triangles Theorem

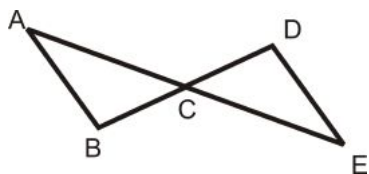
Review

For each pair of triangles, write what needs to be congruent in order for the triangles to be congruent. Then, write the congruence statement for the triangles.

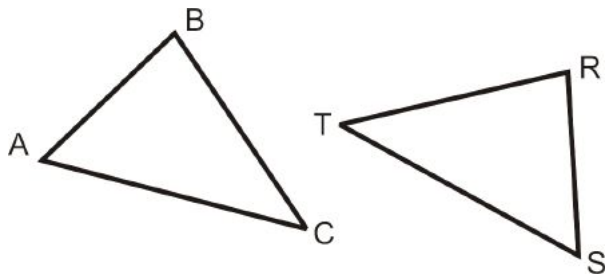
1. HL



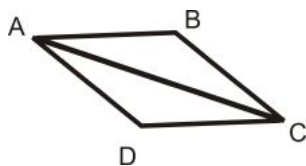
2. ASA



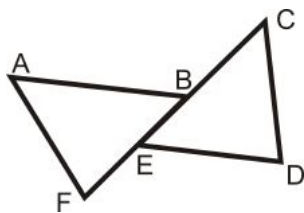
3. AAS



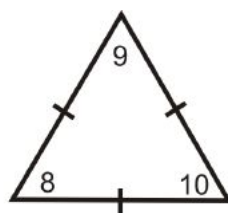
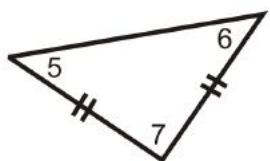
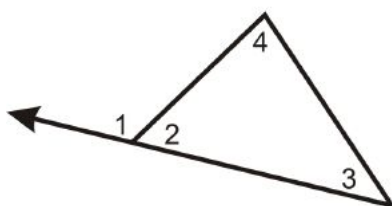
4. SSS



5. SAS



Using the pictures below, determine which theorem, postulate or definition that supports each statement below.



6. $m\angle 1 + m\angle 2 = 180^\circ$
7. $\angle 5 \cong \angle 6$
8. $m\angle 1 + m\angle 4 + m\angle 3$
9. $m\angle 8 = 60^\circ$
10. $m\angle 5 + m\angle 6 + m\angle 7 = 180^\circ$
11. $\angle 8 \cong \angle 9 \cong \angle 10$
12. If $m\angle 7 = 90^\circ$, then $m\angle 5 = m\angle 6 = 45^\circ$

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9689>.

3.7 Study Guide

Keywords: Define, write theorems, and/or draw a diagram for each word below.

1st Section: Triangle Sums

Interior Angles

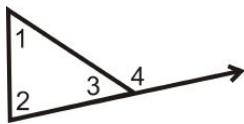
Vertex

Triangle Sum Theorem

Exterior Angle Exterior Angle Sum Theorem

Remote Interior Angles

Exterior Angle Theorem



Homework:

2nd Section: Congruent Figures

Congruent Triangles

Congruence Statements

Third Angle Theorem

Reflexive Property of Congruence

Symmetric Property of Congruence

Transitive Property of Congruence

Homework:

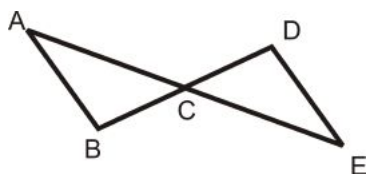
3rd Section: Triangle Congruence using SSS and SAS

Side-Side-Side (SSS) Triangle Congruence Postulate

Included Angle

Side-Angle-Side (SAS) Triangle Congruence Postulate

Distance Formula



Homework:

4th Section: Triangle Congruence using ASA, AAS, and HL

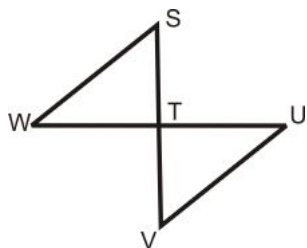
Angle-Side-Angle (ASA) Congruence Postulate

Angle-Angle-Side (AAS) Congruence Theorem

Hypotenuse

Legs (of a right triangle)

HL Congruence Theorem



Homework:

5th Section: Isosceles and Equilateral Triangles

Base

Base Angles

Vertex Angle

Legs (of an isosceles triangle)

Base Angles Theorem

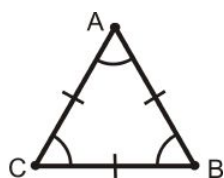
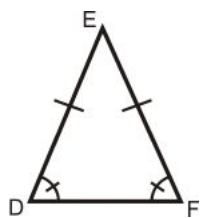
Isosceles Triangle Theorem

Base Angles Theorem Converse

Isosceles Triangle Theorem Converse

Equilateral Triangle Theorem

Homework:



Chapter 4

Right Triangle Trigonometry

Chapter 8 takes a look at right triangles. A right triangle is a triangle with exactly one right angle. In this chapter, we will prove the Pythagorean Theorem and its converse. Then, we will introduce trigonometry ratios. Finally, there is an extension about the Law of Sines and the Law of Cosines.

4.1 The Pythagorean Theorem

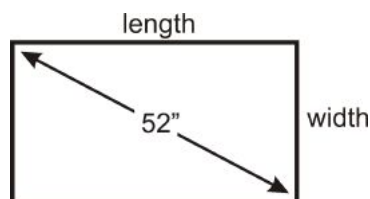
Learning Objectives

- Review simplifying and reducing radicals.
- Prove and use the Pythagorean Theorem.
- Use the Pythagorean Theorem to derive the distance formula.

Review Queue

1. Draw a right scalene triangle.
2. Draw an isosceles right triangle.
3. List all the factors of 75.
4. Write the prime factorization of 75.

Know What? For a 52" TV, 52" is the length of the diagonal. High Definition Televisions (HDTVs) have sides in a ratio of 16:9. What are the length and width of a 52" HDTV?



Simplifying and Reducing Radicals

In algebra, you learned how to simplify radicals. Let's review it here.

Example 1: Simplify the radical.

- a) $\sqrt{50}$
- b) $\sqrt{27}$
- c) $\sqrt{272}$

Solution: For each radical, find the square number(s) that are factors.

- a) $\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$
- b) $\sqrt{27} = \sqrt{9 \cdot 3} = 3\sqrt{3}$
- c) $\sqrt{272} = \sqrt{16 \cdot 17} = 4\sqrt{17}$

When adding radicals, you can only combine radicals with the same number underneath it. For example, $2\sqrt{5} + 3\sqrt{6}$ cannot be combined, because 5 and 6 are not the same number.

Example 2: Simplify the radicals.

- a) $2\sqrt{10} + \sqrt{160}$
- b) $5\sqrt{6} \cdot 4\sqrt{18}$
- c) $\sqrt{8} \cdot 12\sqrt{2}$
- d) $(5\sqrt{2})^2$

Solution:

- a) Simplify $\sqrt{160}$ before adding: $2\sqrt{10} + \sqrt{160} = 2\sqrt{10} + \sqrt{16 \cdot 10} = 2\sqrt{10} + 4\sqrt{10} = 6\sqrt{10}$
- b) To multiply two radicals, multiply what is under the radicals and what is in front.
 $5\sqrt{6} \cdot 4\sqrt{18} = 5 \cdot 4 \sqrt{6 \cdot 18} = 20\sqrt{108} = 20\sqrt{36 \cdot 3} = 20 \cdot 6\sqrt{3} = 120\sqrt{3}$
- c) $\sqrt{8} \cdot 12\sqrt{2} = 12\sqrt{8 \cdot 2} = 12\sqrt{16} = 12 \cdot 4 = 48$
- d) $(5\sqrt{2})^2 = 5^2 (\sqrt{2})^2 = 25 \cdot 2 = 50 \rightarrow$ the $\sqrt{}$ and the 2 cancel each other out

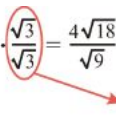
Lastly, to divide radicals, you need to simplify the denominator, which means multiplying the top and bottom of the fraction by the radical in the denominator.

Example 3: Divide and simplify the radicals.

- a) $4\sqrt{6} \div \sqrt{3}$
- b) $\frac{\sqrt{30}}{\sqrt{8}}$
- c) $\frac{8\sqrt{2}}{6\sqrt{7}}$

Solution: Rewrite all division problems like a fraction.

a) $4\sqrt{6} \div \sqrt{3} = \frac{4\sqrt{6}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{18}}{\sqrt{9}} = \frac{4\sqrt{9 \cdot 2}}{3} = \frac{4 \cdot 3\sqrt{2}}{3} = 4\sqrt{2}$


 like multiplying by 1, $\frac{\sqrt{3}}{\sqrt{3}}$ does not change the value of the fraction

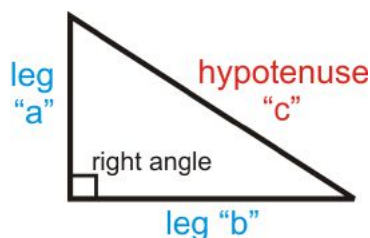
b) $\frac{\sqrt{30}}{\sqrt{8}} \cdot \frac{\sqrt{8}}{\sqrt{8}} = \frac{\sqrt{240}}{\sqrt{64}} = \frac{\sqrt{16 \cdot 15}}{8} = \frac{4\sqrt{15}}{8} = \frac{\sqrt{15}}{2}$

c) $\frac{8\sqrt{2}}{6\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{8\sqrt{14}}{6 \cdot 7} = \frac{4\sqrt{14}}{3 \cdot 7} = \frac{4\sqrt{14}}{21}$

Notice, we do not really “divide” radicals, but get them out of the denominator of a fraction.

The Pythagorean Theorem

We have used the Pythagorean Theorem already in this text, but have not proved it. Recall that the sides of a right triangle are the **legs** (the sides of the right angle) and the **hypotenuse** (the side opposite the right angle). For the Pythagorean Theorem, the legs are “ a ” and “ b ” and the hypotenuse is “ c ”.

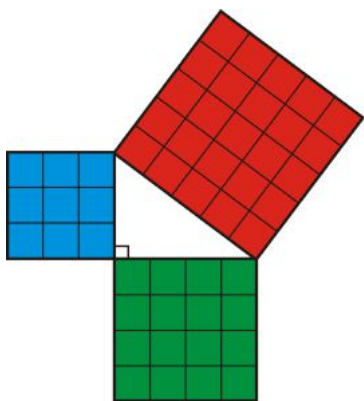


Pythagorean Theorem: Given a right triangle with legs of lengths a and b and a hypotenuse of length c , then $a^2 + b^2 = c^2$.

Investigation 8-1: Proof of the Pythagorean Theorem

Tools Needed: pencil, 2 pieces of graph paper, ruler, scissors, colored pencils (optional)

1. On the graph paper, draw a 3 in. square, a 4 in. square, a 5 in. square and a right triangle with legs of 3 in. and 4 in.
2. Cut out the triangle and square and arrange them like the picture on the right.



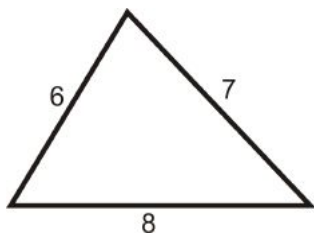
3. This theorem relies on area. Recall that the area of a square is $side^2$. In this case, we have three squares with sides 3 in., 4 in., and 5 in. What is the area of each square?
4. Now, we know that $9 + 16 = 25$, or $3^2 + 4^2 = 5^2$. Cut the smaller squares to fit into the larger square, thus proving the areas are equal.

For two more proofs, go to: <http://www.mathsisfun.com/pythagoras.html> and scroll down to “And You Can Prove the Theorem Yourself.”

Using the Pythagorean Theorem

Here are several examples of the Pythagorean Theorem in action.

Example 4: Do 6, 7, and 8 make the sides of a right triangle?



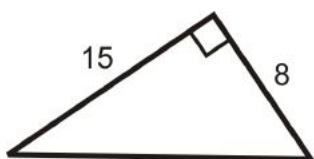
Solution: Plug in the three numbers to the Pythagorean Theorem. *The largest length will always be the hypotenuse.* If $6^2 + 7^2 = 8^2$, then they are the sides of a right triangle.

$$6^2 + 7^2 = 36 + 49 = 85$$

$$8^2 = 64$$

$85 \neq 64$, so the lengths are not the sides of a right triangle.

Example 5: Find the length of the hypotenuse.



Solution: Use the Pythagorean Theorem. Set $a = 8$ and $b = 15$. Solve for c .

$$8^2 + 15^2 = c^2$$

$$64 + 225 = c^2$$

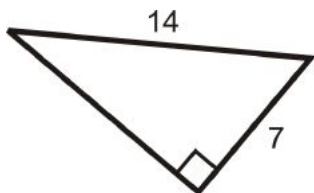
$$289 = c^2$$

$$17 = c$$

Take the square root of both sides.

When you take the square root of an equation, the answer is 17 or -17. *Length is never negative*, which makes 17 the answer.

Example 6: Find the missing side of the right triangle below.



Solution: Here, we are given the hypotenuse and a leg. Let's solve for b .

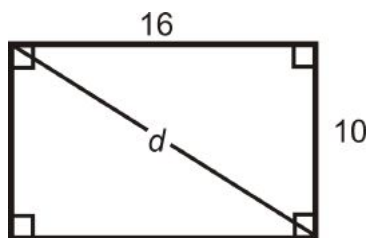
$$7^2 + b^2 = 14^2$$

$$49 + b^2 = 196$$

$$b^2 = 147$$

$$b = \sqrt{147} = \sqrt{49 \cdot 3} = 7\sqrt{3}$$

Example 7: What is the diagonal of a rectangle with sides 10 and 16?



Solution: For any square and rectangle, you can use the Pythagorean Theorem to find the length of a diagonal. Plug in the sides to find d .

$$10^2 + 16^2 = d^2$$

$$100 + 256 = d^2$$

$$356 = d^2$$

$$d = \sqrt{356} = 2\sqrt{89} \approx 18.87$$

Pythagorean Triples

In Example 5, the sides of the triangle were 8, 15, and 17. This combination of numbers is called a *Pythagorean triple*.

Pythagorean Triple: A set of three whole numbers that makes the Pythagorean Theorem true.

3, 4, 5

5, 12, 13

7, 24, 25

8, 15, 17

9, 12, 15

10, 24, 26

Any multiple of a Pythagorean triple is also considered a triple because it would still be three whole numbers. Multiplying 3, 4, 5 by 2 gives 6, 8, 10, which is another triple. To see if a set of numbers makes a triple, plug them into the Pythagorean Theorem.

Example 8: Is 20, 21, 29 a Pythagorean triple?

Solution: If $20^2 + 21^2 = 29^2$, then the set is a Pythagorean triple.

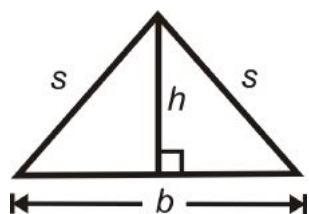
$$20^2 + 21^2 = 400 + 441 = 841$$

$$29^2 = 841$$

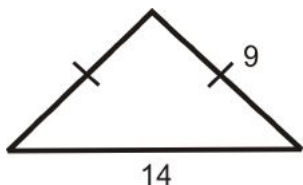
Therefore, 20, 21, and 29 is a Pythagorean triple.

Height of an Isosceles Triangle

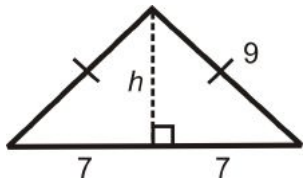
One way to use The Pythagorean Theorem is to find the height of an isosceles triangle.



Example 9: What is the height of the isosceles triangle?



Solution: Draw the altitude from the vertex between the congruent sides, which bisect the base.



$$7^2 + h^2 = 9^2$$

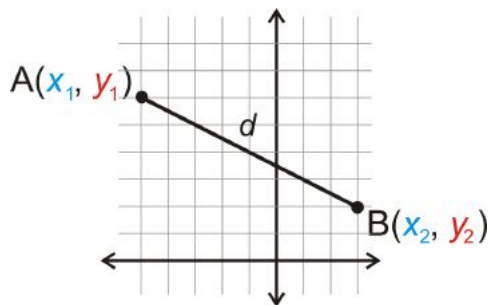
$$49 + h^2 = 81$$

$$h^2 = 32$$

$$h = \sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$$

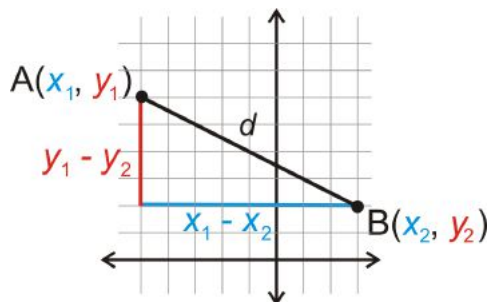
The Distance Formula

Another application of the Pythagorean Theorem is the Distance Formula. We will prove it here.



Let's start with point $A(x_1, y_1)$ and point $B(x_2, y_2)$, to the left. We will call the distance between A and B , d .

Draw the vertical and horizontal lengths to make a right triangle.



Now that we have a right triangle, we can use the Pythagorean Theorem to find the hypotenuse, d .

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Distance Formula: The distance $A(x_1, y_1)$ and $B(x_2, y_2)$ is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Example 10: Find the distance between $(1, 5)$ and $(5, 2)$.

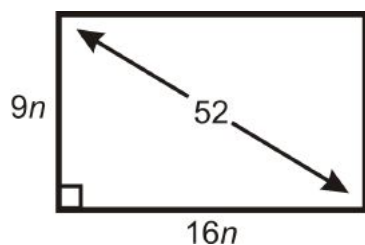
Solution: Make $A(1, 5)$ and $B(5, 2)$. Plug into the distance formula.

$$\begin{aligned}d &= \sqrt{(1 - 5)^2 + (5 - 2)^2} \\&= \sqrt{(-4)^2 + (3)^2} \\&= \sqrt{16 + 9} = \sqrt{25} = 5\end{aligned}$$

Just like the lengths of the sides of a triangle, distances are always positive.

Know What? Revisited To find the length and width of a 52" HDTV, plug in the ratios and 52 into the Pythagorean Theorem. We know that the sides are going to be a multiple of 16 and 9, which we will call n .

$$\begin{aligned}(16n)^2 + (9n)^2 &= 52^2 \\256n^2 + 81n^2 &= 2704 \\337n^2 &= 2704 \\n^2 &= 8.024 \\n &= 2.83\end{aligned}$$



The dimensions of the TV are $16(2.83'') \times 9(2.83'')$, or $45.3'' \times 25.5''$.

Review Questions

- Questions 1-9 are similar to Examples 1-3.
- Questions 10-15 are similar to Example 5 and 6.
- Questions 16-19 are similar to Example 7.
- Questions 20-25 are similar to Example 8.
- Questions 26-28 are similar to Example 9.
- Questions 29-31 are similar to Example 10.
- Questions 32 and 33 are similar to the Know What?
- Question 34 and 35 are a challenge and similar to Example 9.

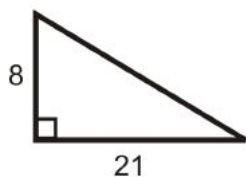
Simplify the radicals.

1. $2\sqrt{5} + \sqrt{20}$
2. $\sqrt{24}$
3. $(6\sqrt{3})^2$
4. $8\sqrt{8} \cdot \sqrt{10}$

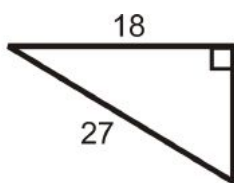
5. $(2\sqrt{30})^2$
6. $\sqrt{320}$
7. $\frac{4\sqrt{5}}{\sqrt{6}}$
8. $\frac{12}{\sqrt{10}}$
9. $\frac{21\sqrt{5}}{9\sqrt{15}}$

Find the length of the missing side. Simplify all radicals.

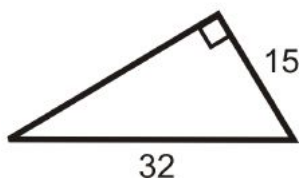
10.



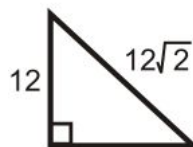
11.



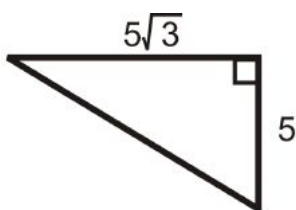
12.



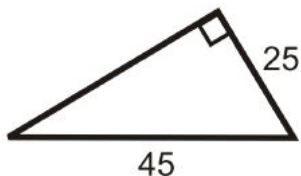
13.



14.



15.



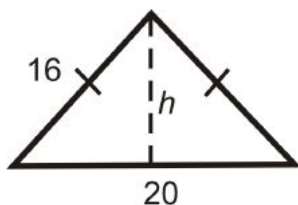
16. If the legs of a right triangle are 10 and 24, then the hypotenuse is _____.
17. If the sides of a rectangle are 12 and 15, then the diagonal is _____.
18. If the sides of a square are 16, then the diagonal is _____.
19. If the sides of a square are 9, then the diagonal is _____.

Determine if the following sets of numbers are Pythagorean Triples.

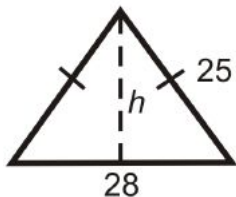
20. 12, 35, 37
21. 9, 17, 18
22. 10, 15, 21
23. 11, 60, 61
24. 15, 20, 25
25. 18, 73, 75

Find the height of each isosceles triangle below. Simplify all radicals.

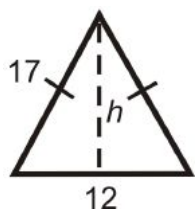
26.



27.

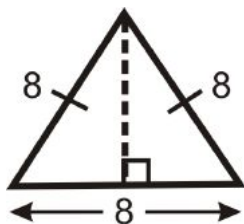


28.



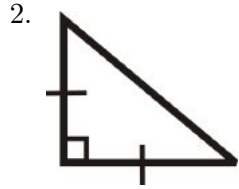
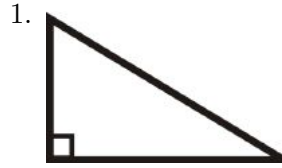
Find the length between each pair of points.

29. $(-1, 6)$ and $(7, 2)$
30. $(10, -3)$ and $(-12, -6)$
31. $(1, 3)$ and $(-8, 16)$
32. What are the length and width of a 42" HDTV? Round your answer to the nearest tenth.
33. Standard definition TVs have a length and width ratio of 4:3. What are the length and width of a 42" Standard definition TV? Round your answer to the nearest tenth.
34. **Challenge** An equilateral triangle is an isosceles triangle. If all the sides of an equilateral triangle are 8, find the height. Leave your answer in simplest radical form.



35. If the sides are length s , what would the height be?

Review Queue Answers



3. Factors of 75: 1, 3, 5, 15, 25, 75
4. Prime Factorization of 75: $3 \cdot 5 \cdot 5$

4.2 Converse of the Pythagorean Theorem

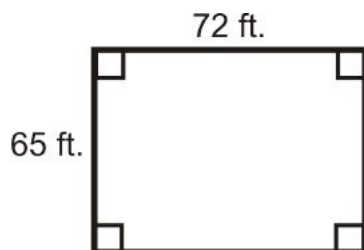
Learning Objectives

- Understand the converse of the Pythagorean Theorem.
- Determine if a triangle is acute or obtuse from side measures.

Review Queue

1. Determine if the following sets of numbers are Pythagorean triples.
 - (a) 14, 48, 50
 - (b) 9, 40, 41
 - (c) 12, 43, 44
 - (d) 12, 35, 37
2. Simplify the radicals.
 - (a) $(5\sqrt{12})^2$
 - (b) $\frac{14}{\sqrt{2}}$
 - (c) $\frac{18}{\sqrt{3}}$

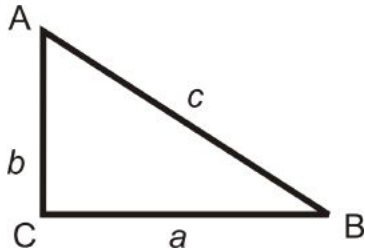
Know What? A friend of yours is designing a building and wants it to be rectangular. One wall 65 ft. long and the other is 72 ft. long. How can he ensure the walls are going to be perpendicular?



Converse of the Pythagorean Theorem

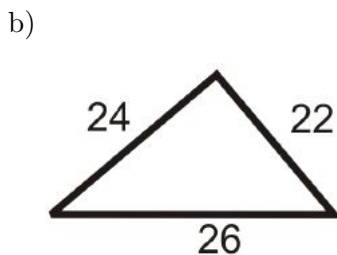
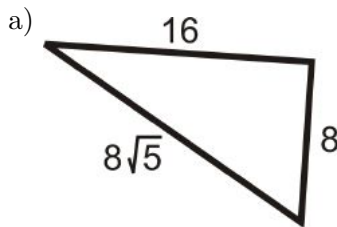
Pythagorean Theorem Converse: If the square of the longest side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.

If $a^2 + b^2 = c^2$, then $\triangle ABC$ is a right triangle.



With this converse, you can use the Pythagorean Theorem to prove that a triangle is a right triangle, even if you do not know any angle measures.

Example 1: Determine if the triangles below are right triangles.



Solution: Check to see if the three lengths satisfy the Pythagorean Theorem. Let the longest side represent c .

$$\begin{aligned} \text{a) } a^2 + b^2 &= c^2 \\ 8^2 + 16^2 &\stackrel{?}{=} (8\sqrt{5})^2 \\ 64 + 256 &\stackrel{?}{=} 64 \cdot 5 \\ 320 &= 320 \quad \text{Yes} \end{aligned}$$

$$\begin{aligned} \text{b) } a^2 + b^2 &= c^2 \\ 22^2 + 24^2 &\stackrel{?}{=} 26^2 \\ 484 + 576 &\stackrel{?}{=} 676 \\ 1060 &\neq 676 \quad \text{No} \end{aligned}$$

Example 2: Do the following lengths make a right triangle?

- a) $\sqrt{5}, 3, \sqrt{14}$
- b) $6, 2\sqrt{3}, 8$
- c) $3\sqrt{2}, 4\sqrt{2}, 5\sqrt{2}$

Solution: Even though there is no picture, you can still use the Pythagorean Theorem. Again, the longest length will be c .

a) $(\sqrt{5})^2 + 3^2 = \sqrt{14}^2$

$$5 + 9 = 14$$

Yes

b) $6^2 + (2\sqrt{3})^2 = 8^2$

$$36 + (4 \cdot 3) = 64$$

$$36 + 12 \neq 64$$

c) This is a multiple of $\sqrt{2}$ of a 3, 4, 5 right triangle. Yes, this is a right triangle.

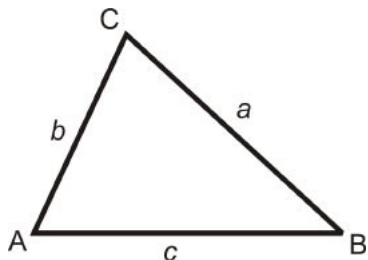
Identifying Acute and Obtuse Triangles

We can extend the converse of the Pythagorean Theorem to determine if a triangle is an obtuse or acute triangle.

Theorem 8-3: If the sum of the squares of the two shorter sides in a right triangle is *greater* than the square of the longest side, then the triangle is *acute*.

$$b < c \text{ and } a < c$$

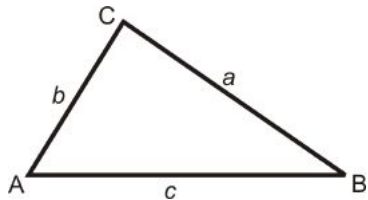
If $a^2 + b^2 > c^2$, then the triangle is acute.



Theorem 8-4: If the sum of the squares of the two shorter sides in a right triangle is *less* than the square of the longest side, then the triangle is *obtuse*.

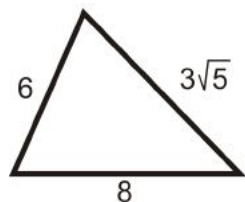
$$b < c \text{ and } a < c$$

If $a^2 + b^2 < c^2$, then the triangle is obtuse.

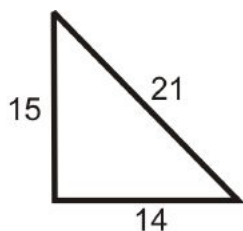


Example 3: Determine if the following triangles are acute, right or obtuse.

a)



b)



Solution: Set the longest side equal to c .

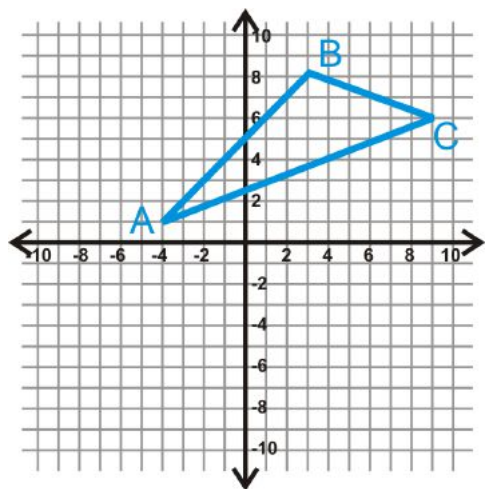
$$\begin{aligned} \text{a) } 6^2 + (3\sqrt{5})^2 &? 8^2 \\ 36 + 45 &? 64 \\ 81 &> 64 \end{aligned}$$

The triangle is acute.

$$\begin{aligned} \text{b) } 15^2 + 14^2 &? 21^2 \\ 225 + 196 &? 441 \\ 421 &< 441 \end{aligned}$$

The triangle is obtuse.

Example 4: Graph $A(-4, 1)$, $B(3, 8)$, and $C(9, 6)$. Determine if $\triangle ABC$ is acute, obtuse, or right.



Solution: Use the distance formula to find the length of each side.

$$\begin{aligned} AB &= \sqrt{(-4 - 3)^2 + (1 - 8)^2} = \sqrt{49 + 49} = \sqrt{98} = 7\sqrt{2} \\ BC &= \sqrt{(3 - 9)^2 + (8 - 6)^2} = \sqrt{36 + 4} = \sqrt{40} = 2\sqrt{10} \\ AC &= \sqrt{(-4 - 9)^2 + (1 - 6)^2} = \sqrt{169 + 25} = \sqrt{194} \end{aligned}$$

Plug these lengths into the Pythagorean Theorem.

$$\begin{aligned} (\sqrt{98})^2 + (\sqrt{40})^2 &? (\sqrt{194})^2 \\ 98 + 40 &? 194 \\ 138 &< 194 \end{aligned}$$

$\triangle ABC$ is an obtuse triangle.

Know What? Revisited Find the length of the diagonal.

$$65^2 + 72^2 = c^2$$

$$4225 + 5184 = c^2$$

$$9409 = c^2$$

$$97 = c$$

To make the building rectangular, both diagonals must be 97 feet.

Review Questions

- Questions 1-6 are similar to Examples 1 and 2.
- Questions 7-15 are similar to Example 3.
- Questions 16-20 are similar to Example 4.
- Questions 21-24 use the Pythagorean Theorem.
- Question 25 uses the definition of similar triangles.

Determine if the following lengths make a right triangle.

1. 7, 24, 25
2. $\sqrt{5}$, $2\sqrt{10}$, $3\sqrt{5}$
3. $2\sqrt{3}$, $\sqrt{6}$, 8
4. 15, 20, 25
5. 20, 25, 30
6. $8\sqrt{3}$, 6, $2\sqrt{39}$

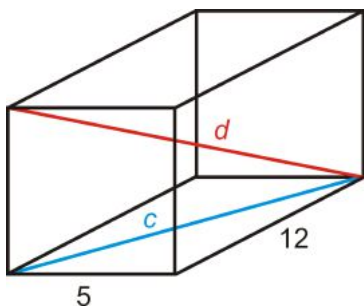
Determine if the following triangles are acute, right or obtuse.

7. 7, 8, 9
8. 14, 48, 50
9. 5, 12, 15
10. 13, 84, 85
11. 20, 20, 24
12. 35, 40, 51
13. 39, 80, 89
14. 20, 21, 38
15. 48, 55, 76

Graph each set of points and determine if $\triangle ABC$ is acute, right, or obtuse, using the distance formula.

16. $A(3, -5)$, $B(-5, -8)$, $C(-2, 7)$
17. $A(5, 3)$, $B(2, -7)$, $C(-1, 5)$
18. $A(1, 6)$, $B(5, 2)$, $C(-2, 3)$
19. $A(-6, 1)$, $B(-4, -5)$, $C(5, -2)$
20. Show that #18 is a right triangle by using the slopes of the sides of the triangle.

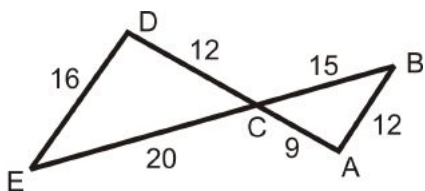
The figure to the right is a rectangular prism. All sides (or faces) are either squares (the front and back) or rectangles (the four around the middle). All faces are perpendicular.



21. Find c .
22. Find d .

Now, the figure is a cube, where all the sides are squares. If all the sides have length 4, find:

23. Find c .
24. Find d .



25. **Writing** Explain why $m\angle A = 90^\circ$.

Review Queue Answers

1. (a) Yes
(b) Yes
(c) No
(d) Yes
2. (a) $(5\sqrt{12})^2 = 5^2 \cdot 12 = 25 \cdot 12 = 300$
(b) $\frac{14}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{14\sqrt{2}}{2} = 7\sqrt{2}$
(c) $\frac{18}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{18\sqrt{3}}{3} = 6\sqrt{3}$

4.3 Using Similar Right Triangles

Learning Objectives

- Identify similar triangles inscribed in a larger triangle.
- Use proportions in similar right triangles.

Review Queue

1. Solve the following ratios.
 - (a) $\frac{3}{x} = \frac{x}{27}$
 - (b) $\frac{\sqrt{6}}{x} = \frac{x}{9\sqrt{6}}$

(c) $\frac{x}{15} = \frac{12}{x}$

2. If the legs of an isosceles right triangle are 4, find the length of the hypotenuse. Draw a picture and simplify the radical.

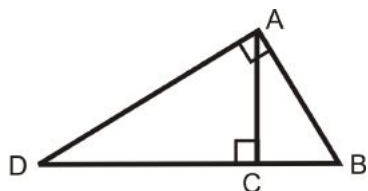
Know What? The bridge to the right is called a truss bridge. It is a steel bridge with a series of right triangles that are connected as support. All the red right triangles are similar. Can you find x , y and z ?



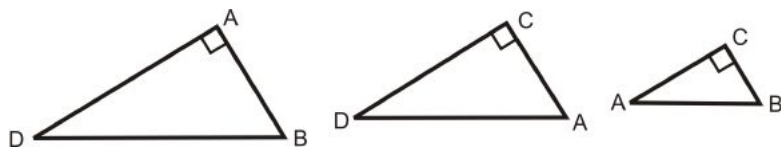
Inscribed Similar Triangles

You may recall that if two objects are similar, corresponding angles are congruent and their sides are proportional in length.

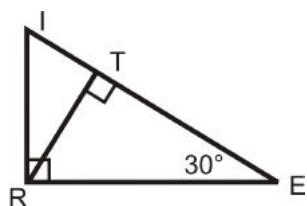
Theorem 8-5: If an altitude is drawn from the right angle of any right triangle, then the two triangles formed are similar to the original triangle and all three triangles are similar to each other.



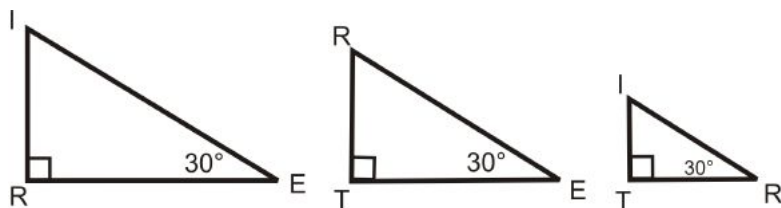
In $\triangle ADB$, $m\angle A = 90^\circ$ and $\overline{AC} \perp \overline{DB}$, then $\triangle ADB \sim \triangle CDA \sim \triangle CAB$.



Example 1: Write the similarity statement for the triangles below.



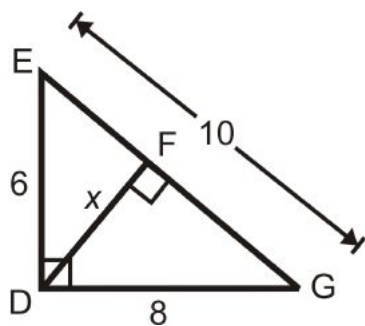
Solution: Separate out the three triangles.



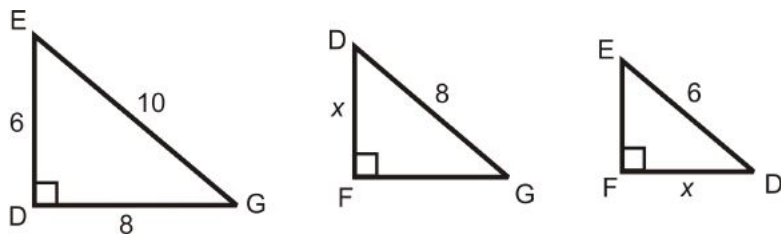
Line up the congruent angles: $\triangle IRE \sim \triangle ITR \sim \triangle RTE$

We can also use the side proportions to find the length of the altitude.

Example 2: Find the value of x .



Solution: Separate the triangles to find the corresponding sides.



Set up a proportion.

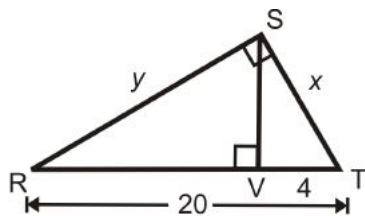
$$\frac{\text{shorter leg in } \triangle EDG}{\text{shorter leg in } \triangle DFG} = \frac{\text{hypotenuse in } \triangle EDG}{\text{hypotenuse in } \triangle DFG}$$

$$\frac{6}{x} = \frac{10}{8}$$

$$48 = 10x$$

$$4.8 = x$$

Example 3: Find the value of x .



Solution: Set up a proportion.

$$\begin{aligned}\frac{\text{shorter leg in } \triangle SVT}{\text{shorter leg in } \triangle RST} &= \frac{\text{hypotenuse in } \triangle SVT}{\text{hypotenuse in } \triangle RST} \\ \frac{4}{x} &= \frac{x}{20} \\ x^2 &= 80 \\ x &= \sqrt{80} = 4\sqrt{5}\end{aligned}$$

Example 4: Find the value of y in $\triangle RST$ above.

Solution: Use the Pythagorean Theorem.

$$\begin{aligned}y^2 + (4\sqrt{5})^2 &= 20^2 \\ y^2 + 80 &= 400 \\ y^2 &= 320 \\ y &= \sqrt{320} = 8\sqrt{5}\end{aligned}$$

The Geometric Mean

Geometric Mean: The geometric mean of two positive numbers a and b is the positive number x , such that $\frac{a}{x} = \frac{x}{b}$ or $x^2 = ab$ and $x = \sqrt{ab}$.

Example 5: Find the geometric mean of 24 and 36.

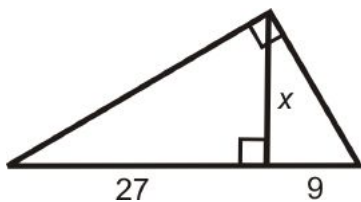
Solution: $x = \sqrt{24 \cdot 36} = \sqrt{12 \cdot 2 \cdot 12 \cdot 3} = 12\sqrt{6}$

Example 6: Find the geometric mean of 18 and 54.

Solution: $x = \sqrt{18 \cdot 54} = \sqrt{18 \cdot 18 \cdot 3} = 18\sqrt{3}$

In both of these examples, we did not multiply the numbers together. This makes it easier to simplify the radical. A practical application of the geometric mean is to find the altitude of a right triangle.

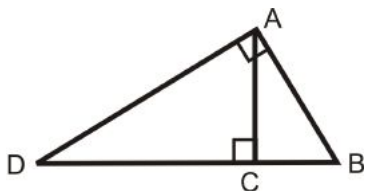
Example 7: Find the value of x .



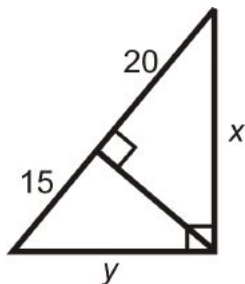
Solution: Set up a proportion.

$$\begin{aligned}\frac{\text{shortest leg of smallest } \triangle}{\text{shortest leg of middle } \triangle} &= \frac{\text{longer leg of smallest } \triangle}{\text{longer leg of middle } \triangle} \\ \frac{9}{x} &= \frac{x}{27} \\ x^2 &= 243 \\ x &= \sqrt{243} = 9\sqrt{3}\end{aligned}$$

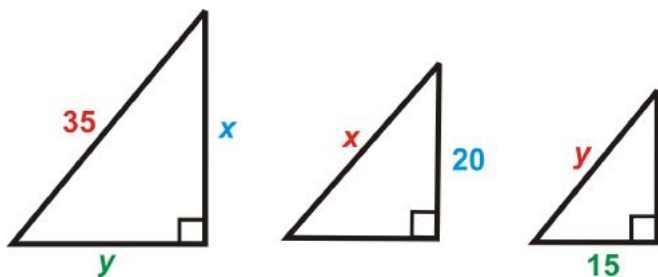
In Example 7, $\frac{9}{x} = \frac{x}{27}$ is in the definition of the geometric mean. So, the altitude is the geometric mean of the two segments that it divides the hypotenuse into. In other words, $\frac{BC}{AC} = \frac{AC}{DC}$. Two other true proportions are $\frac{BC}{AB} = \frac{AB}{DB}$ and $\frac{DC}{AD} = \frac{AD}{DB}$.



Example 8: Find the value of x and y .



Solution: Separate the triangles. Write a proportion for x .



$$\begin{aligned}\frac{20}{x} &= \frac{x}{35} \\ x^2 &= 20 \cdot 35 \\ x &= \sqrt{20 \cdot 35} \\ x &= 10\sqrt{7}\end{aligned}$$

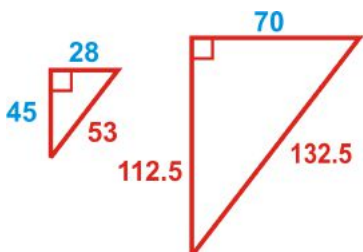
Set up a proportion for y . Or, you can use the Pythagorean Theorem to solve for y .

$$\begin{aligned}\frac{15}{y} &= \frac{y}{35} \\ y^2 &= 15 \cdot 35 \\ y &= \sqrt{15 \cdot 35} \\ y &= 5\sqrt{21}\end{aligned}$$

$$\begin{aligned}(10\sqrt{7})^2 + y^2 &= 35^2 \\ 700 + y^2 &= 1225 \\ y &= \sqrt{525} = 5\sqrt{21}\end{aligned}$$

Use the method you feel most comfortable with.

Know What? Revisited To find the hypotenuse of the smallest triangle, do the Pythagorean Theorem.



$$45^2 + 28^2 = x^2$$

$$2809 = x^2$$

$$53 = x$$

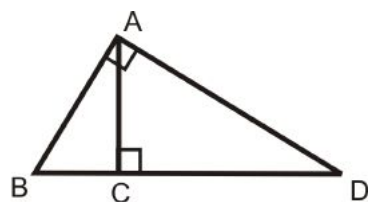
Because the triangles are similar, find the scale factor of $\frac{70}{28} = 2.5$.

$$y = 45 \cdot 2.5 = 112.5 \text{ and } z = 53 \cdot 2.5 = 135.5$$

Review Questions

- Questions 1-4 use the ratios of similar right triangles.
- Questions 5-8 are similar to Example 1.
- Questions 9-11 are similar to Examples 2-4
- Questions 12-17 are similar to Examples 5 and 6.
- Questions 18-29 are similar to Examples 2, 3, 4, 7, and 8.
- Question 30 is a proof of theorem 8-5.

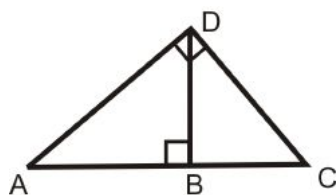
Fill in the blanks.



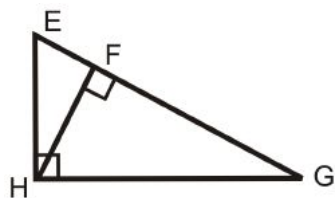
1. $\triangle BAD \sim \triangle \underline{\hspace{1cm}} \sim \triangle \underline{\hspace{1cm}}$
2. $\frac{BC}{?} = \frac{?}{CD}$
3. $\frac{BC}{AB} = \frac{AB}{?}$
4. $\frac{?}{AD} = \frac{AD}{BD}$

Write the similarity statement for the right triangles in each diagram.

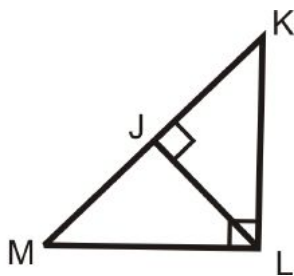
5.



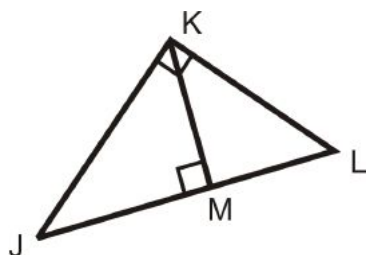
6.



7.



Use the diagram to answer questions 8-11.



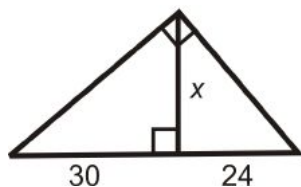
8. Write the similarity statement for the three triangles in the diagram.
9. If $JM = 12$ and $ML = 9$, find KM .
10. Find JK .
11. Find KL .

Find the geometric mean between the following two numbers. Simplify all radicals.

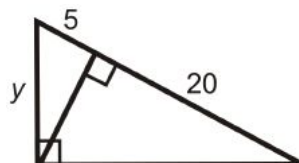
12. 16 and 32
13. 45 and 35
14. 10 and 14
15. 28 and 42
16. 40 and 100
17. 51 and 8

Find the length of the missing variable(s). Simplify all radicals.

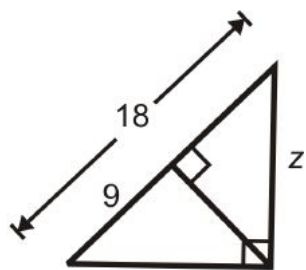
18.



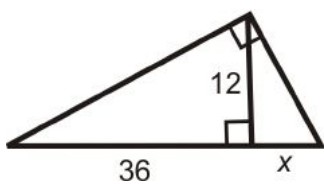
19.



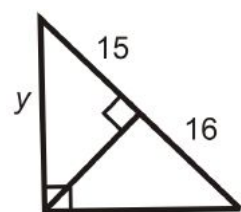
20.



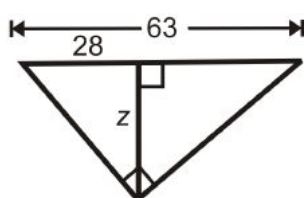
21.



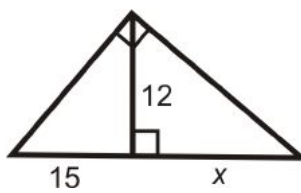
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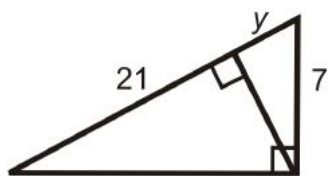
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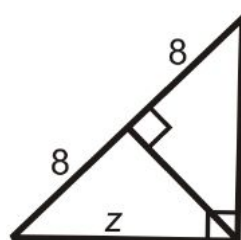
24.



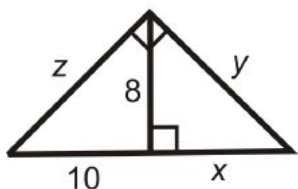
25.



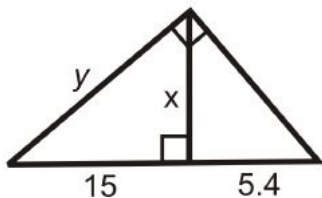
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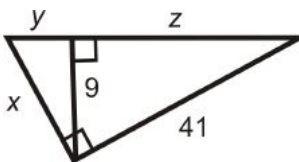
27.



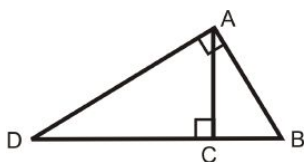
28.



29.



30. Fill in the blanks of the proof for Theorem 8-5.



Given: $\triangle ABD$ with $\overline{AC} \perp \overline{DB}$ and $\angle DAB$ is a right angle.

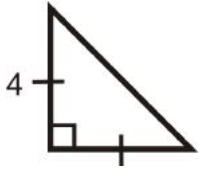
Prove: $\triangle ABD \sim \triangle CBA \sim \triangle CAD$

Table 4.1:

<i>Statement</i>	<i>Reason</i>
1.	Given
2. $\angle DCA$ and $\angle ACB$ are right angles	
3. $\angle DAB \cong \angle DCA \cong \angle ACB$	
4.	Reflexive PoC
5.	AA Similarity Postulate
6. $B \cong \angle B$	
7. $\triangle CBA \cong \triangle ABD$	
8. $\triangle CAD \cong \triangle CBA$	

Review Queue Answers

- $\frac{3}{x} = \frac{x}{27} \rightarrow x^2 = 81 \rightarrow x = 9$
 - $\frac{\sqrt{6}}{x} = \frac{x}{9\sqrt{6}} \rightarrow x^2 = 54 \rightarrow x = \sqrt{54} = \sqrt{9 \cdot 6} = 3\sqrt{6}$
 - $\frac{x}{15} = \frac{12}{x} \rightarrow x^2 = 180 \rightarrow x = \sqrt{180} = \sqrt{4 \cdot 9 \cdot 5} = 2 \cdot 3\sqrt{5} = 6\sqrt{5}$
- $4^2 + 4^2 = h^2$
 $h = \sqrt{32} = 4\sqrt{2}$



4.4 Special Right Triangles

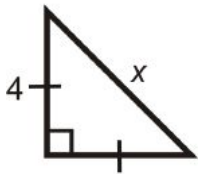
Learning Objectives

- Learn and use the 45-45-90 triangle ratio.
- Learn and use the 30-60-90 triangle ratio.

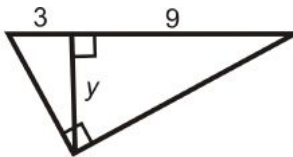
Review Queue

Find the value of the missing variables. Simplify all radicals.

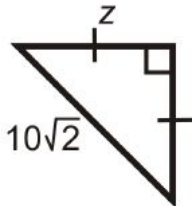
1.



2.

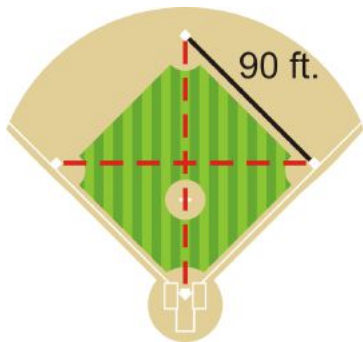


3.



- Is 9, 12, and 15 a right triangle?
- Is 3 , $3\sqrt{3}$, and 6 a right triangle?

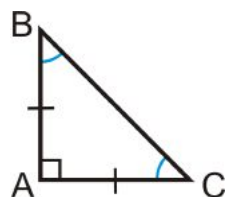
Know What? A baseball diamond is a square with sides that are 90 feet long. Each base is a corner of the square. What is the length between 1st and 3rd base and between 2nd base and home plate? (the red dotted lines in the diagram).



Isosceles Right Triangles

There are two special right triangles. The first is an isosceles right triangle.

Isosceles Right Triangle: A right triangle with congruent legs and acute angles. This triangle is also called a 45-45-90 triangle (after the angle measures).



$\triangle ABC$ is a right triangle with:

$$m\angle A = 90^\circ$$

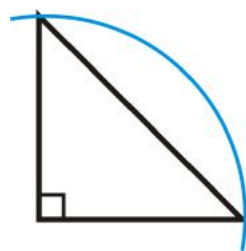
$$\overline{AB} \cong \overline{AC}$$

$$m\angle B = m\angle C = 45^\circ$$

Investigation 8-2: Properties of an Isosceles Right Triangle

Tools Needed: Pencil, paper, compass, ruler, protractor

1. Draw an isosceles right triangle with 2 inch legs and the 90° angle between them.



2. Find the measure of the hypotenuse, using the Pythagorean Theorem. Simplify the radical.

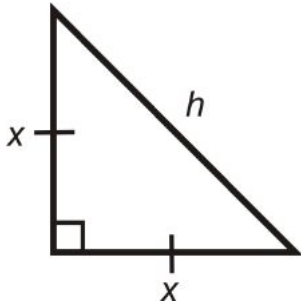
$$2^2 + 2^2 = c^2$$

$$8 = c^2$$

$$c = \sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$$

What do you notice about the length of the legs and hypotenuse?

3. Now, let's say the legs are of length x and the hypotenuse is h . Use the Pythagorean Theorem to find the hypotenuse. How is it similar to your answer in #2?



$$x^2 + x^2 = h^2$$

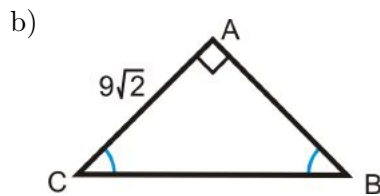
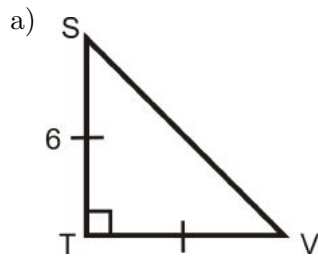
$$2x^2 = h^2$$

$$x\sqrt{2} = h$$

45-45-90 Theorem: If a right triangle is isosceles, then its sides are $x : x : x\sqrt{2}$.

For any isosceles right triangle, the legs are x and *the hypotenuse is always* $x\sqrt{2}$. Because the three angles are always $45^\circ, 45^\circ$, and 90° , *all isosceles right triangles are similar*.

Example 1: Find the length of the missing sides.

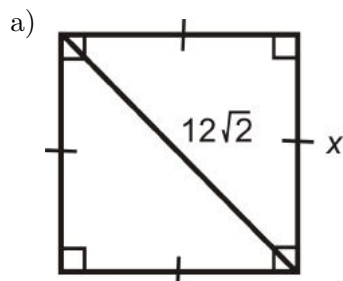


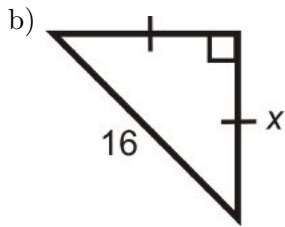
Solution: Use the $x : x : x\sqrt{2}$ ratio.

a) $TV = 6$ because it is equal to ST . So, $SV = 6 \cdot \sqrt{2} = 6\sqrt{2}$.

b) $AB = 9\sqrt{2}$ because it is equal to AC . So, $BC = 9\sqrt{2} \cdot \sqrt{2} = 9 \cdot 2 = 18$.

Example 2: Find the length of x .





Solution: Use the $x : x : x\sqrt{2}$ ratio.

a) $12\sqrt{2}$ is the diagonal of the square. Remember that the diagonal of a square bisects each angle, so it splits the square into two 45-45-90 triangles. $12\sqrt{2}$ would be the hypotenuse, or equal to $x\sqrt{2}$.

$$12\sqrt{2} = x\sqrt{2}$$

$$12 = x$$

b) Here, we are given the hypotenuse. Solve for x in the ratio.

$$x\sqrt{2} = 16$$

$$x = \frac{16}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{16\sqrt{2}}{2} = 8\sqrt{2}$$

In part b, we *rationalized the denominator* which we learned in the first section.

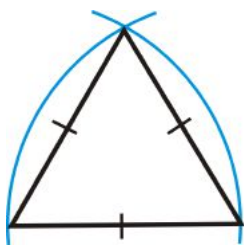
30-60-90 Triangles

The second special right triangle is called a 30-60-90 triangle, after the three angles. To draw a 30-60-90 triangle, start with an equilateral triangle.

Investigation 8-3: Properties of a 30-60-90 Triangle

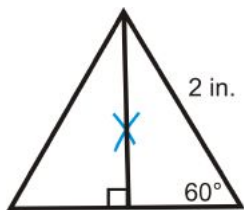
Tools Needed: Pencil, paper, ruler, compass

1. Construct an equilateral triangle with 2 inch sides.



<http://www.mathsisfun.com/geometry/construct-equitriangle.html>

2. Draw or construct the altitude from the top vertex to form two congruent triangles.



3. Find the measure of the two angles at the top vertex and the length of the shorter leg.

The top angles are each 30° and the shorter leg is 1 in because the altitude of an equilateral triangle is also the angle and perpendicular bisector.

4. Find the length of the longer leg, using the Pythagorean Theorem. Simplify the radical.

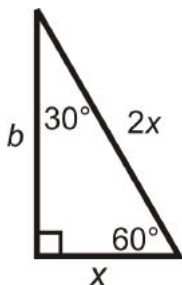
$$1^2 + b^2 = 2^2$$

$$1 + b^2 = 4$$

$$b^2 = 3$$

$$b = \sqrt{3}$$

5. Now, let's say the shorter leg is length x and the hypotenuse is $2x$. Use the Pythagorean Theorem to find the longer leg. How is this similar to your answer in #4?



$$x^2 + b^2 = (2x)^2$$

$$x^2 + b^2 = 4x^2$$

$$b^2 = 3x^2$$

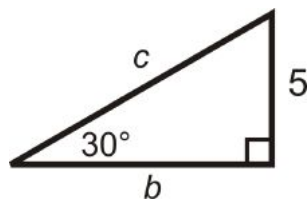
$$b = x\sqrt{3}$$

30-60-90 Theorem: If a triangle has angle measures 30° , 60° and 90° , then the sides are $x : x\sqrt{3} : 2x$.

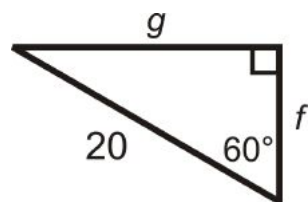
The shortest leg is always x , the longest leg is always $x\sqrt{3}$, and the hypotenuse is always $2x$. If you ever forget these theorems, you can still use the Pythagorean Theorem.

Example 3: Find the length of the missing sides.

a)



b)



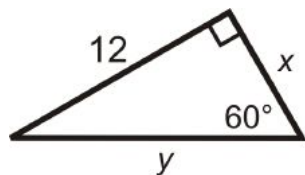
Solution: In part a, we are given the shortest leg and in part b, we are given the hypotenuse.

a) If $x = 5$, then the longer leg, $b = 5\sqrt{3}$, and the hypotenuse, $c = 2(5) = 10$.

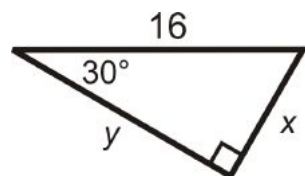
b) Now, $2x = 20$, so the shorter leg, $f = \frac{20}{2} = 10$, and the longer leg, $g = 10\sqrt{3}$.

Example 4: Find the value of x and y .

a)



b)



Solution: In part a, we are given the longer leg and in part b, we are given the hypotenuse.

$$a) x \sqrt{3} = 12$$

$$x = \frac{12}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$$

The hypotenuse is

$$y = 2(4\sqrt{3}) = 8\sqrt{3}$$

$$b) 2x = 16$$

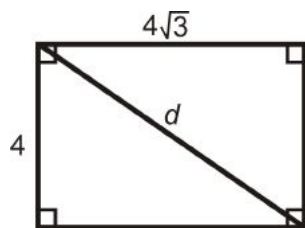
$$x = 8$$

The longer leg is

$$y = 8 \cdot \sqrt{3} = 8\sqrt{3}$$

Example 5: A rectangle has sides 4 and $4\sqrt{3}$. What is the length of the diagonal?

Solution: If you are not given a picture, draw one.



The two lengths are $x, x\sqrt{3}$, so the diagonal would be $2x$, or $2(4) = 8$.

If you did not recognize this is a 30-60-90 triangle, you can use the Pythagorean Theorem too.

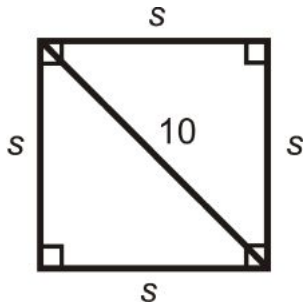
$$4^2 + (4\sqrt{3})^2 = d^2$$

$$16 + 48 = d^2$$

$$d = \sqrt{64} = 8$$

Example 6: A square has a diagonal with length 10, what are the sides?

Solution: Draw a picture.

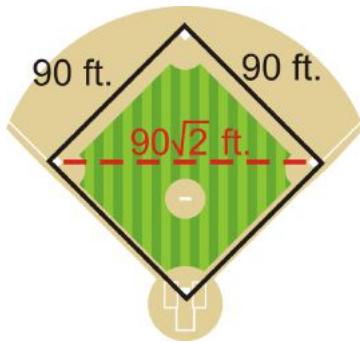


We know half of a square is a 45-45-90 triangle, so $10 = s\sqrt{2}$.

$$s\sqrt{2} = 10$$

$$s = \frac{10}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$$

Know What? Revisited The distance between 1st and 3rd base is one of the diagonals of the square. So, it would be the same as the hypotenuse of a 45-45-90 triangle. Using our ratios, the distance is $90\sqrt{2} \approx 127.3$ ft. The distance between 2nd base and home plate is the same length.

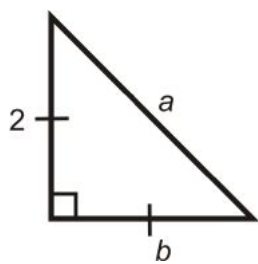


Review Questions

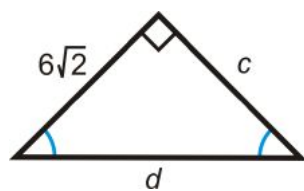
- Questions 1-4 are similar to Example 1-4.
 - Questions 5-8 are similar to Examples 5 and 6.
 - Questions 9-23 are similar to Examples 1-4.
 - Questions 24 and 25 are a challenge.
1. In an isosceles right triangle, if a leg is 4, then the hypotenuse is _____.
 2. In a 30-60-90 triangle, if the shorter leg is 5, then the longer leg is _____ and the hypotenuse is _____.
 3. In an isosceles right triangle, if a leg is x , then the hypotenuse is _____.
 4. In a 30-60-90 triangle, if the shorter leg is x , then the longer leg is _____ and the hypotenuse is _____.
 5. A square has sides of length 15. What is the length of the diagonal?
 6. A square's diagonal is 22. What is the length of each side?
 7. A rectangle has sides of length 6 and $6\sqrt{3}$. What is the length of the diagonal?
 8. Two (opposite) sides of a rectangle are 10 and the diagonal is 20. What is the length of the other two sides?

For questions 9-23, find the lengths of the missing sides. Simplify all radicals.

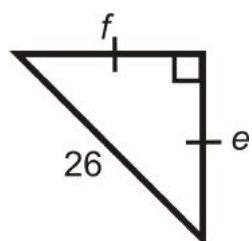
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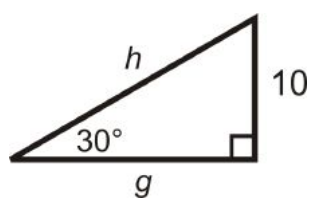
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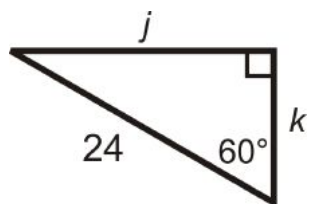
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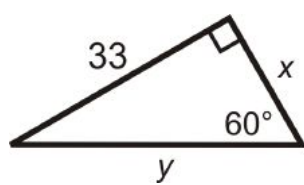
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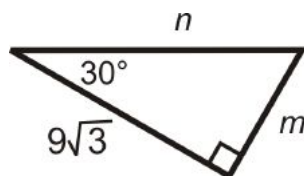
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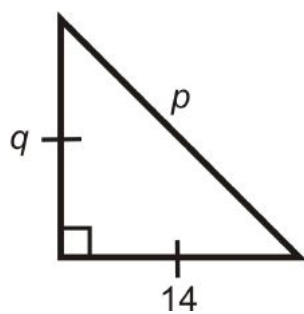
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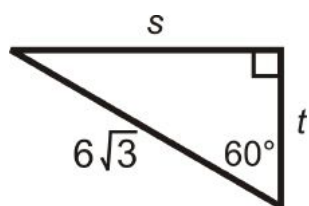
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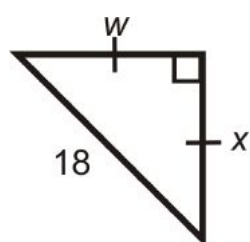
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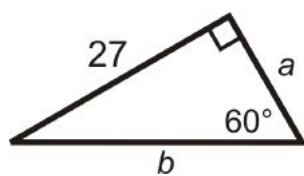
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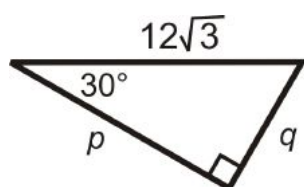
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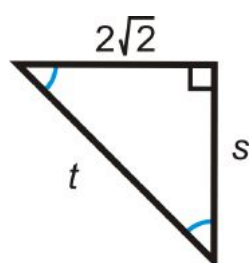
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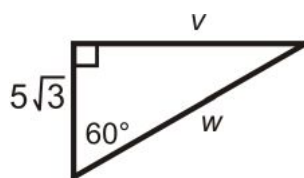
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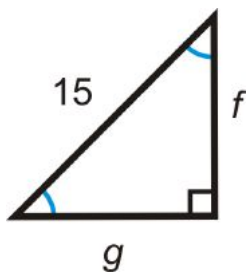
21.



22.

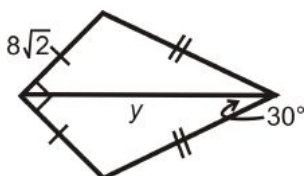


23.

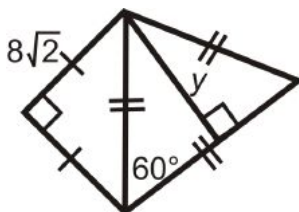


Challenge For 24 and 25, find the value of y . You may need to draw in additional lines. Round all answers to the nearest hundredth.

24.



25.



Review Queue Answers

1. $4^2 + 4^2 = x^2$
 $32 = x^2$
 $x = 4\sqrt{2}$
2. $3^2 + y^2 = 6^2$
 $y^2 = 27$
 $y = 3\sqrt{3}$
3. $x^2 + x^2 = (10\sqrt{2})^2$
 $2x^2 = 200$
 $x^2 = 100$
 $x = 10$
4. Yes, $9^2 + 12^2 = 15^2 \rightarrow 81 + 144 = 225$
5. Yes, $3^2 + (3\sqrt{3})^2 = 6^2 \rightarrow 9 + 27 = 36$

4.5 Tangent, Sine and Cosine

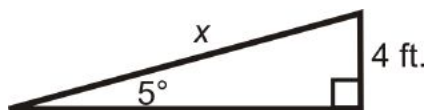
Learning Objectives

- Use the tangent, sine and cosine ratios.
- Use a scientific calculator to find sine, cosine and tangent.
- Use trigonometric ratios in real-life situations.

Review Queue

1. The legs of an isosceles right triangle have length 14. What is the hypotenuse?
2. Do the lengths 8, 16, 20 make a right triangle? If not, is the triangle obtuse or acute?
3. In a 30-60-90 triangle, what do the 30, 60, and 90 refer to?

Know What? A restaurant is building a wheelchair ramp. The angle of elevation for the ramp is 5° . If the vertical distance from the sidewalk to the front door is 4 feet, how long will the ramp be (x)? Round your answers to the nearest hundredth.

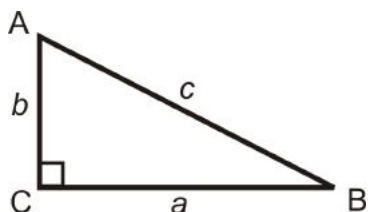


What is Trigonometry?

In this lesson we will define three trigonometric (or trig) ratios. Once we have defined these ratios, we will be able to solve problems like the **Know What?** above.

Trigonometry: The study of the relationships between the sides and angles of right triangles.

The legs are called *adjacent* or *opposite* depending on which *acute* angle is being used.



a is *adjacent* to $\angle B$ a is *opposite* $\angle A$
 b is *adjacent* to $\angle A$ b is *opposite* $\angle B$

c is the *hypotenuse*

Sine, Cosine, and Tangent Ratios

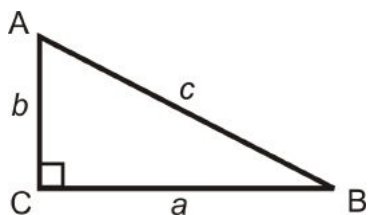
The three basic trig ratios are called, sine, cosine and tangent. For now, we will only take the sine, cosine and tangent of *acute* angles. However, you can use these ratios with obtuse angles as well.

For right triangle $\triangle ABC$, we have:

Sine Ratio: $\frac{\text{opposite leg}}{\text{hypotenuse}}$ $\sin A = \frac{a}{c}$ or $\sin B = \frac{b}{c}$

Cosine Ratio: $\frac{\text{adjacent leg}}{\text{hypotenuse}}$ $\cos A = \frac{b}{c}$ or $\cos B = \frac{a}{c}$

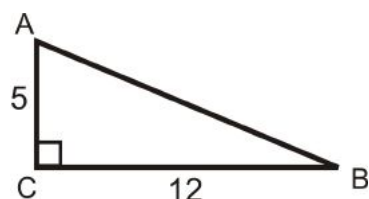
Tangent Ratio: $\frac{\text{opposite leg}}{\text{adjacent leg}}$ $\tan A = \frac{a}{b}$ or $\tan B = \frac{b}{a}$



An easy way to remember ratios is to use SOH-CAH-TOA.

$$\text{Sine} = \frac{\text{Opposite}}{\text{Hypotenuse}} \quad \text{Cosine} = \frac{\text{Adjacent}}{\text{Hypotenuse}} \quad \text{Tangent} = \frac{\text{Opposite}}{\text{Adjacent}}$$

Example 1: Find the sine, cosine and tangent ratios of $\angle A$.



Solution: First, we need to use the Pythagorean Theorem to find the length of the hypotenuse.

$$5^2 + 12^2 = h^2$$

$$13 = h$$

$$\sin A = \frac{\text{leg opposite } \angle A}{\text{hypotenuse}} = \frac{12}{13}$$

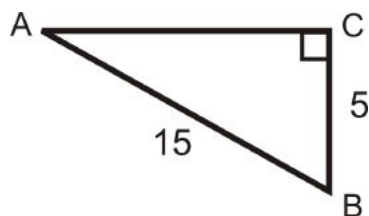
$$\cos A = \frac{\text{leg adjacent to } \angle A}{\text{hypotenuse}} = \frac{5}{13},$$

$$\tan A = \frac{\text{leg opposite } \angle A}{\text{leg adjacent to } \angle A} = \frac{12}{5}$$

A few important points:

- Always **reduce ratios** (fractions) when you can.
- Use the Pythagorean Theorem to find the missing side (if there is one).
- If there is a radical in the denominator, **rationalize the denominator**.

Example 2: Find the sine, cosine, and tangent of $\angle B$.



Solution: Find the length of the missing side.

$$AC^2 + 5^2 = 15^2$$

$$AC^2 = 200$$

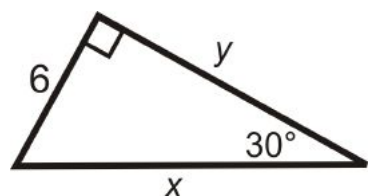
$$AC = 10\sqrt{2}$$

$$\sin B = \frac{10\sqrt{2}}{15} = \frac{2\sqrt{2}}{3}$$

$$\cos B = \frac{5}{15} = \frac{1}{3}$$

$$\tan B = \frac{10\sqrt{2}}{5} = 2\sqrt{2}$$

Example 3: Find the sine, cosine and tangent of 30° .



Solution: This is a 30-60-90 triangle. The short leg is 6, $y = 6\sqrt{3}$ and $x = 12$.

$$\sin 30^\circ = \frac{6}{12} = \frac{1}{2} \qquad \cos 30^\circ = \frac{6\sqrt{3}}{12} = \frac{\sqrt{3}}{2} \qquad \tan 30^\circ = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Sine, Cosine, and Tangent with a Calculator

From Example 3, we can conclude that *there is a fixed sine, cosine, and tangent value for every angle, from 0° to 90°* . Your scientific (or graphing) calculator knows all the trigonometric values for any angle. Your calculator, should have [SIN], [COS], and [TAN] buttons.

Example 4: Find the trigonometric value, using your calculator. Round to 4 decimal places.

- a) $\sin 78^\circ$
- b) $\cos 60^\circ$
- c) $\tan 15^\circ$

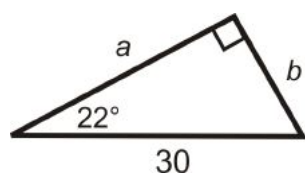
Solution: Depending on your calculator, you enter the degree and then press the trig button or the other way around. Also, make sure the mode of your calculator is in *DEGREES*.

- a) $\sin 78^\circ = 0.9781$
- b) $\cos 60^\circ = 0.5$
- c) $\tan 15^\circ = 0.2679$

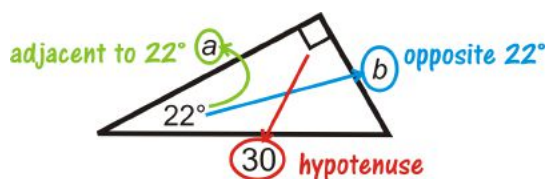
Finding the Sides of a Triangle using Trig Ratios

One application of the trigonometric ratios is to use them to find the missing sides of a right triangle.

Example 5: Find the value of each variable. Round your answer to the nearest tenth.



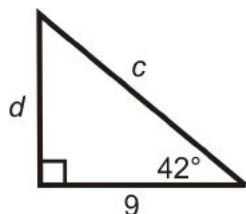
Solution: We are given the hypotenuse. Use *sine* to find b , and *cosine* to find a .



$$\begin{aligned} \sin 22^\circ &= \frac{b}{30} \\ 30 \cdot \sin 22^\circ &= b \\ b &\approx 11.2 \end{aligned}$$

$$\begin{aligned} \cos 22^\circ &= \frac{a}{30} \\ 30 \cdot \cos 22^\circ &= a \\ a &\approx 27.8 \end{aligned}$$

Example 6: Find the value of each variable. Round your answer to the nearest tenth.



Solution: We are given the adjacent leg to 42° . To find c , use *cosine* and *tangent* to find d .

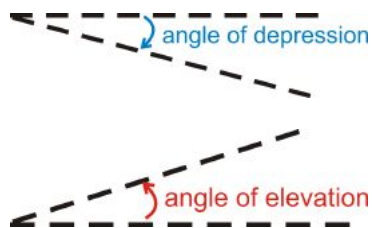
$$\begin{aligned}\cos 42^\circ &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{9}{c} & \tan 42^\circ &= \frac{\text{opposite}}{\text{adjacent}} = \frac{d}{9} \\ c \cdot \cos 42^\circ &= 9 & 9 \cdot \tan 42^\circ &= d \\ c &= \frac{9}{\cos 42^\circ} \approx 12.1 & d &\approx 8.1\end{aligned}$$

Anytime you use trigonometric ratios, only use the information that you are given. This will give the most accurate answers.

Angles of Depression and Elevation

Another application of the trigonometric ratios is to find lengths that you cannot measure. Very frequently, angles of depression and elevation are used in these types of problems.

Angle of Depression: The angle measured from the horizon or horizontal line, down.



Angle of Elevation: The angle measure from the horizon or horizontal line, up.

Example 7: A math student is standing 25 feet from the base of the Washington Monument. The angle of elevation from her horizontal line of sight is 87.4° . If her “eye height” is 5 ft, how tall is the monument?



Solution: We can find the height of the monument by using the tangent ratio.

$$\tan 87.4^\circ = \frac{h}{25}$$

$$h = 25 \cdot \tan 87.4^\circ = 550.54$$

Adding 5 ft, the total height of the Washington Monument is 555.54 ft.

Know What? Revisited To find the length of the ramp, we need to use sine.

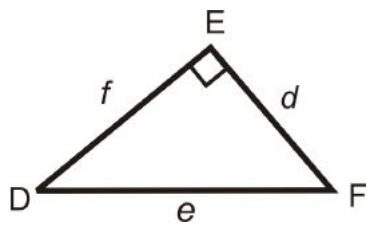
$$\sin 5^\circ = \frac{4}{x}$$

$$y = \frac{2}{\sin 5^\circ} = 22.95$$

Review Questions

- Questions 1-8 use the definitions of sine, cosine and tangent.
- Questions 9-16 are similar to Example 4.
- Questions 17-22 are similar to Examples 1-3.
- Questions 23-28 are similar to Examples 5 and 6.
- Questions 29 and 30 are similar to Example 7.

Use the diagram to fill in the blanks below.



1. $\tan D = \frac{?}{?}$
2. $\sin F = \frac{?}{?}$
3. $\tan F = \frac{?}{?}$
4. $\cos F = \frac{?}{?}$
5. $\sin D = \frac{?}{?}$
6. $\cos D = \frac{?}{?}$

From questions 1-6, we can conclude the following. Fill in the blanks.

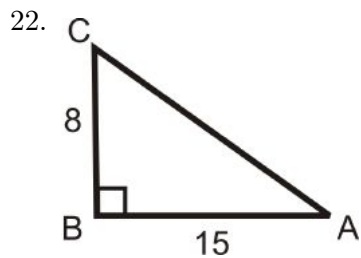
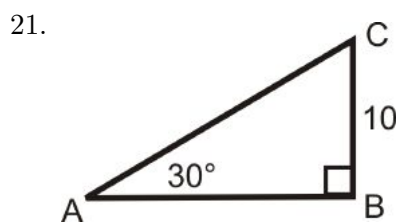
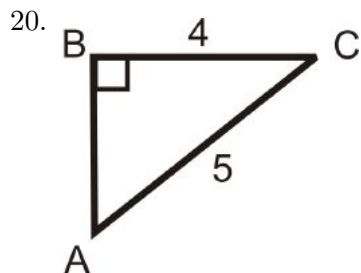
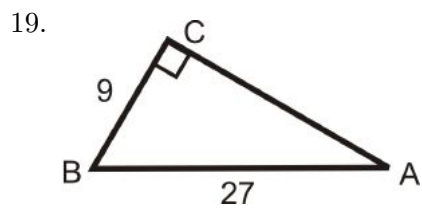
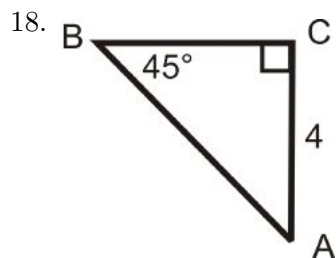
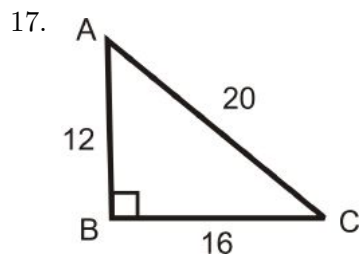
7. $\cos \underline{\hspace{1cm}} = \sin F$ and $\sin \underline{\hspace{1cm}} = \cos F$.
8. $\tan D$ and $\tan F$ are of each other.

Use your calculator to find the value of each trig function below. Round to four decimal places.

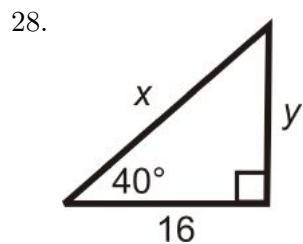
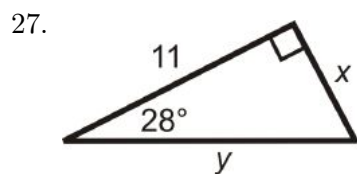
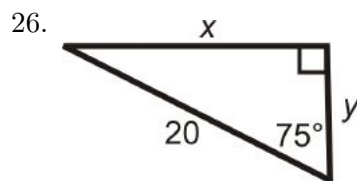
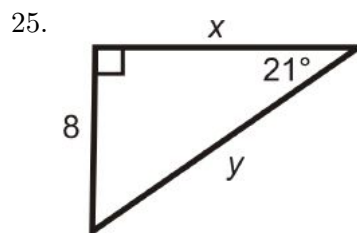
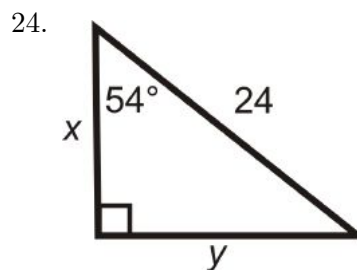
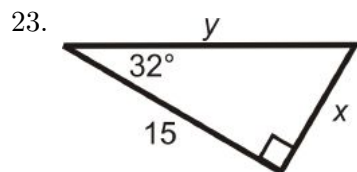
9. $\sin 24^\circ$
10. $\cos 45^\circ$
11. $\tan 88^\circ$
12. $\sin 43^\circ$

13. $\tan 12^\circ$
14. $\cos 79^\circ$
15. $\sin 82^\circ$
16. $\tan 45^\circ$

Find the sine, cosine and tangent of $\angle A$. Reduce all fractions and radicals.



Find the length of the missing sides. Round your answers to the nearest tenth.



29. Kristin is swimming in the ocean and notices a coral reef below her. The angle of depression is 35° and the depth of the ocean, at that point is 250 feet. How far away is she from the reef?



30. The Leaning Tower of Piza currently “leans” at a 4° angle and has a vertical height of 55.86 meters. How tall was the tower when it was originally built?



Review Queue Answers

1. The hypotenuse is $14\sqrt{2}$.
2. No, $8^2 + 16^2 < 20^2$, the triangle is obtuse.
3. 30° , 60° , and 90° refer to the angle measures in the special right triangle.

4.6 Inverse Trigonometric Ratios

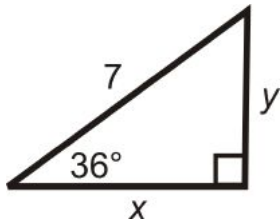
Learning Objectives

- Use the inverse trigonometric ratios to find an angle in a right triangle.
- Solve a right triangle.

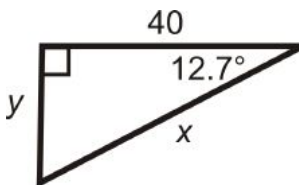
Review Queue

Find the lengths of the missing sides. Round your answer to the nearest tenth.

1.



2.



3. Draw an isosceles right triangle with legs of length 3. What is the hypotenuse?
4. Use the triangle from #3, to find the sine, cosine, and tangent of 45° .

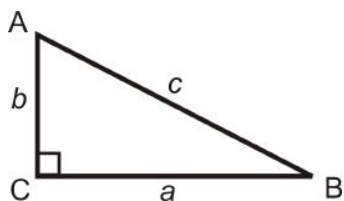
Know What? The longest escalator in North America is at the Wheaton Metro Station in Maryland. It is 230 feet long and is 115 ft. high. What is the angle of elevation, x° , of this escalator?



Inverse Trigonometric Ratios

In mathematics, the word inverse means “undo.” For example, addition and subtraction are inverses of each other because one undoes the other. When we apply inverses to the trigonometric ratios, we can find acute angle measures as long as we are given two sides.

Inverse Tangent: Labeled \tan^{-1} , the “-1” means inverse.



$$\tan^{-1}\left(\frac{b}{a}\right) = m\angle B$$

$$\tan^{-1}\left(\frac{a}{b}\right) = m\angle A$$

Inverse Sine: Labeled \sin^{-1} .

$$\sin^{-1}\left(\frac{b}{c}\right) = m\angle B$$

$$\sin^{-1}\left(\frac{a}{c}\right) = m\angle A$$

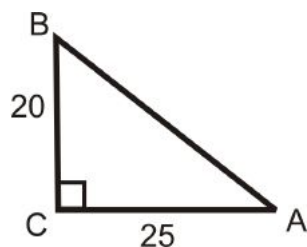
Inverse Cosine: Labeled \cos^{-1} .

$$\cos^{-1}\left(\frac{a}{c}\right) = m\angle B$$

$$\cos^{-1}\left(\frac{b}{c}\right) = m\angle A$$

In order to find the measure of the angles, you will need you use your calculator. On most scientific and graphing calculators, the buttons look like $[\text{SIN}^{-1}]$, $[\text{COS}^{-1}]$, and $[\text{TAN}^{-1}]$. You might also have to hit a shift or 2^{nd} button to access these functions.

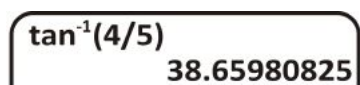
Example 1: Use the sides of the triangle and your calculator to find the value of $\angle A$. Round your answer to the nearest tenth of a degree.



Solution: In reference to $\angle A$, we are given the *opposite* leg and the *adjacent* leg. This means we should use the *tangent* ratio.

$\tan A = \frac{20}{25} = \frac{4}{5}$. So, $\tan^{-1} \frac{4}{5} = m\angle A$. Now, use your calculator.

If you are using a TI-83 or 84, the keystrokes would be: $[2^{nd}][\text{TAN}](\frac{4}{5})$ [ENTER] and the screen looks like:



$$m\angle A = 38.7^\circ$$

Example 2: $\angle A$ is an acute angle in a right triangle. Find $m\angle A$ to the nearest tenth of a degree.

a) $\sin A = 0.68$

b) $\cos A = 0.85$

c) $\tan A = 0.34$

Solution:

a) $m\angle A = \sin^{-1} 0.68 = 42.8^\circ$

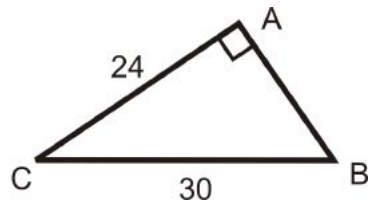
b) $m\angle A = \cos^{-1} 0.85 = 31.8^\circ$

c) $m\angle A = \tan^{-1} 0.34 = 18.8^\circ$

Solving Triangles

To solve a right triangle, you need to find all sides and angles in a right triangle, using sine, cosine or tangent, inverse sine, inverse cosine, or inverse tangent, or the Pythagorean Theorem.

Example 3: Solve the right triangle.



Solution: To solve this right triangle, we need to find AB , $m\angle C$ and $m\angle B$. Only use the values you are given.

AB : Use the Pythagorean Theorem.

$$\begin{aligned}
 24^2 + AB^2 &= 30^2 \\
 576 + AB^2 &= 900 \\
 AB^2 &= 324 \\
 AB &= \sqrt{324} = 18
 \end{aligned}$$

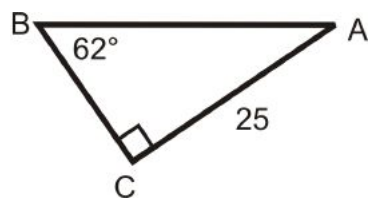
$m\angle B$: Use the inverse sine ratio.

$$\begin{aligned}
 \sin B &= \frac{24}{30} = \frac{4}{5} \\
 \sin^{-1}\left(\frac{4}{5}\right) &= 53.1^\circ = m\angle B
 \end{aligned}$$

$m\angle C$: Use the inverse cosine ratio.

$$\cos C = \frac{24}{30} = \frac{4}{5} \rightarrow \cos^{-1}\left(\frac{4}{5}\right) = 36.9^\circ = m\angle C$$

Example 4: Solve the right triangle.



Solution: To solve this right triangle, we need to find AB , BC and $m\angle A$.

AB : Use sine ratio.

$$\begin{aligned}
 \sin 62^\circ &= \frac{25}{AB} \\
 AB &= \frac{25}{\sin 62^\circ} \\
 AB &\approx 28.31
 \end{aligned}$$

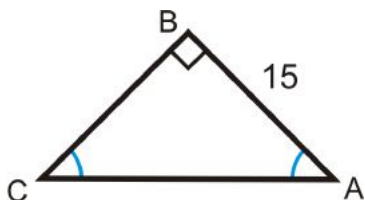
BC : Use tangent ratio.

$$\begin{aligned}
 \tan 62^\circ &= \frac{25}{BC} \\
 BC &= \frac{25}{\tan 62^\circ} \\
 BC &\approx 13.30
 \end{aligned}$$

$m\angle A$: Use Triangle Sum Theorem

$$\begin{aligned}
 62^\circ + 90^\circ + m\angle A &= 180^\circ \\
 m\angle A &= 28^\circ
 \end{aligned}$$

Example 5: Solve the right triangle.



Solution: The two acute angles are congruent, making them both 45° . This is a 45-45-90 triangle. You can use the trigonometric ratios or the special right triangle ratios.

Trigonometric Ratios

$$\begin{aligned}\tan 45^\circ &= \frac{15}{BC} \\ BC &= \frac{15}{\tan 45^\circ} = 15\end{aligned}$$

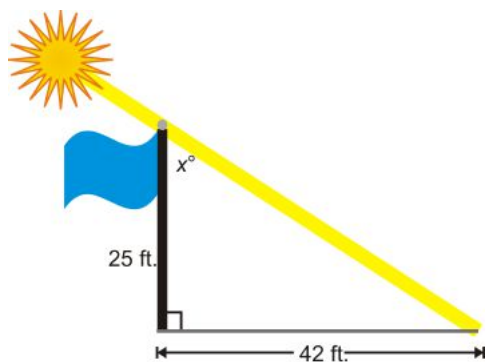
$$\begin{aligned}\sin 45^\circ &= \frac{15}{AC} \\ AC &= \frac{15}{\sin 45^\circ} \approx 21.21\end{aligned}$$

45-45-90 Triangle Ratios

$$BC = AB = 15, AC = 15\sqrt{2} \approx 21.21$$

Real-Life Situations

Example 6: A 25 foot tall flagpole casts a 42 feet shadow. What is the angle that the sun hits the flagpole?



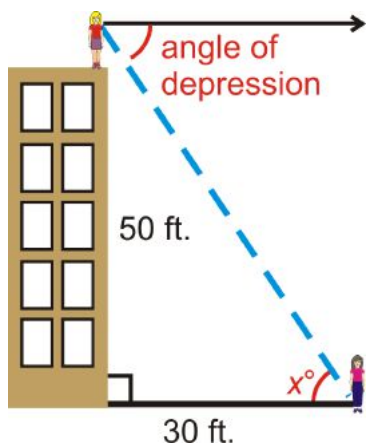
Solution: Draw a picture. The angle that the sun hits the flagpole is x° . We need to use the inverse tangent ratio.

$$\begin{aligned}\tan x &= \frac{42}{25} \\ \tan^{-1} \frac{42}{25} &\approx 59.2^\circ = x\end{aligned}$$

Example 7: Elise is standing on top of a 50 foot building and sees her friend, Molly. If Molly is 35 feet away from the base of the building, what is the angle of depression from Elise to Molly? Elise's eye height is 4.5 feet.

Solution: Because of parallel lines, the angle of depression is equal to the angle at Molly, or x° . We can use the inverse tangent ratio.

$$\tan^{-1} \left(\frac{54.5}{30} \right) = 61.2^\circ = x$$



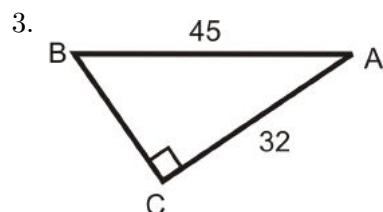
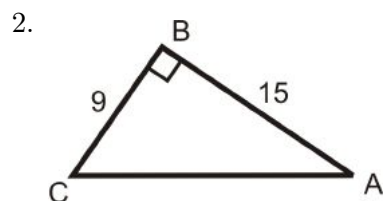
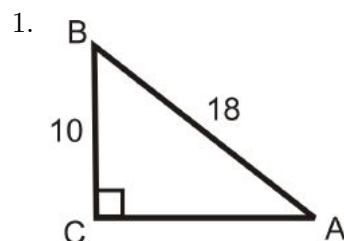
Know What? Revisited To find the escalator's angle of elevation, use the inverse sine.

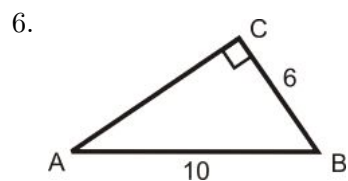
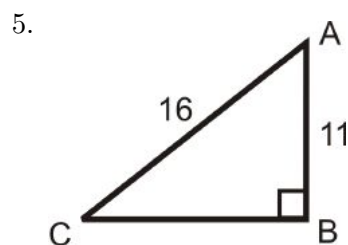
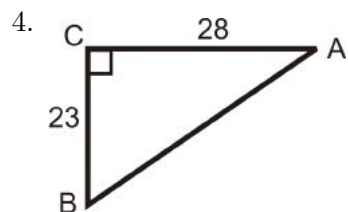
$$\sin^{-1}\left(\frac{115}{230}\right) = 30^\circ \quad \text{The angle of elevation is } 30^\circ.$$

Review Questions

- Questions 1-6 are similar to Example 1.
- Questions 7-12 are similar to Example 2.
- Questions 13-21 are similar to Examples 3 and 4.
- Questions 22-24 are similar to Examples 6 and 7.
- Questions 25-30 are a review of the trigonometric ratios.

Use your calculator to find $m\angle A$ to the nearest tenth of a degree.

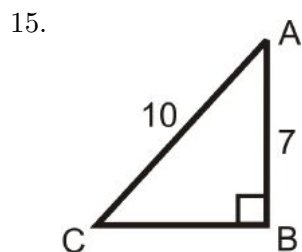
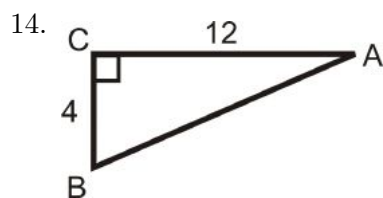
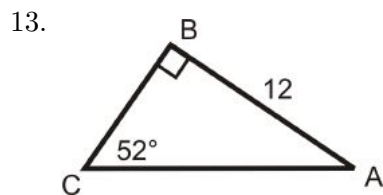


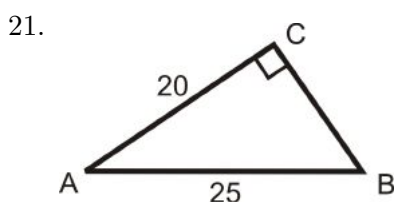
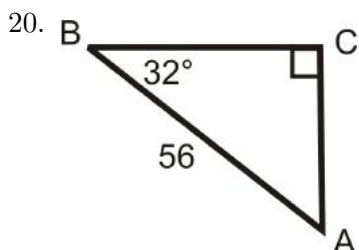
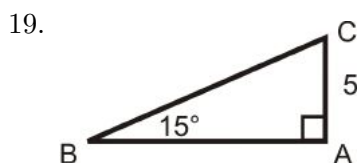
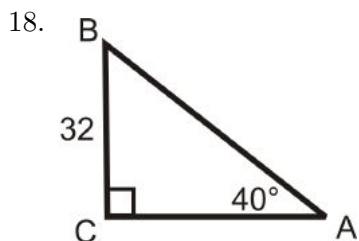
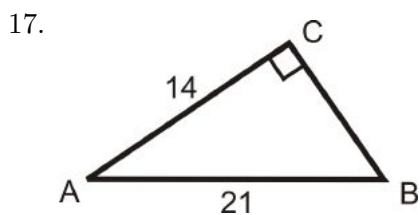
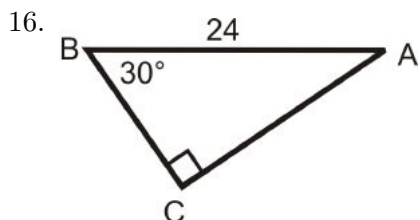


Let $\angle A$ be an acute angle in a right triangle. Find $m\angle A$ to the nearest tenth of a degree.

7. $\sin A = 0.5684$
8. $\cos A = 0.1234$
9. $\tan A = 2.78$
10. $\cos^{-1} 0.9845$
11. $\tan^{-1} 15.93$
12. $\sin^{-1} 0.7851$

Solving the following right triangles. Find all missing sides and angles. Round any decimal answers to the nearest tenth.





Real-Life Situations Use what you know about right triangles to solve for the missing angle. If needed, draw a picture. Round all answers to the nearest tenth of a degree.

22. A 75 foot building casts an 82 foot shadow. What is the angle that the sun hits the building?
23. Over 2 miles (horizontal), a road rises 300 feet (vertical). What is the angle of elevation?
24. A boat is sailing and spots a shipwreck 650 feet below the water. A diver jumps from the boat and swims 935 feet to reach the wreck. What is the angle of depression from the boat to the shipwreck?

Examining Patterns Below is a table that shows the sine, cosine, and tangent values for eight different angle measures. Answer the following questions.

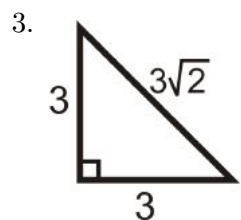
Table 4.2:

	10°	20°	30°	40°	50°	60°	70°	80°
Sine	0.1736	0.3420	0.5	0.6428	0.7660	0.8660	0.9397	0.9848
Cosine	0.9848	0.9397	0.8660	0.7660	0.6428	0.5	0.3420	0.1736
Tangent	0.1763	0.3640	0.5774	0.8391	1.1918	1.7321	2.7475	5.6713

25. What value is equal to $\sin 40^\circ$?
26. What value is equal to $\cos 70^\circ$?
27. Describe what happens to the sine values as the angle measures increase.
28. Describe what happens to the cosine values as the angle measures increase.
29. What two numbers are the sine and cosine values between?
30. Find $\tan 85^\circ$, $\tan 89^\circ$, and $\tan 89.5^\circ$ using your calculator. Now, describe what happens to the tangent values as the angle measures increase.

Review Queue Answers

1. $\sin 36^\circ = \frac{y}{7}$ $\cos 36^\circ = \frac{x}{7}$
 $y = 4.11$ $x = 5.66$
2. $\cos 12.7^\circ = \frac{40}{x}$ $\tan 12.7^\circ = \frac{y}{40}$
 $x = 41.00$ $y = 9.01$



4. $\sin 45^\circ = \frac{3}{3\sqrt{2}} = \frac{\sqrt{2}}{2}$
 $\cos 45^\circ = \frac{3}{3\sqrt{2}} = \frac{\sqrt{2}}{2}$
 $\tan 45^\circ = \frac{3}{3} = 1$

4.7 Chapter 8 Review

Keywords & Theorems

The Pythagorean Theorem

- Pythagorean Theorem
- Pythagorean Triple
- Distance Formula

The Pythagorean Theorem Converse

- Pythagorean Theorem Converse
- Theorem 8-3

- Theorem 8-4

Similar Right Triangles

- Theorem 8-5
- Geometric Mean

Special Right Triangles

- Isosceles Right (45-45-90) Triangle
- 30-60-90 Triangle
- 45-45-90 Theorem
- 30-60-90 Theorem

Tangent, Sine and Cosine Ratios

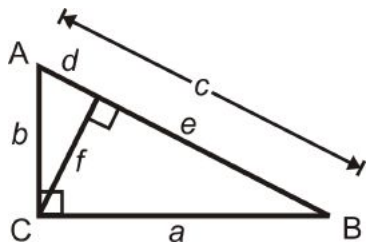
- Trigonometry
- Adjacent (Leg)
- Opposite (Leg)
- Sine Ratio
- Cosine Ratio
- Tangent Ratio
- Angle of Depression
- Angle of Elevation

Solving Right Triangles

- Inverse Tangent
- Inverse Sine
- Inverse Cosine

Review

Fill in the blanks using right triangle $\triangle ABC$.

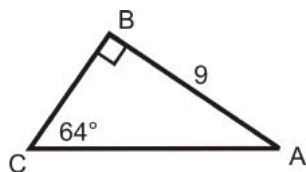


1. $a^2 + \underline{\hspace{1cm}}^2 = c^2$
2. $\sin \underline{\hspace{1cm}} = \frac{b}{c}$
3. $\tan \underline{\hspace{1cm}} = \frac{f}{d}$
4. $\cos \underline{\hspace{1cm}} = \frac{b}{c}$
5. $\tan^{-1}\left(\frac{f}{e}\right) = \underline{\hspace{1cm}}$

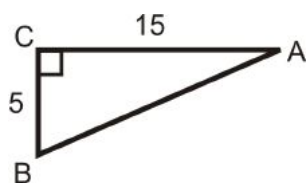
6. $\sin^{-1}\left(\frac{f}{b}\right) = \underline{\hspace{2cm}}$
7. $\frac{\hspace{1cm}}{\hspace{1cm}}^2 + d^2 = b^2$
8. $\frac{\hspace{1cm}}{b} = \frac{b}{c}$
9. $\frac{e}{\hspace{1cm}} = \frac{\hspace{1cm}}{c}$
10. $\frac{d}{f} = \frac{f}{\hspace{1cm}}$

Solve the following right triangles using the Pythagorean Theorem, the trigonometric ratios, and the inverse trigonometric ratios. When possible, simplify the radical. If not, round all decimal answers to the nearest tenth.

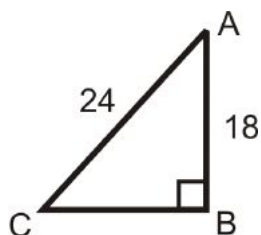
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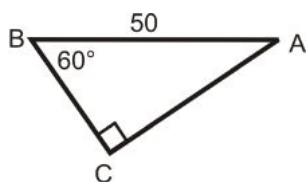
12.



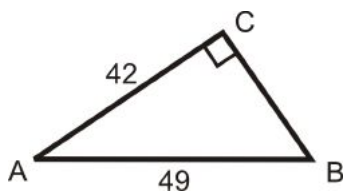
13.



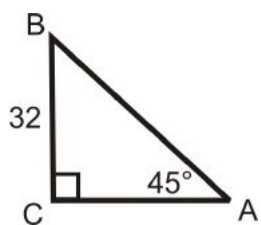
14.



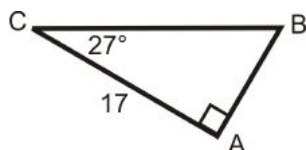
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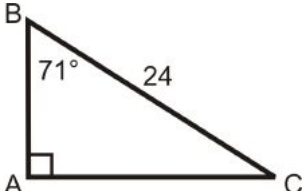
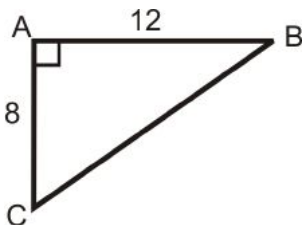


16.



17.

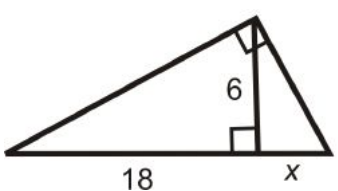
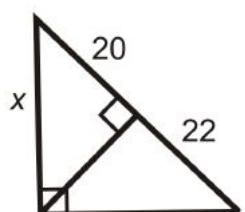
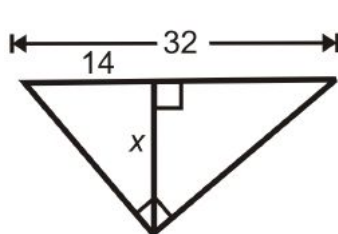


18. 
19. 

Determine if the following lengths make an acute, right, or obtuse triangle. If they make a right triangle, determine if the lengths are a Pythagorean triple.

20. 11, 12, 13
 21. 16, 30, 34
 22. 20, 25, 42
 23. $10\sqrt{6}$, 30, $10\sqrt{15}$
 24. 22, 25, 31
 25. 47, 27, 35

Find the value of x .

26. 
27. 
28. 

29. The angle of elevation from the base of a mountain to its peak is 76° . If its height is 2500 feet, what is the length to reach the top? Round the answer to the nearest tenth.
30. Taylor is taking an aerial tour of San Francisco in a helicopter. He spots AT&T Park (baseball stadium) at a horizontal distance of 850 feet and down (vertical) 475 feet. What is the angle of depression from the helicopter to the park? Round the answer to the nearest tenth.

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9693>.

4.8 Study Guide

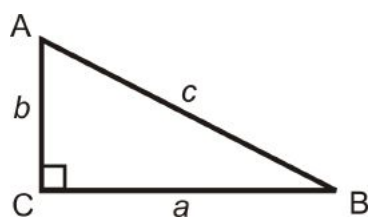
Keywords: Define, write theorems, and/or draw a diagram for each word below.

1st Section: The Pythagorean Theorem

Pythagorean Theorem

Pythagorean Triple

Distance Formula



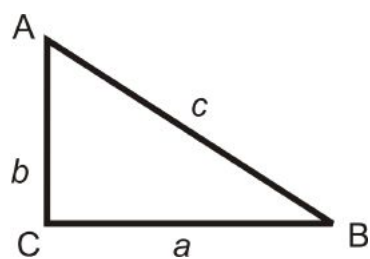
Homework:

2nd Section: The Pythagorean Theorem Converse

Pythagorean Theorem Converse

Theorem 8-3

Theorem 8-4

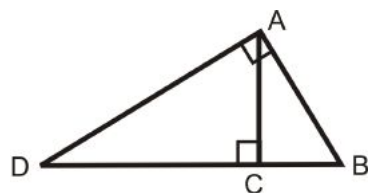


Homework:

3rd Section: Similar Right Triangles

Theorem 8-5

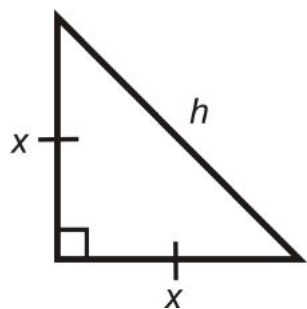
Geometric Mean



Homework:

4th Section: Special Right Triangles

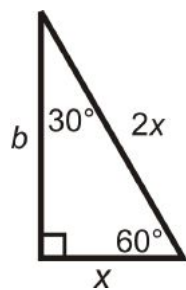
Isosceles Right (45-45-90) Triangle



30-60-90 Triangle

45-45-90 Theorem

30-60-90 Theorem



Homework:

5th Section: Tangent, Sine and Cosine Ratios

Trigonometry

Adjacent (Leg)

Opposite (Leg)

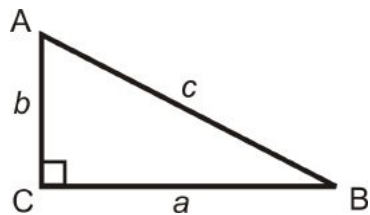
Sine Ratio

Cosine Ratio

Tangent Ratio

Angle of Depression

Angle of Elevation



Homework:

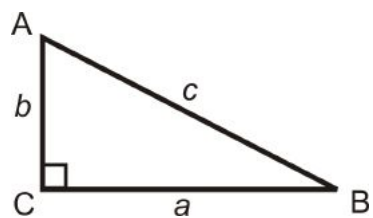
6th Section: Solving Right Triangles

Inverse Tangent

Inverse Sine

Inverse Cosine

Solving Right Triangles



Homework:

Chapter 5

Circles

First, we will define all the parts of circles and explore the properties of tangent lines, arcs, inscribed angles, and chords. Next, we will learn about angles and segments that are formed by chords, tangents and secants. Lastly, we will place circles in the coordinate plane and find the equation of and graph circles.

5.1 Parts of Circles & Tangent Lines

Learning Objectives

- Define the parts of a circle.
- Discover the properties of tangent lines.

Review Queue

1. Find the equation of the line with $m = \frac{2}{5}$ and y-intercept of 4.
2. Find the equation of the line with $m = -2$ and passes through (4, -5).
3. Find the equation of the line that passes through (6, 2) and (-3, -1).
4. Find the equation of the line perpendicular to the line in #2 and passes through (-8, 11).

Know What? The clock to the right is an ancient astronomical clock in Prague. It has a large background circle that tells the local time and the “ancient time” and the smaller circle rotates to show the current astrological sign. The yellow point is the center of the larger clock. How does the orange line relate to the small and large circle? How does the hand with the moon on it relate to both circles?

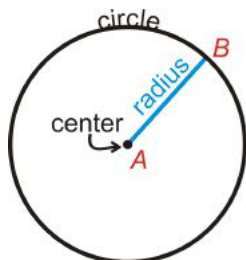


Defining Terms

Circle: The set of all points that are the same distance away from a specific point, called the *center*.

The center of the circle is point A . We call this circle, “circle A ,” and it is labeled $\odot A$.

Radii (the plural of radius) are line segments. There are infinitely many radii in any circle and they are all equal.

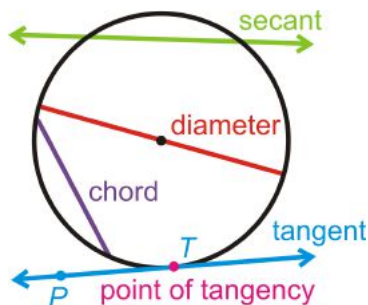


Radius: The distance from the center to the circle.

Chord: A line segment whose endpoints are on a circle.

Diameter: A chord that passes through the center of the circle.

Secant: A line that intersects a circle in two points.



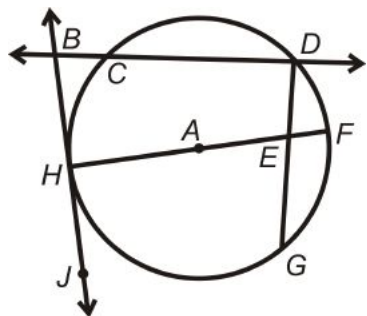
The tangent ray \overrightarrow{TP} and tangent segment \overline{TP} are also called tangents.

The length of a diameter is two times the length of a radius.

Tangent: A line that intersects a circle in exactly one point.

Point of Tangency: The point where the tangent line touches the circle.

Example 1: Find the parts of $\odot A$ that best fit each description.



- a) A radius
- b) A chord

- c) A tangent line
- d) A point of tangency
- e) A diameter
- f) A secant

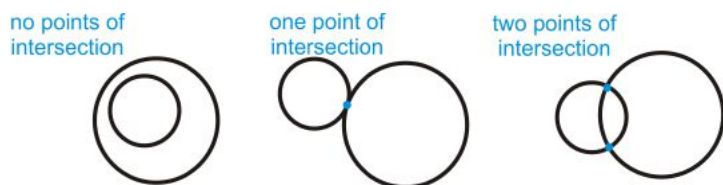
Solution:

- a) \overline{HA} or \overline{AF}
- b) \overline{CD} , \overline{HF} , or \overline{DG}
- c) \overleftrightarrow{BJ}
- d) *Point H*
- e) \overline{HF}
- f) \overleftrightarrow{BD}

Coplanar Circles

Example 2: Draw an example of how two circles can intersect with no, one and two points of intersection. You will make three separate drawings.

Solution:



Tangent Circles: When two circles intersect at one point.

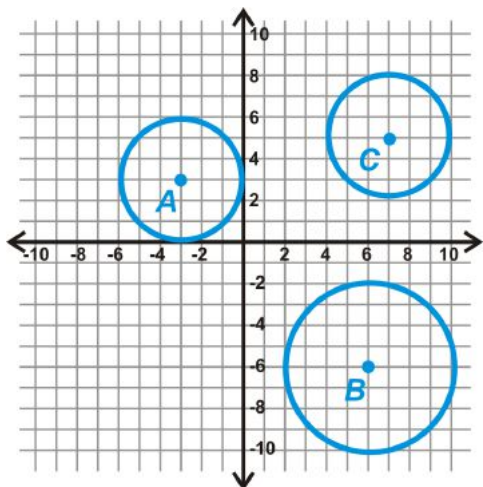


Concentric Circles: When two circles have the same center, but different radii.

Congruent Circles: Two circles with the same radius, but different centers.

If two circles have different radii, they are similar. ***All circles are similar.***

Example 3: Determine if any of the following circles are congruent.



Solution: From each center, count the units to the circle. It is easiest to count vertically or horizontally. Doing this, we have:

Radius of $\odot A = 3 \text{ units}$

Radius of $\odot B = 4 \text{ units}$

Radius of $\odot C = 3 \text{ units}$

From these measurements, we see that $\odot A \cong \odot C$.

Notice the *circles are congruent*. The *lengths of the radii* are *equal*.

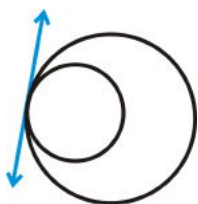
Internally & Externally Tangent

If two circles are tangent to each other, then they are *internally* or *externally* tangent.

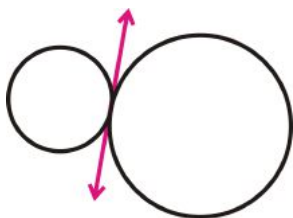
Internally Tangent Circles: When two circles are tangent and one is inside the other.

Externally Tangent Circles: When two circles are tangent and next to each other.

Internally Tangent



Externally Tangent

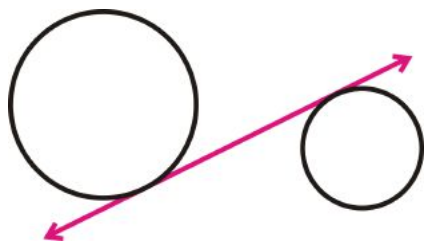


If circles are not tangent, they can still share a tangent line, called a *common* tangent.

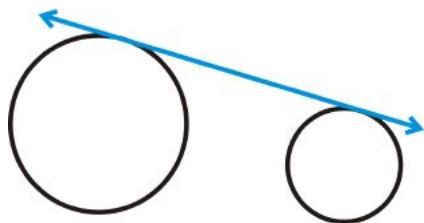
Common Internal Tangent: A line that is tangent to two circles and passes between the circles.

Common External Tangent: A line that is tangent to two circles and stays on the top or bottom of both circles.

Common Internal Tangent



Common External Tangent



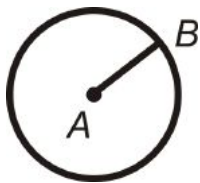
Tangents and Radii

Let's investigate a tangent line and the radius drawn to the point of tangency.

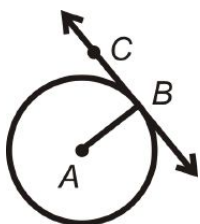
Investigation 9-1: Tangent Line and Radius Property

Tools Needed: compass, ruler, pencil, paper, protractor

1. Using your compass, draw a circle. Locate the center and draw a radius. Label the radius \overline{AB} , with A as the center.

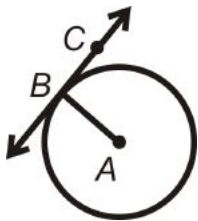


2. Draw a tangent line, \overleftrightarrow{BC} , where B is the point of tangency. To draw a tangent line, take your ruler and line it up with point B . B must be the only point on the circle that the line passes through.



3. Find $m\angle ABC$.

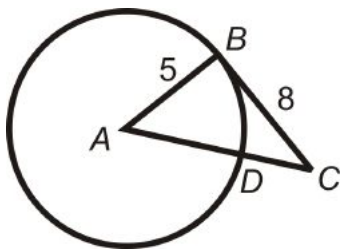
Tangent to a Circle Theorem: A line is tangent to a circle if and only if the line is perpendicular to the radius drawn to the point of tangency.



\overleftrightarrow{BC} is tangent at point B if and only if $\overleftrightarrow{BC} \perp \overline{AB}$.

This theorem uses the words “if and only if,” making it a biconditional statement, which means the converse of this theorem is also true.

Example 4: In $\odot A$, \overline{CB} is tangent at point B . Find AC . Reduce any radicals.



Solution: \overline{CB} is tangent, so $\overline{AB} \perp \overline{CB}$ and $\triangle ABC$ a right triangle. Use the Pythagorean Theorem to find AC .

$$5^2 + 8^2 = AC^2$$

$$25 + 64 = AC^2$$

$$89 = AC^2$$

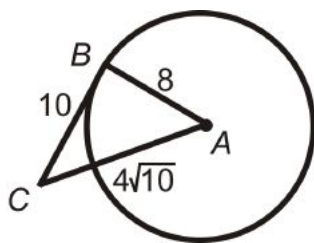
$$AC = \sqrt{89}$$

Example 5: Find DC , in $\odot A$. Round your answer to the nearest hundredth.

Solution: $DC = AC - AD$

$$DC = \sqrt{89} - 5 \approx 4.43$$

Example 6: Determine if the triangle below is a right triangle.



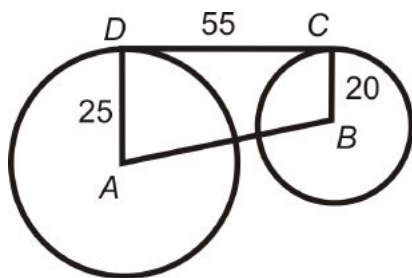
Solution: Again, use the Pythagorean Theorem. $4\sqrt{10}$ is the longest side, so it will be c .

$$8^2 + 10^2 ? (4\sqrt{10})^2$$

$$64 + 100 \neq 160$$

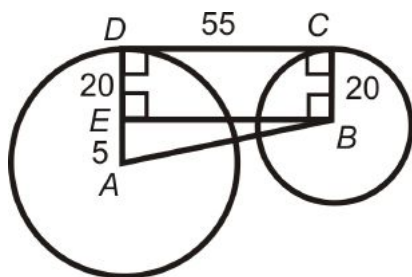
$\triangle ABC$ is not a right triangle. From this, we also find that \overline{CB} is not tangent to $\odot A$.

Example 7: Find AB in $\odot A$ and $\odot B$. Reduce the radical.



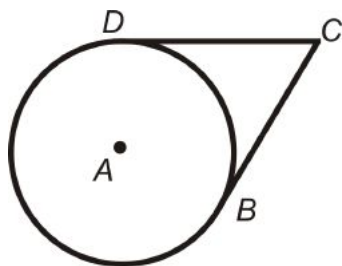
Solution: $\overline{AD} \perp \overline{DC}$ and $\overline{DC} \perp \overline{CB}$. Draw in \overline{BE} , so $EDCB$ is a rectangle. Use the Pythagorean Theorem to find AB .

$$\begin{aligned} 5^2 + 55^2 &= AC^2 \\ 25 + 3025 &= AC^2 \\ 3050 &= AC^2 \\ AC &= \sqrt{3050} = 5\sqrt{122} \end{aligned}$$

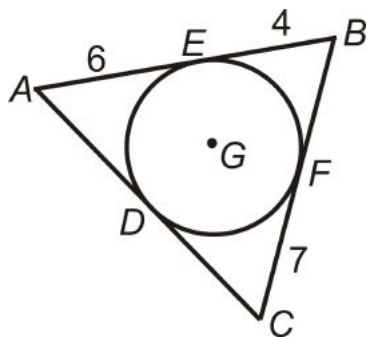


Tangent Segments

Theorem 9-2: If two tangent segments are drawn from the same external point, then they are equal.
 \overline{BC} and \overline{DC} have C as an endpoint and are tangent; $\overline{BC} \cong \overline{DC}$.



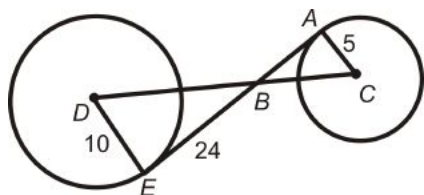
Example 8: Find the perimeter of $\triangle ABC$.



Solution: $AE = AD$, $EB = BF$, and $CF = CD$. Therefore, the perimeter of $\triangle ABC = 6+6+4+4+7+7 = 34$.

$\odot G$ is *inscribed* in $\triangle ABC$. A circle is inscribed in a polygon, if every side of the polygon is tangent to the circle.

Example 9: If D and A are the centers and AE is tangent to both circles, find DC .



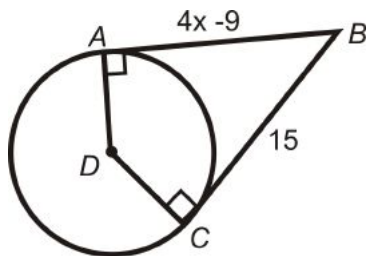
Solution: $\overline{AE} \perp \overline{DE}$ and $\overline{AE} \perp \overline{AC}$ and $\triangle ABC \sim \triangle DBE$.

To find DB , use the Pythagorean Theorem.

$$\begin{aligned} 10^2 + 24^2 &= DB^2 \\ 100 + 576 &= 676 \\ DB &= \sqrt{676} = 26 \end{aligned}$$

To find BC , use similar triangles. $\frac{5}{10} = \frac{BC}{26} \rightarrow BC = 13$. $DC = AB + BC = 26 + 13 = 39$

Example 10: Algebra Connection Find the value of x .



Solution: $\overline{AB} \cong \overline{CB}$ by Theorem 9-2. Set $AB = CB$ and solve for x .

$$\begin{aligned} 4x - 9 &= 15 \\ 4x &= 24 \\ x &= 6 \end{aligned}$$

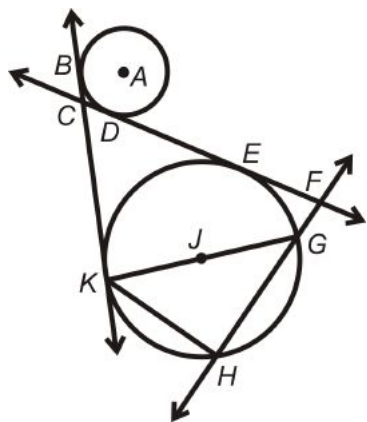
Know What? Revisited The orange line is a diameter of the smaller circle. Since this line passes through the center of the larger circle (yellow point), it is part of one of its diameters. The “moon” hand is a diameter of the larger circle, but a secant of the smaller circle.

Review Questions

- Questions 1-9 are similar to Example 1.
- Questions 10-12 are similar to Example 2.
- Questions 13-17 are similar to Example 3.
- Questions 18-20 are similar to Example 6.
- Questions 21-26 are similar to Example 4, 5, 7, and 10.
- Questions 27-31 are similar to Example 9.
- Questions 32-37 are similar to Example 8.

- Question 38 and 39 use the proof of Theorem 9-2.
- Question 40 uses Theorem 9-2.

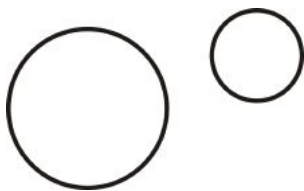
Determine which term best describes each of the following parts of $\odot P$.



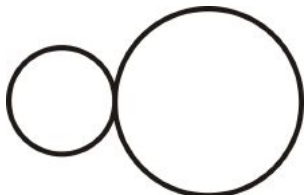
1. \overline{KG}
2. \overleftrightarrow{FH}
3. \overline{KH}
4. \overline{E}
5. \overleftrightarrow{BK}
6. \overleftrightarrow{CF}
7. A
8. \overline{JG}
9. What is the longest chord in any circle?

Copy each pair of circles. Draw in all common tangents.

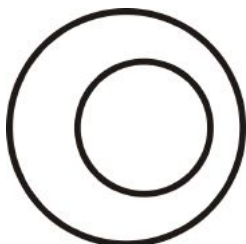
10.



11.

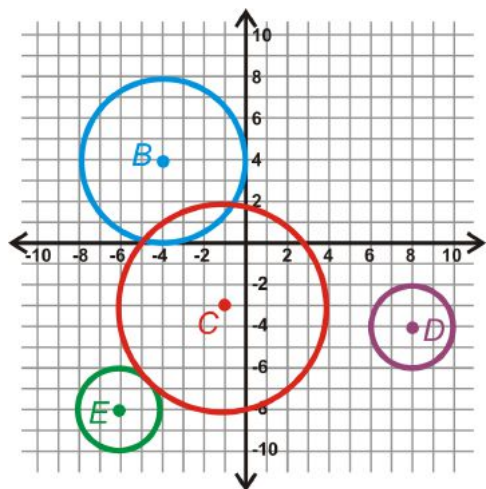


12.



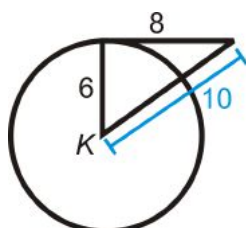
Coordinate Geometry Use the graph below to answer the following questions.

13. Find the radius of each circle.
14. Are any circles congruent? How do you know?
15. Find all the common tangents for $\odot B$ and $\odot C$.
16. $\odot C$ and $\odot E$ are externally tangent. What is CE ?
17. Find the equation of \overline{CE} .

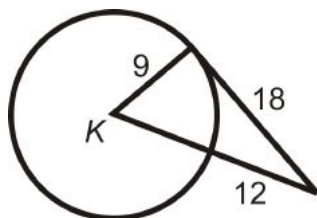


Determine whether the given segment is tangent to $\odot K$.

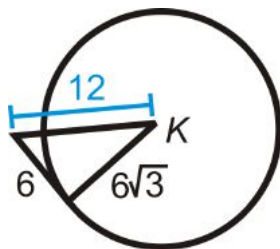
18.



19.

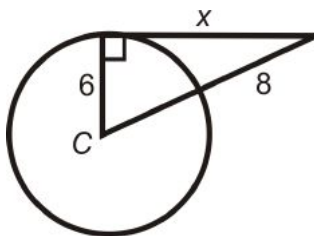


20.

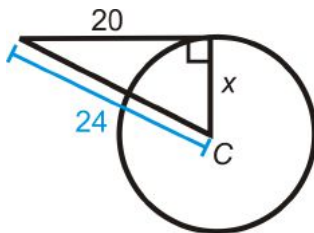


Algebra Connection Find the value of the indicated length(s) in $\odot C$. A and B are points of tangency. Simplify all radicals.

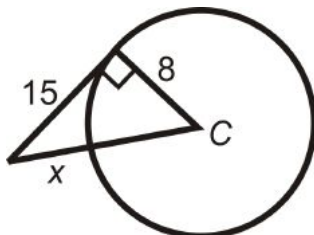
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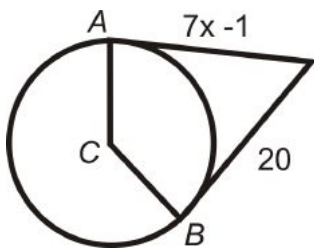
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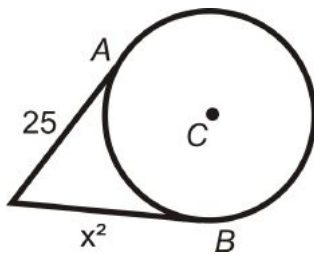
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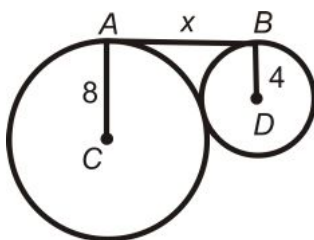
24.



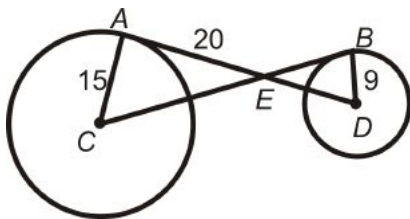
25.



26.

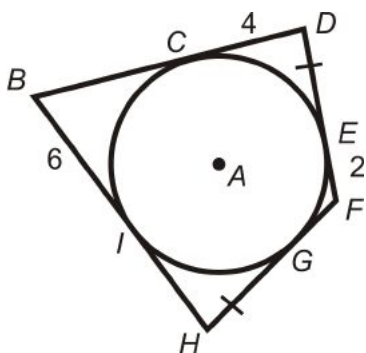


A and B are points of tangency for $\odot C$ and $\odot D$.



27. Is $\triangle AEC \sim \triangle BED$? Why?
28. Find CE .
29. Find BE .
30. Find ED .
31. Find BC and AD .

⊙ A is inscribed in $BDFH$.



32. Find the perimeter of $BDFH$.
33. What type of quadrilateral is $BDFH$? How do you know?
34. Draw a circle inscribed in a square. If the radius of the circle is 5, what is the perimeter of the square?
35. Can a circle be inscribed in a rectangle? If so, draw it. If not, explain.
36. Draw a triangle with two sides tangent to a circle, but the third side is not.
37. Can a circle be inscribed in an obtuse triangle? If so, draw it. If not, explain.
38. Fill in the blanks in the proof of Theorem 9-2.

Given: \overline{AB} and \overline{CB} with points of tangency at A and C .

\overline{AD} and \overline{DC} are radii.

Prove: $\overline{AB} \cong \overline{CB}$

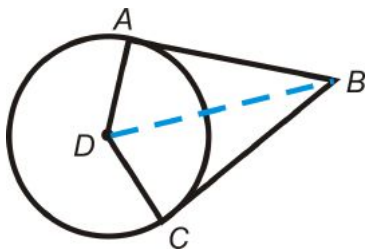


Table 5.1:

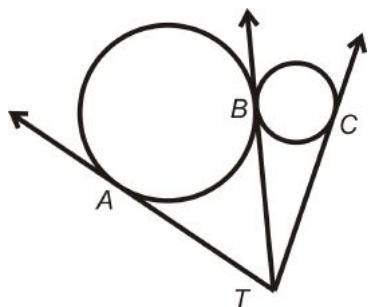
<i>Statement</i>	<i>Reason</i>
1.	
2. $\overline{AD} \cong \overline{DC}$	
3. $\overline{DA} \perp \overline{AB}$ and $\overline{DC} \perp \overline{CB}$	
4.	Definition of perpendicular lines
5.	Connecting two existing points
6. $\triangle ADB$ and $\triangle DCB$ are right triangles	
7. $\overline{DB} \cong \overline{DB}$	
8. $\triangle ABD \cong \triangle CBD$	
9. $\overline{AB} \cong \overline{CB}$	

39. Fill in the blanks, using the proof from #38.

(a) $ABCD$ is a _____ (type of quadrilateral).

(b) The line that connects the _____ and the external point B _____ $\angle ABC$.

40. Points A , B , and C are points of tangency for the three tangent circles. *Explain* why $\overline{AT} \cong \overline{BT} \cong \overline{CT}$.



Review Queue Answers

- $y = \frac{2}{5}x + 4$
- $y = -2x + 3$
- $$m = \frac{2 - (-1)}{6 - (-3)} = \frac{3}{9} = \frac{1}{3}$$

$$y = \frac{1}{3}x + b \rightarrow \text{plug in } (6, 2)$$

$$2 = \frac{1}{3}(6) + b$$

$$2 = 2 + b \rightarrow b = 0$$

$$y = \frac{1}{3}x$$
- $$m_{\perp} = -3$$

$$11 = -3(-8) + b$$

$$11 = 24 + b \rightarrow b = -13$$

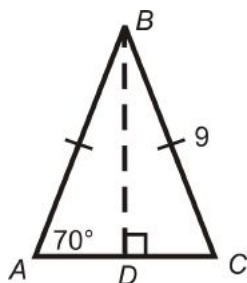
$$y = -3x - 13$$

5.2 Properties of Arcs

Learning Objectives

- Define and measure central angles, minor arcs, and major arcs.

Review Queue



1. What kind of triangle is $\triangle ABC$?
2. How does \overline{BD} relate to $\triangle ABC$?
3. Find $m\angle ABC$ and $m\angle ABD$.

Round to the nearest tenth. *Use the trig ratios.*

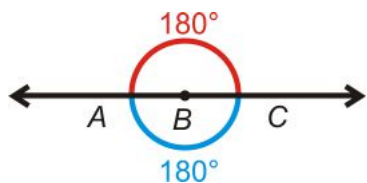
4. Find AD .
5. Find AC .

Know What? The Ferris wheel to the right has equally spaced seats, such that the central angle is 20° . How many seats are on this ride? Why do you think it is important to have equally spaced seats on a Ferris wheel?



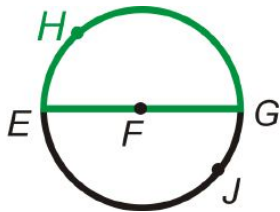
Central Angles & Arcs

Recall that a straight angle is 180° . If take two straight angles and put one on top of the other, we would have a circle. This means that a circle has 360° , $180^\circ + 180^\circ$. This also means that a semicircle, or half circle, is 180° .



Arc: A section of the circle.

Semicircle: An arc that measures 180° .



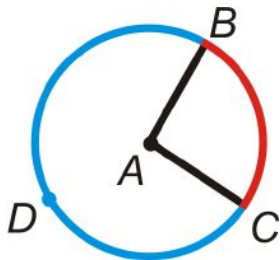
To label an arc, place a curve above the endpoints. You may want to use 3 points to clarify.

\widehat{EHG} and \widehat{EJG} are semicircles $m\widehat{EHG} = 180^\circ$

Central Angle: The angle formed by two radii and its vertex at the center of the circle.

Minor Arc: An arc that is less than 180°

Major Arc: An arc that is greater than 180° . *Always* use 3 letters to label a major arc.



The central angle is $\angle BAC$.

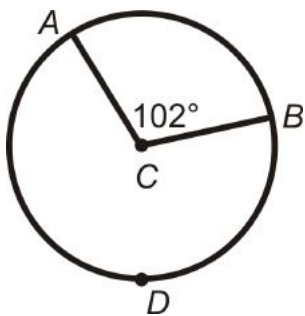
The minor arc is \widehat{BC} .

The major arc is \widehat{BDC} .

Every central angle divides a circle into two arcs.

An arc can be measured in degrees or in a linear measure (cm, ft, etc.). In this chapter we will use degree measure. *The measure of the minor arc is the same as the measure of the central angle* that corresponds to it. The measure of the major arc is 360° minus the measure of the minor arc.

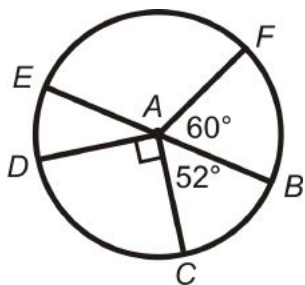
Example 1: Find $m\widehat{AB}$ and $m\widehat{ADB}$ in $\odot C$.



Solution: $m\widehat{AB} = m\widehat{ACB}$. So, $m\widehat{AB} = 102^\circ$.

$$m\widehat{ADB} = 360^\circ - m\widehat{AB} = 360^\circ - 102^\circ = 258^\circ$$

Example 2: Find the measures of the arcs in $\odot A$. \overline{EB} is a diameter.



Solution: Because \overline{EB} is a diameter, $m\angle EAB = 180^\circ$. Each arc is the same as its corresponding central angle.

$$m\widehat{BF} = m\angle FAB = 60^\circ$$

$$m\widehat{EF} = m\angle EAF = 120^\circ \rightarrow 180^\circ - 60^\circ$$

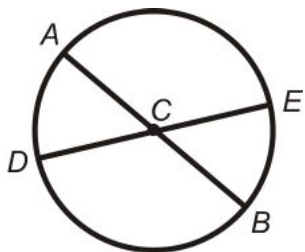
$$m\widehat{ED} = m\angle EAD = 38^\circ \rightarrow 180^\circ - 90^\circ - 52^\circ$$

$$m\widehat{DC} = m\angle DAC = 90^\circ$$

$$m\widehat{BC} = m\angle BAC = 52^\circ$$

Congruent Arcs: Two arcs are congruent if their central angles are congruent.

Example 3: List the congruent arcs in $\odot C$ below. \overline{AB} and \overline{DE} are diameters.

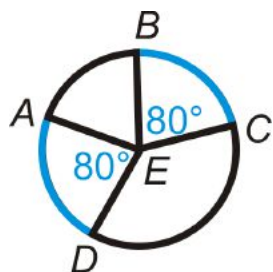


Solution: $\angle ACD = \angle ECB$ because they are vertical angles. $\angle DCB = \angle ACE$ because they are also vertical angles.

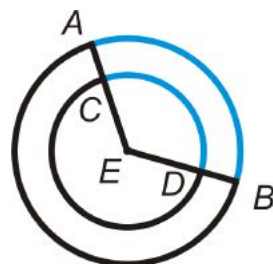
$$\widehat{AD} \cong \widehat{EB} \text{ and } \widehat{AE} \cong \widehat{DB}$$

Example 4: Are the blue arcs congruent? Explain why or why not.

a)



b)



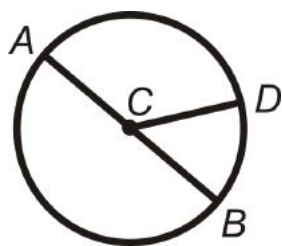
Solution:

- a) $\widehat{AD} \cong \widehat{BC}$ because they have the same central angle measure and in the same circle.
 b) The two arcs have the same measure, but are not congruent because the circles have different radii.

Arc Addition Postulate

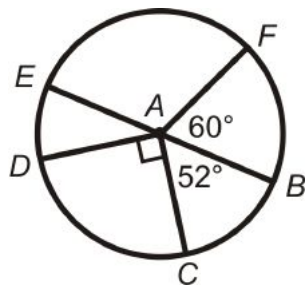
Just like the Angle Addition Postulate and the Segment Addition Postulate, there is an Arc Addition Postulate.

Arc Addition Postulate: The measure of the arc formed by two adjacent arcs is the sum of the measures of the two arcs.



$$m\widehat{AD} + m\widehat{DB} = m\widehat{ADB}$$

Example 5: Find the measure of the arcs in $\odot A$. \overline{EB} is a diameter.

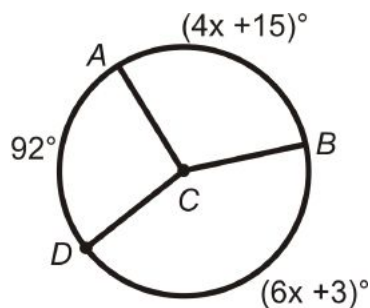


- a) $m\widehat{FED}$
 b) $m\widehat{CDF}$
 c) $m\widehat{DFC}$

Solution: Use the Arc Addition Postulate.

- a) $m\widehat{FED} = m\widehat{FE} + m\widehat{ED} = 120^\circ + 38^\circ = 158^\circ$
 b) $m\widehat{CDF} = m\widehat{CD} + m\widehat{DE} + m\widehat{EF} = 90^\circ + 38^\circ + 120^\circ = 248^\circ$
 c) $m\widehat{DFC} = 38^\circ + 120^\circ + 60^\circ + 52^\circ = 270^\circ$

Example 6: Algebra Connection Find the value of x for $\odot C$ below.



Solution:

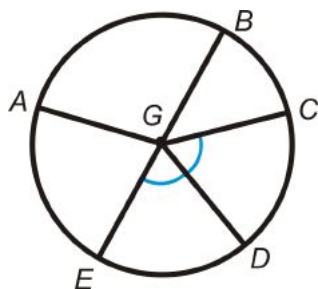
$$\begin{aligned}
 m\widehat{AB} + m\widehat{AD} + m\widehat{DB} &= 360^\circ \\
 (4x + 15)^\circ + 92^\circ + (6x + 3)^\circ &= 360^\circ \\
 10x + 110^\circ &= 360^\circ \\
 10x &= 250^\circ \\
 x &= 25^\circ
 \end{aligned}$$

Know What? Revisited Because the seats are 20° apart, there will be $\frac{360^\circ}{20^\circ} = 18$ seats. It is important to have the seats evenly spaced for the balance of the Ferris wheel.

Review Questions

- Questions 1-6 use the definition of minor arc, major arc, and semicircle.
- Question 7 is similar to Example 3.
- Questions 8 and 9 are similar to Example 5.
- Questions 10-15 are similar to Example 1.
- Questions 16-18 are similar to Example 4.
- Questions 19-26 are similar to Example 2 and 5.
- Questions 27-29 are similar to Example 6.
- Question 30 is a challenge.

Determine if the arcs below are a minor arc, major arc, or semicircle of $\odot G$. \overline{EB} is a diameter.

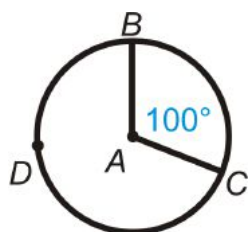


1. \widehat{AB}
2. \widehat{ABD}
3. \widehat{BCE}
4. \widehat{CAE}
5. \widehat{ABC}
6. \widehat{EAB}

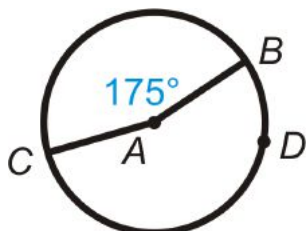
7. Are there any congruent arcs? If so, list them.
8. If $m\widehat{BC} = 48^\circ$, find $m\widehat{CD}$.
9. Using #8, find $m\widehat{CAE}$.

Find the measure of the minor arc and the major arc in each circle below.

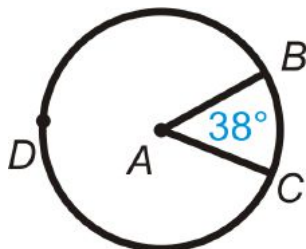
10.



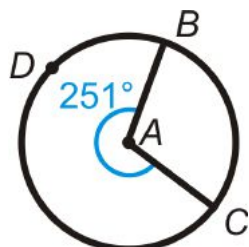
11.



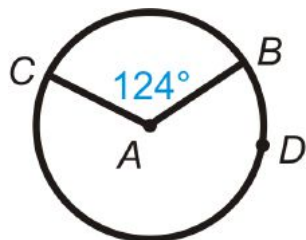
12.



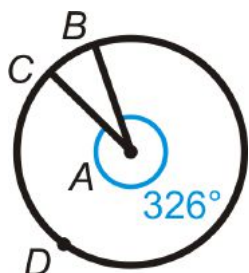
13.



14.

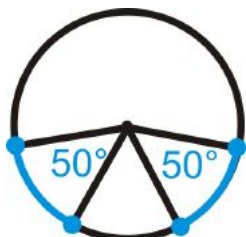


15.

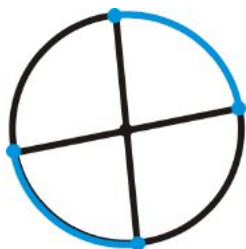


Determine if the blue arcs are congruent. If so, state why.

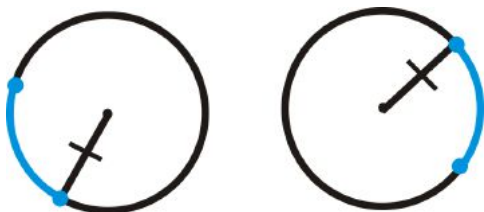
16.



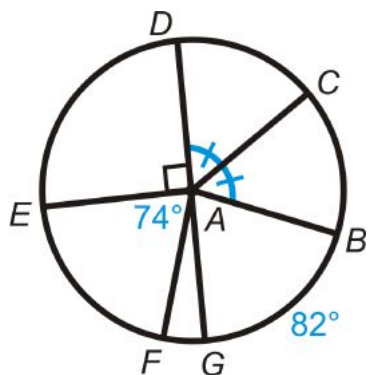
17.



18.



Find the measure of the indicated arcs or central angles in $\odot A$. \overline{DG} is a diameter.



19. \widehat{DE}

20. \widehat{DC}

21. \widehat{GAB}

22. \widehat{FG}

23. \widehat{EDB}

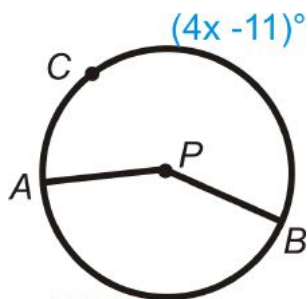
24. \widehat{EAB}

25. \widehat{DCF}

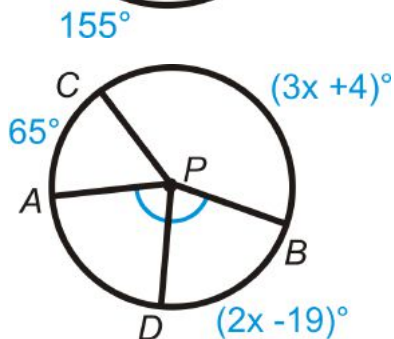
26. \widehat{DBE}

Algebra Connection Find the measure of x in $\odot P$.

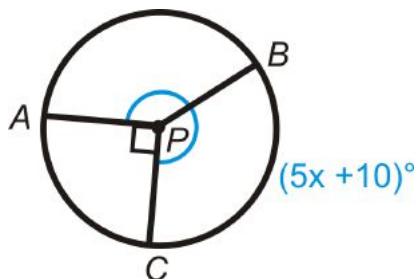
27.



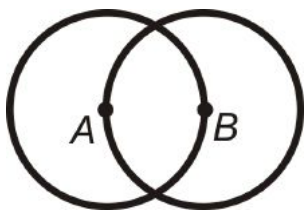
28.



29.



30. **Challenge** What can you conclude about $\odot A$ and $\odot B$?



Review Queue Answers

1. isosceles
2. \overline{BD} is the angle bisector of $\angle ABC$ and the perpendicular bisector of \overline{AC} .
3. $m\angle ABC = 40^\circ, m\angle ABD = 25^\circ$
4. $\cos 70^\circ = \frac{AD}{9} \rightarrow AD = 9 \cdot \cos 70^\circ = 3.1$
5. $AC = 2 \cdot AD = 2 \cdot 3.1 = 6.2$

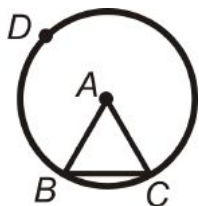
5.3 Properties of Chords

Learning Objectives

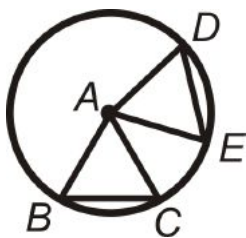
- Find the lengths of chords in a circle.
- Discover properties of chords and arcs.

Review Queue

1. Draw a chord in a circle.
2. Draw a diameter in the circle from #1. Is a diameter a chord?
3. $\triangle ABC$ is an equilateral triangle in $\odot A$. Find $m\widehat{BC}$ and $m\widehat{BDC}$.



4. $\triangle ABC$ and $\triangle ADE$ are equilateral triangles in $\odot A$. List a pair of congruent arcs and chords.



Know What? To the right is the Gran Teatro Falla, in Cadiz, Andalucía, Spain. Notice the five windows, $A - E$. $\odot A \cong \odot E$ and $\odot B \cong \odot C \cong \odot D$. Each window is topped with a 240° arc. The gold chord in each circle connects the rectangular portion of the window to the circle. Which chords are congruent?

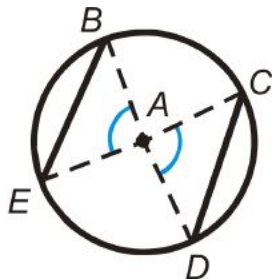
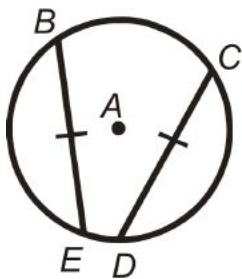


Recall from the first section, a chord is a line segment whose endpoints are on a circle. A diameter is the longest chord in a circle.

Congruent Chords & Congruent Arcs

From #4 in the Review Queue above, we noticed that $\overline{BC} \cong \overline{DE}$ and $\widehat{BC} \cong \widehat{DE}$.

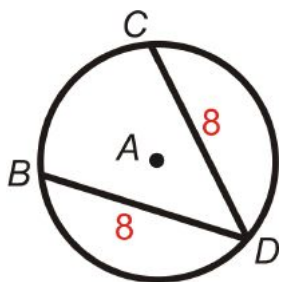
Theorem 9-3: In the same circle or congruent circles, minor arcs are congruent if and only if their corresponding chords are congruent.



In both of these pictures, $\overline{BE} \cong \overline{CD}$ and $\widehat{BE} \cong \widehat{CD}$.

In the second circle, $\triangle BAE \cong \triangle CAD$ by SAS.

Example 1: Use $\odot A$ to answer the following.



a) If $m\widehat{BD} = 125^\circ$, find $m\widehat{CD}$.

b) If $m\widehat{BC} = 80^\circ$, find $m\widehat{CD}$.

Solution:

a) $BD = CD$, which means the arcs are equal too. $m\widehat{CD} = 125^\circ$.

b) $m\widehat{CD} \cong m\widehat{BD}$ because $BD = CD$.

$$m\widehat{BC} + m\widehat{CD} + m\widehat{BD} = 360^\circ$$

$$80^\circ + 2m\widehat{CD} = 360^\circ$$

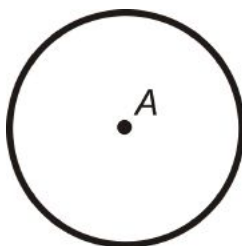
$$2m\widehat{CD} = 280^\circ$$

$$m\widehat{CD} = 140^\circ$$

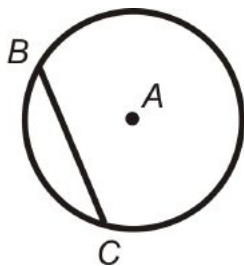
Investigation 9-2: Perpendicular Bisector of a Chord

Tools Needed: paper, pencil, compass, ruler

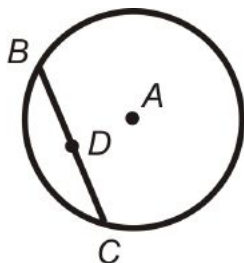
1. Draw a circle. Label the center A.



2. Draw a chord. Label it \overline{BC} .

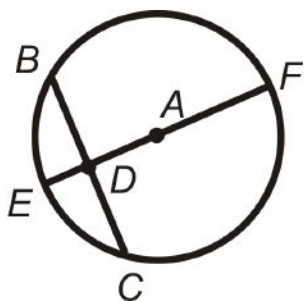


3. Find the midpoint of \overline{BC} using a ruler. Label it D .



4. Connect A and D to form a diameter. How does \overline{AD} relate to \overline{BC} ?

Theorem 9-4: The perpendicular bisector of a chord is also a diameter.

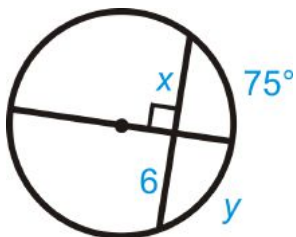


If $\overline{AD} \perp \overline{BC}$ and $\overline{BD} \cong \overline{DC}$ then \overline{EF} is a diameter.

If $\overline{EF} \perp \overline{BC}$, then $\overline{BD} \cong \overline{DC}$ and $\widehat{BE} \cong \widehat{EC}$.

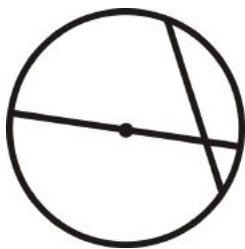
Theorem 9-5: If a diameter is perpendicular to a chord, then the diameter bisects the chord and its corresponding arc.

Example 2: Find the value of x and y .



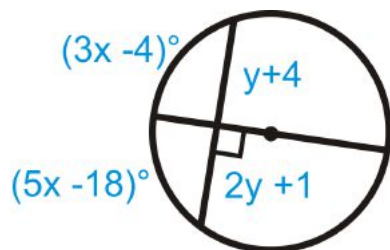
Solution: The diameter is perpendicular to the chord. From Theorem 9-5, $x = 6$ and $y = 75^\circ$.

Example 3: Is the converse of Theorem 9-4 true?



Solution: The converse of Theorem 9-4 would be: *A diameter is also the perpendicular bisector of a chord.* This is not true, a diameter cannot always be a perpendicular bisector to every chord. See the picture.

Example 4: Algebra Connection Find the value of x and y .



Solution: The diameter is perpendicular to the chord, which means it bisects the chord and the arc. Set up an equation for x and y .

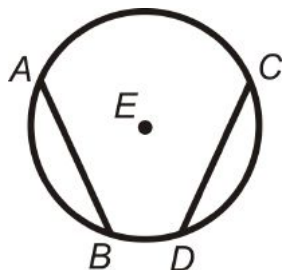
$$\begin{aligned} (3x - 4)^\circ &= (5x - 18)^\circ & y + 4 &= 2y + 1 \\ 14^\circ &= 2x & 3 &= y \\ 7^\circ &= x & & \end{aligned}$$

Equidistant Congruent Chords

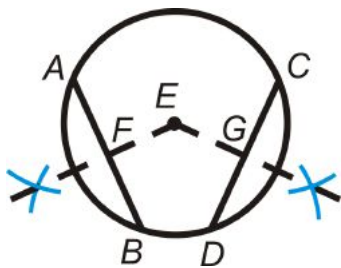
Investigation 9-3: Properties of Congruent Chords

Tools Needed: pencil, paper, compass, ruler

1. Draw a circle with a radius of 2 inches and two chords that are both 3 inches. Label like the picture to the right. *This diagram is drawn to scale.*

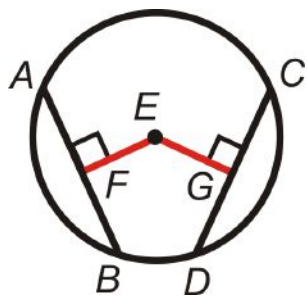


2. From the center, draw the perpendicular segment to \overline{AB} and \overline{CD} . You can use Investigation 3-2



3. Erase the arc marks and lines beyond the points of intersection, leaving \overline{FE} and \overline{EG} . Find the measure of these segments. What do you notice?

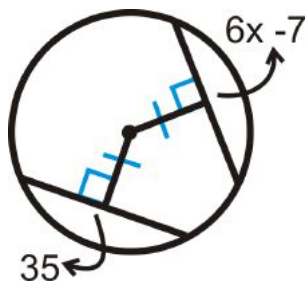
Theorem 9-6: In the same circle or congruent circles, two chords are congruent if and only if they are equidistant from the center.



The shortest distance from any point to a line is the perpendicular line between them.

If $FE = EG$ and $\overline{EF} \perp \overline{EG}$, then \overline{AB} and \overline{CD} are equidistant to the center and $\overline{AB} \cong \overline{CD}$.

Example 5: Algebra Connection Find the value of x .



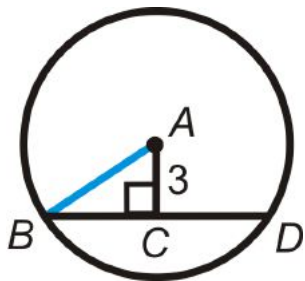
Solution: Because the distance from the center to the chords is congruent and perpendicular to the chords, the chords are equal.

$$6x - 7 = 35$$

$$6x = 42$$

$$x = 7$$

Example 6: $BD = 12$ and $AC = 3$ in $\odot A$. Find the radius.



Solution: First find the radius. \overline{AB} is a radius, so we can use the right triangle $\triangle ABC$, so \overline{AB} is the hypotenuse. From Theorem 9-5, $BC = 6$.

$$\begin{aligned} 3^2 + 6^2 &= AB^2 \\ 9 + 36 &= AB^2 \\ AB &= \sqrt{45} = 3\sqrt{5} \end{aligned}$$

Example 7: Find $m\widehat{BD}$ from Example 6.

Solution: First, find the corresponding central angle, $\angle BAD$. We can find $m\angle BAC$ using the tangent ratio. Then, multiply $m\angle BAC$ by 2 for $m\angle BAD$ and $m\widehat{BD}$.

$$\begin{aligned} \tan^{-1}\left(\frac{6}{3}\right) &= m\angle BAC \\ m\angle BAC &\approx 63.43^\circ \\ m\angle BAD &\approx 2 \cdot 63.43^\circ \approx 126.86^\circ \approx m\widehat{BD} \end{aligned}$$

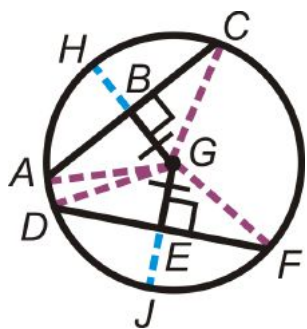
Know What? Revisited In the picture, the chords from $\odot A$ and $\odot E$ are congruent and the chords from $\odot B$, $\odot C$, and $\odot D$ are also congruent. We know this from Theorem 9-3.

Review Questions

- Questions 1-3 use the theorems from this section and similar to Example 3.
- Questions 4-10 use the definitions and theorems from this section.
- Questions 11-16 are similar to Example 1 and 2.
- Questions 17-25 are similar to Examples 2, 4, 5, and 6.
- Questions 26 and 27 are similar to Example 7.
- Questions 28-30 use the theorems from this section.

1. Two chords in a circle are perpendicular and congruent. Does one of them have to be a diameter? Why or why not?
2. Write the converse of Theorem 9-5. Is it true? If not, draw a counterexample.
3. Write the original and converse statements from Theorem 9-3 and Theorem 9-6.

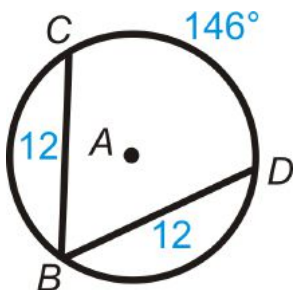
Fill in the blanks.



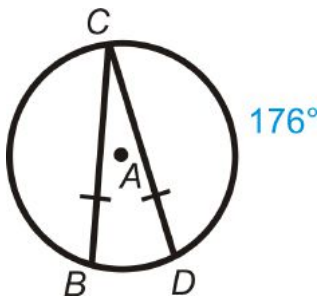
4. $\overline{\quad} \cong \overline{DF}$
5. $\widehat{AC} \cong \widehat{\quad}$
6. $\widehat{DJ} \cong \widehat{\quad}$
7. $\overline{\quad} \cong \overline{EJ}$
8. $\angle AGH \cong \angle \quad$
9. $\angle DGF \cong \angle \quad$
10. List all the congruent radii in $\odot G$.

Find the value of the indicated arc in $\odot A$.

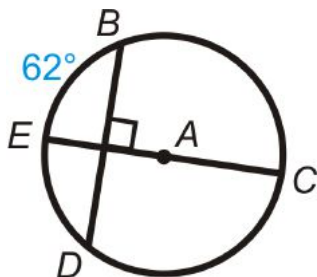
11. $m\widehat{BC}$



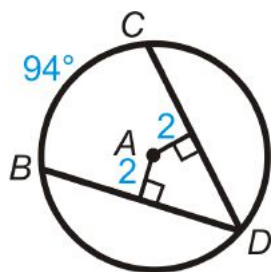
12. $m\widehat{BD}$



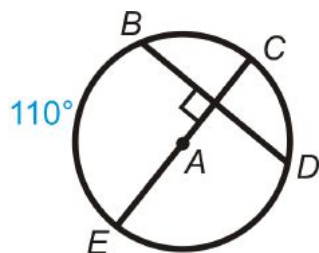
13. $m\widehat{BC}$



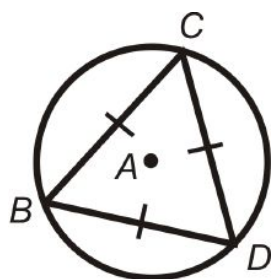
14. $m\widehat{BD}$



15. $m\widehat{BD}$

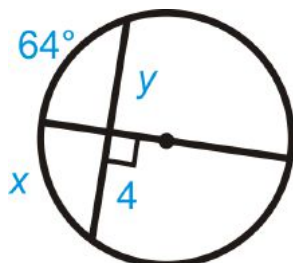


16. $m\widehat{BD}$

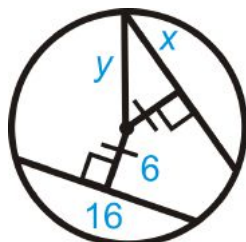


Algebra Connection Find the value of x and/or y .

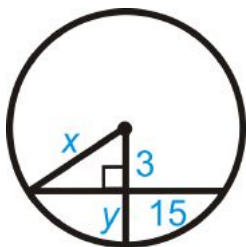
17.



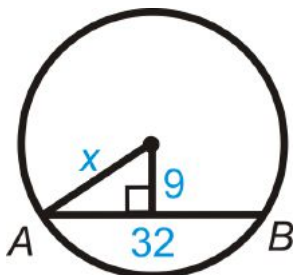
18.



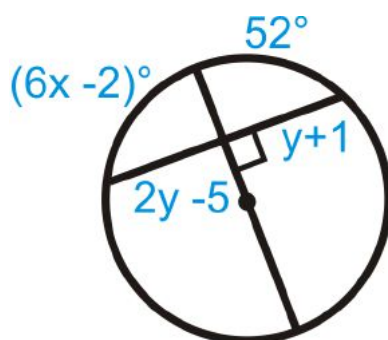
19.



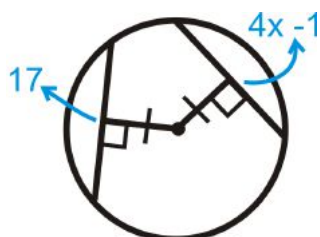
20. $AB = 32$



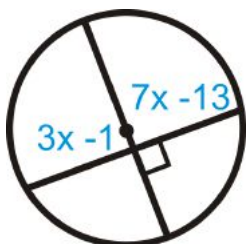
21.



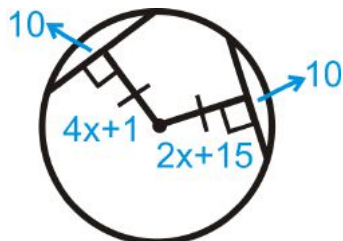
22.



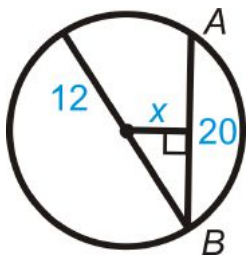
23.



24.

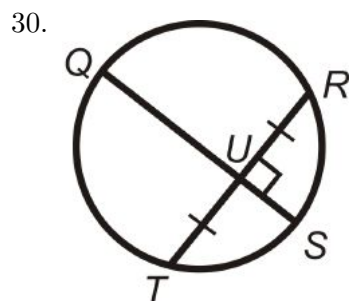
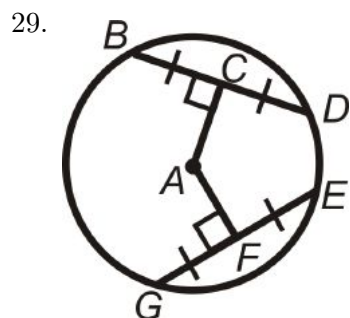
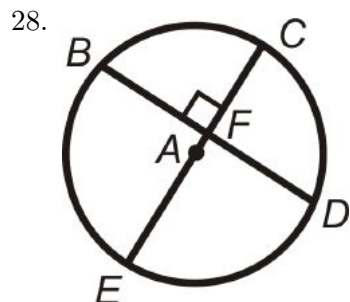


25. $AB = 20$



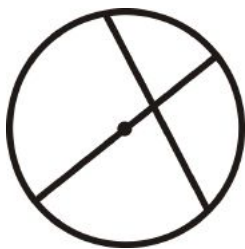
26. Find $m\widehat{AB}$ in Question 20. Round your answer to the nearest tenth of a degree.
 27. Find $m\widehat{AB}$ in Question 25. Round your answer to the nearest tenth of a degree.

In problems 28-30, what can you conclude about the picture? State a theorem that justifies your answer. You may assume that A is the center of the circle.



Review Queue Answers

1 & 2. Answers will vary



3. $m\widehat{BC} = 60^\circ, m\widehat{BDC} = 300^\circ$

4. $\overline{BC} \cong \overline{DE}$ and $\widehat{BC} \cong \widehat{DE}$

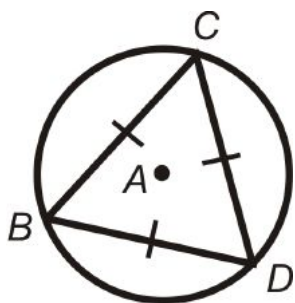
5.4 Inscribed Angles

Learning Objectives

- Find the measure of inscribed angles and the arcs they intercept.

Review Queue

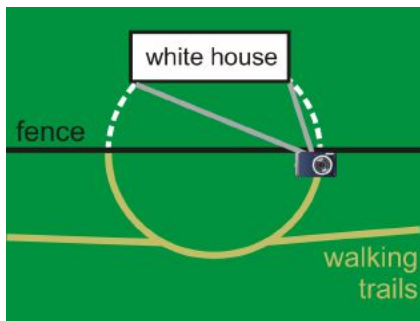
We are going to use #14 from the homework in the previous section.



- What is the measure of each angle in the triangle? How do you know?
- What do you know about the three arcs?
- What is the measure of each arc?

Know What? The closest you can get to the White House are the walking trails on the far right. You want to get as close as you can (on the trail) to the fence to take a picture (you were not allowed to walk on the grass). Where else can you take a picture from to get the same frame of the White House? *Your line of sight in the camera is marked in the picture as the grey lines.*

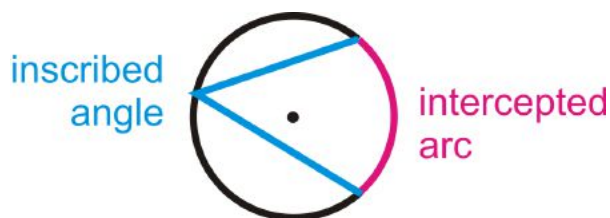




Inscribed Angles

In addition to central angles, we will now learn about inscribed angles in circles.

Inscribed Angle: An angle with its vertex on the circle and sides are chords.



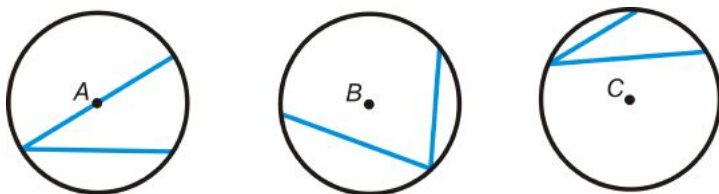
Intercepted Arc: The arc that is inside the inscribed angle and endpoints are on the angle.

The vertex of an inscribed angle can be anywhere on the circle as long as its sides intersect the circle to form an intercepted arc.

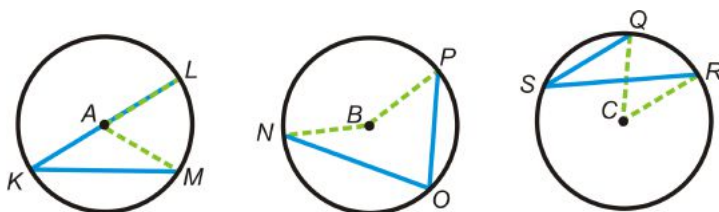
Investigation 9-4: Measuring an Inscribed Angle

Tools Needed: pencil, paper, compass, ruler, protractor

1. Draw three circles with three different inscribed angles. Try to make all the angles different sizes.



2. Using your ruler, draw in the corresponding central angle for each angle and label each set of endpoints.



3. Using your protractor measure the six angles and determine if there is a relationship between the central angle, the inscribed angle, and the intercepted arc.

$$m\angle LAM = \underline{\hspace{2cm}}$$

$$m\angle NBP = \underline{\hspace{2cm}}$$

$$m\angle QCR = \underline{\hspace{2cm}}$$

$$m\widehat{LM} = \underline{\hspace{2cm}}$$

$$m\widehat{NP} = \underline{\hspace{2cm}}$$

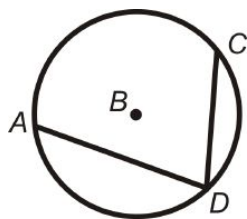
$$m\widehat{QR} = \underline{\hspace{2cm}}$$

$$m\angle LKM = \underline{\hspace{2cm}}$$

$$m\angle NOP = \underline{\hspace{2cm}}$$

$$m\angle QSR = \underline{\hspace{2cm}}$$

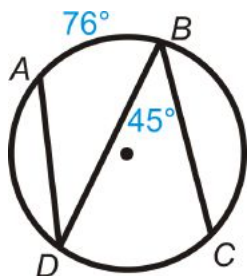
Inscribed Angle Theorem: The measure of an inscribed angle is half the measure of its intercepted arc.



$$m\angle ADC = \frac{1}{2}m\widehat{AC}$$

$$m\widehat{AC} = 2m\angle ADC$$

Example 1: Find $m\widehat{DC}$ and $m\angle ADB$.

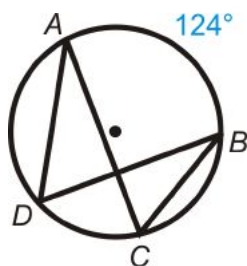


Solution: From the Inscribed Angle Theorem:

$$m\widehat{DC} = 2 \cdot 45^\circ = 90^\circ$$

$$m\angle ADB = \frac{1}{2} \cdot 76^\circ = 38^\circ$$

Example 2: Find $m\angle ADB$ and $m\angle ACB$.



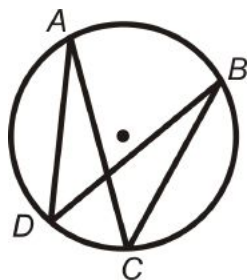
Solution: The intercepted arc for both angles is \widehat{AB} . Therefore,

$$m\angle ADB = \frac{1}{2} \cdot 124^\circ = 62^\circ$$

$$m\angle ACB = \frac{1}{2} \cdot 124^\circ = 62^\circ$$

This example leads us to our next theorem.

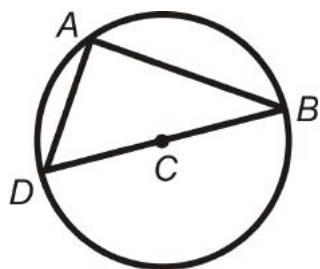
Theorem 9-8: Inscribed angles that intercept the same arc are congruent.



$\angle ADB$ and $\angle ACB$ intercept \widehat{AB} , so $m\angle ADB = m\angle ACB$.

$\angle DAC$ and $\angle DBC$ intercept \widehat{DC} , so $m\angle DAC = m\angle DBC$.

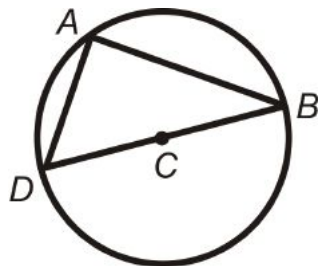
Example 3: Find $m\angle DAB$ in $\odot C$.



Solution: C is the center, so \overline{DB} is a diameter. $\angle DAB$ endpoints are on the diameter, so the central angle is 180° .

$$m\angle DAB = \frac{1}{2} \cdot 180^\circ = 90^\circ.$$

Theorem 9-9: An angle intercepts a semicircle if and only if it is a right angle.

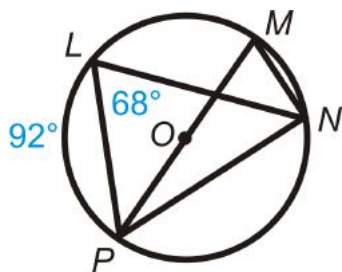


$\angle DAB$ intercepts a semicircle, so $m\angle DAB = 90^\circ$.

$\angle DAB$ is a right angle, so \widehat{DB} is a semicircle.

Anytime a right angle is inscribed in a circle, the endpoints of the angle are the endpoints of a diameter and the diameter is the hypotenuse.

Example 4: Find $m\angle PMN$, $m\widehat{PN}$, $m\angle MNP$, and $m\angle LNP$.

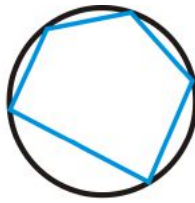
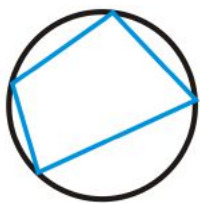
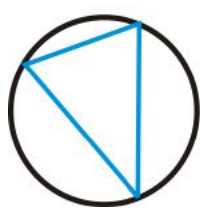


Solution:

$$\begin{aligned}
 m\angle PMN &= m\angle PLN = 68^\circ && \text{by Theorem 9-8.} \\
 m\widehat{PN} &= 2 \cdot 68^\circ = 136^\circ && \text{from the Inscribed Angle Theorem.} \\
 m\angle MNP &= 90^\circ && \text{by Theorem 9-9.} \\
 m\angle LNP &= \frac{1}{2} \cdot 92^\circ = 46^\circ && \text{from the Inscribed Angle Theorem.}
 \end{aligned}$$

Inscribed Quadrilaterals

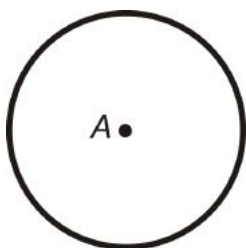
Inscribed Polygon: A polygon where every vertex is on a circle.



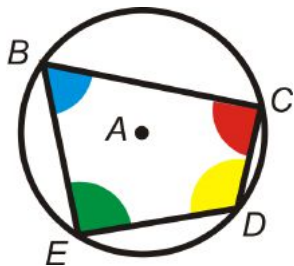
Investigation 9-5: Inscribing Quadrilaterals

Tools Needed: pencil, paper, compass, ruler, colored pencils, scissors

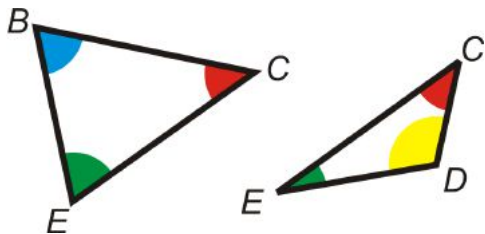
1. Draw a circle. Mark the center point A.



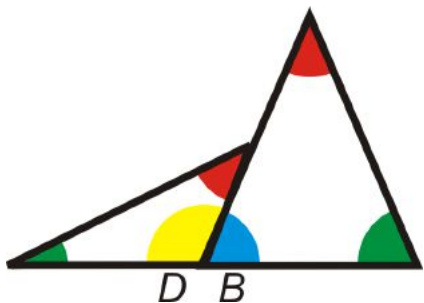
2. Place four points on the circle. Connect them to form a quadrilateral. Color in the 4 angles.



3. Cut out the quadrilateral. Then cut the diagonal \overline{CE} , making two triangles.

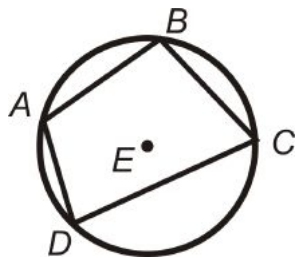


4. Line up $\angle B$ and $\angle D$ so that they are next to each other. What do you notice?



By cutting the quadrilateral in half, we are able to show that $\angle B$ and $\angle D$ form a linear pair when they are placed next to each other, making $\angle B$ and $\angle D$ supplementary.

Theorem 9-10: A quadrilateral is inscribed in a circle if and only if the opposite angles are supplementary.

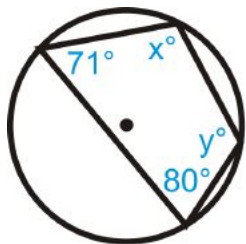


If $ABCD$ is inscribed in $\odot E$, then $m\angle A + m\angle C = 180^\circ$ and $m\angle B + m\angle D = 180^\circ$.

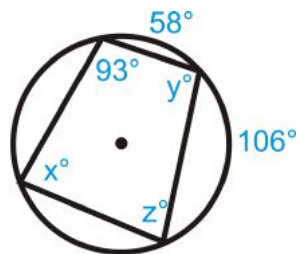
If $m\angle A + m\angle C = 180^\circ$ and $m\angle B + m\angle D = 180^\circ$, then $ABCD$ is inscribed in $\odot E$.

Example 5: Find the value of the missing variables.

a)



b)



Solution:

$$\begin{aligned} \text{a) } x + 80^\circ &= 180^\circ \\ x &= 100^\circ \end{aligned}$$

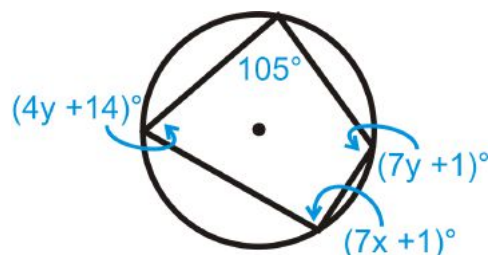
$$\begin{aligned} y + 71^\circ &= 180^\circ \\ y &= 109^\circ \end{aligned}$$

$$\begin{aligned} \text{b) } z + 93^\circ &= 180^\circ \\ z &= 87^\circ \end{aligned}$$

$$\begin{aligned} x &= \frac{1}{2}(58^\circ + 106^\circ) \\ x &= 82^\circ \end{aligned}$$

$$\begin{aligned} y + 82^\circ &= 180^\circ \\ y &= 98^\circ \end{aligned}$$

Example 6: Algebra Connection Find x and y in the picture below.



Solution:

$$(7x + 1)^\circ + 105^\circ = 180^\circ$$

$$7x + 106^\circ = 180^\circ$$

$$7x = 84^\circ$$

$$x = 12^\circ$$

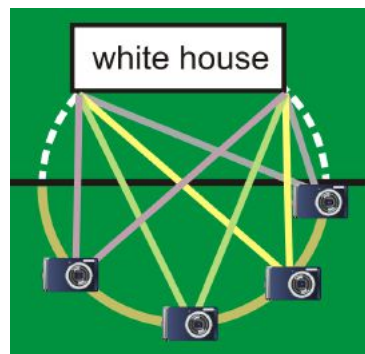
$$(4y + 14)^\circ + (7y + 1)^\circ = 180^\circ$$

$$11y + 15^\circ = 180^\circ$$

$$11y = 165^\circ$$

$$y = 15^\circ$$

Know What? Revisited You can take the picture from anywhere on the semicircular walking path, the frame will be the same.



Review Questions

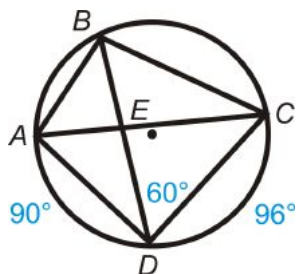
- Questions 1-8 use the vocabulary and theorems learned in this section.
- Questions 9-27 are similar to Examples 1-5.
- Questions 28-33 are similar to Example 6.
- Question 34 is a proof of the Inscribed Angle Theorem.

Fill in the blanks.

1. A(n) _____ polygon has all its vertices on a circle.
2. An inscribed angle is _____ the measure of the intercepted arc.
3. A central angle is _____ the measure of the intercepted arc.
4. An angle inscribed in a _____ is 90° .
5. Two inscribed angles that intercept the same arc are _____.

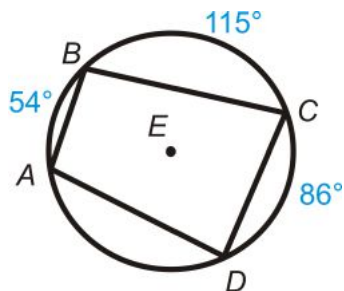
6. The _____ angles of an inscribed quadrilateral are _____.
7. The sides of an inscribed angle are _____.
8. Draw inscribed angle $\angle JKL$ in $\odot M$. Then draw central angle $\angle JML$. How do the two angles relate?

Quadrilateral $ABCD$ is inscribed in $\odot E$. Find:



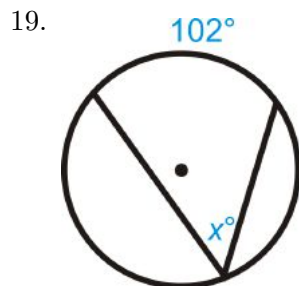
9. $m\angle DBC$
10. $m\widehat{BC}$
11. $m\widehat{AB}$
12. $m\angle ACD$
13. $m\angle ADC$
14. $m\angle ACB$

Quadrilateral $ABCD$ is inscribed in $\odot E$. Find:

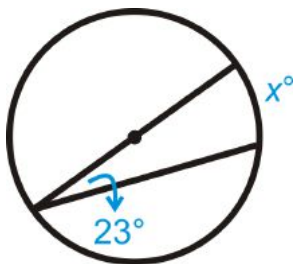


15. $m\angle A$
16. $m\angle B$
17. $m\angle C$
18. $m\angle D$

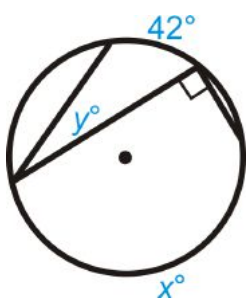
Find the value of x and/or y in $\odot A$.



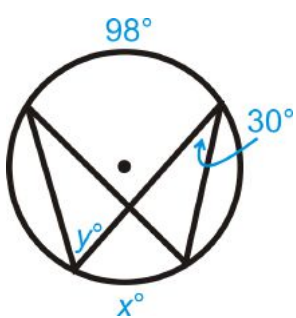
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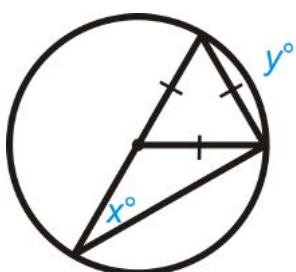
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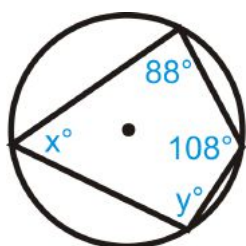
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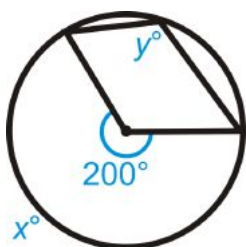
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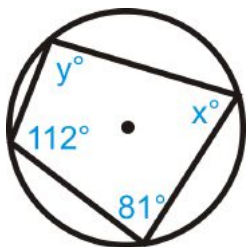
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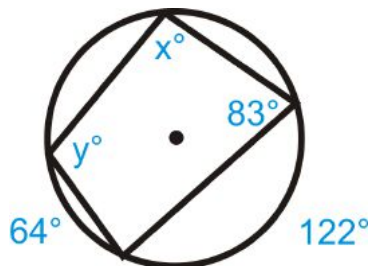
25.



26.

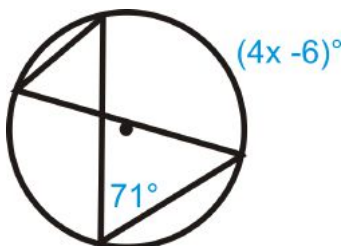


27.

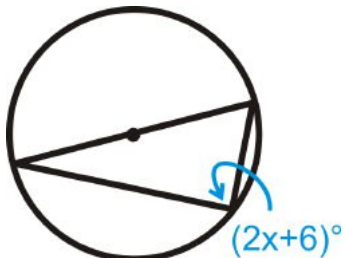


Algebra Connection Solve for x .

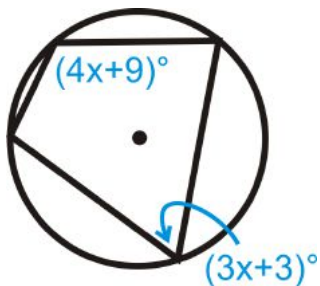
28.



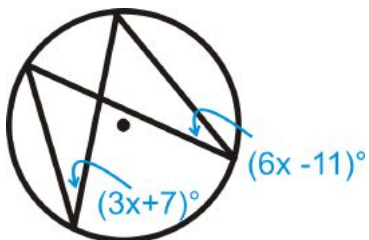
29.



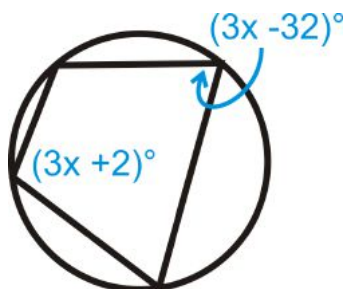
30.



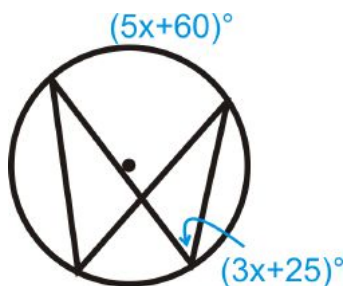
31.



32.



33.



34. Fill in the blanks of the Inscribed Angle Theorem proof.

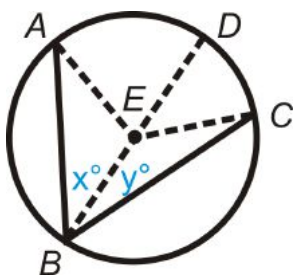
Given: Inscribed $\angle ABC$ and diameter \overline{BD} Prove: $m\angle ABC = \frac{1}{2}m\widehat{AC}$

Table 5.2:

Statement	Reason
1. Inscribed $\angle ABC$ and diameter \overline{BD} $m\angle ABE = x^\circ$ and $m\angle CBE = y^\circ$	
2. $x^\circ + y^\circ = m\angle ABC$	
3.	All radii are congruent
4.	Definition of an isosceles triangle
5. $m\angle EAB = x^\circ$ and $m\angle ECB = y^\circ$	
6. $m\angle AED = 2x^\circ$ and $m\angle CED = 2y^\circ$	
7. $m\widehat{AD} = 2x^\circ$ and $m\widehat{DC} = 2y^\circ$	
8.	Arc Addition Postulate
9. $m\widehat{AC} = 2x^\circ + 2y^\circ$	
10.	Distributive PoE
11. $m\widehat{AC} = 2m\angle ABC$	
12. $m\angle ABC = \frac{1}{2}m\widehat{AC}$	

Review Queue Answers

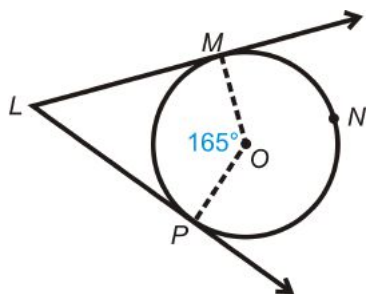
1. 60° , it is an equilateral triangle.
2. They are congruent because the chords are congruent.
3. $\frac{360^\circ}{3} = 120^\circ$

5.5 Angles of Chords, Secants, and Tangents

Learning Objectives

- Find the measures of angles formed by chords, secants, and tangents.

Review Queue



1. What is $m\angle OML$ and $m\angle OPL$? How do you know?
2. Find $m\angle MLP$.
3. Find $m\widehat{MNP}$.

Know What? The sun's rays hit the Earth such that the tangent rays determine when daytime and night time are. If the arc that is exposed to sunlight is 178° , what is the angle at which the sun's rays hit the earth (x°)?



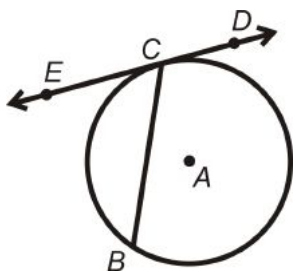
Angle on a Circle

When an angle is **on** a circle, the vertex is on the edge of the circle. One type of angle **on** a circle is the inscribed angle, from the previous section. Another type of angle **on** a circle is one formed by a tangent and a chord.

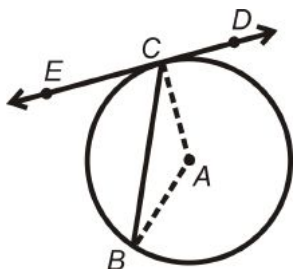
Investigation 9-6: The Measure of an Angle formed by a Tangent and a Chord

Tools Needed: pencil, paper, ruler, compass, protractor

1. Draw $\odot A$ with chord \overline{BC} and tangent line \overleftrightarrow{ED} with point of tangency C .

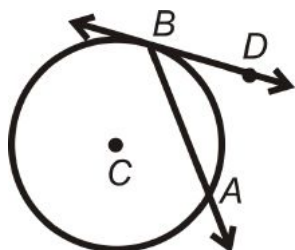


2. Draw in central angle $\angle CAB$. Find $m\angle CAB$ and $m\angle BCE$.



3. Find $m\widehat{BC}$. How does the measure of this arc relate to $m\angle BCE$?

Theorem 9-11: The measure of an angle formed by a chord and a tangent that intersect on the circle is half the measure of the intercepted arc.

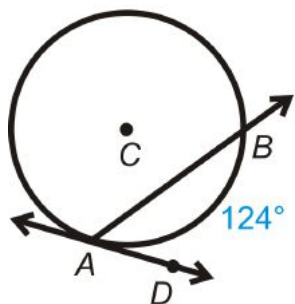


$$m\angle DBA = \frac{1}{2}m\widehat{AB}$$

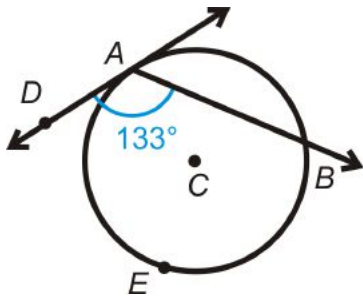
We now know that there are *two* types of angles that are *half* the measure of the intercepted arc; an *inscribed angle* and an *angle formed by a chord and a tangent*.

Example 1: Find:

- a) $m\angle BAD$



- b) $m\widehat{AEB}$

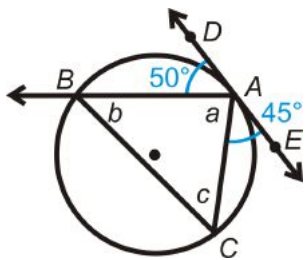


Solution: Use Theorem 9-11.

$$a) m\angle BAD = \frac{1}{2}m\widehat{AB} = \frac{1}{2} \cdot 124^\circ = 62^\circ$$

$$b) m\widehat{AEB} = 2 \cdot m\angle DAB = 2 \cdot 133^\circ = 266^\circ$$

Example 2: Find a , b , and c .



Solution:

$$\begin{aligned} 50^\circ + 45^\circ + m\angle a &= 180^\circ && \text{straight angle} \\ m\angle a &= 85^\circ \end{aligned}$$

$$\begin{aligned} m\angle b &= \frac{1}{2} \cdot m\widehat{AC} \\ m\widehat{AC} &= 2 \cdot m\angle EAC = 2 \cdot 45^\circ = 90^\circ \\ m\angle b &= \frac{1}{2} \cdot 90^\circ = 45^\circ \end{aligned}$$

$$\begin{aligned} 85^\circ + 45^\circ + m\angle c &= 180^\circ && \text{Triangle Sum Theorem} \\ m\angle c &= 50^\circ \end{aligned}$$

From this example, we see that Theorem 9-8 is true for angles formed by a tangent and chord with the vertex on the circle. ***If two angles, with their vertices on the circle, intercept the same arc then the angles are congruent.***

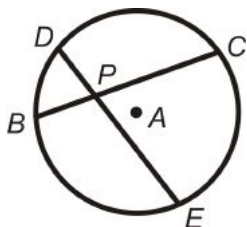
Angles inside a Circle

An angle is *inside* a circle when the vertex anywhere inside the circle, but not on the center.

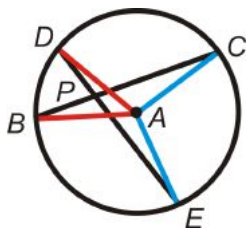
Investigation 9-7: Find the Measure of an Angle *inside* a Circle

Tools Needed: pencil, paper, compass, ruler, protractor, colored pencils (optional)

1. Draw $\odot A$ with chord \overline{BC} and \overline{DE} . Label the point of intersection P .

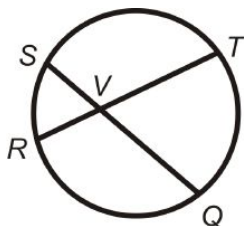


2. Draw central angles $\angle DAB$ and $\angle CAE$. Use colored pencils, if desired.



3. Find $m\angle DPB$, $m\angle DAB$, and $m\angle CAE$. Find $m\widehat{DB}$ and $m\widehat{CE}$.
 4. Find $\frac{m\widehat{DB} + m\widehat{CE}}{2}$.
 5. What do you notice?

Theorem 9-12: The measure of the angle formed by two chords that intersect *inside* a circle is the average of the measure of the intercepted arcs.

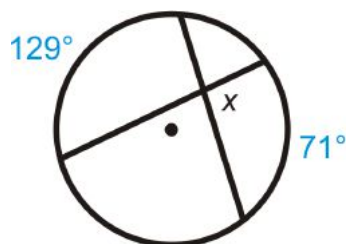


$$m\angle SVR = \frac{1}{2}(m\widehat{SR} + m\widehat{TQ}) = \frac{m\widehat{SR} + m\widehat{TQ}}{2} = m\angle TVQ$$

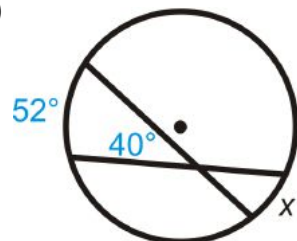
$$m\angle SVT = \frac{1}{2}(m\widehat{ST} + m\widehat{RQ}) = \frac{m\widehat{ST} + m\widehat{RQ}}{2} = m\angle RVQ$$

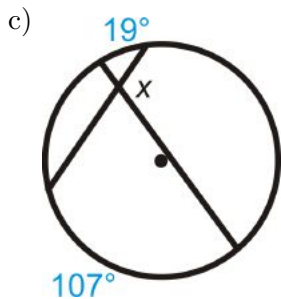
Example 3: Find x .

a)



b)





Solution: Use Theorem 9-12 to write an equation.

a) $x = \frac{129^\circ + 71^\circ}{2} = \frac{200^\circ}{2} = 100^\circ$

b) $40^\circ = \frac{52^\circ + x}{2}$

$80^\circ = 52^\circ + x$

$28^\circ = x$

c) x is supplementary to the angle that the average of the given intercepted arcs, y .

$$y = \frac{19^\circ + 107^\circ}{2} = \frac{126^\circ}{2} = 63^\circ \quad x + 63^\circ = 180^\circ; \quad x = 117^\circ$$

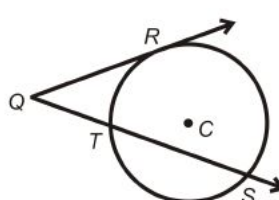
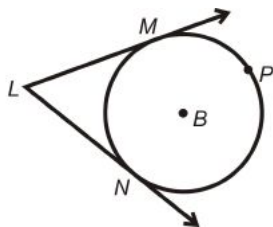
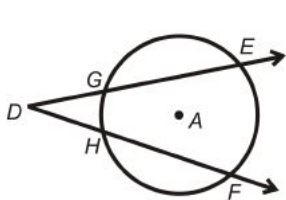
Angles outside a Circle

An angle is *outside* a circle if the vertex of the angle is outside the circle and the sides are tangents or secants. The possibilities are: an angle formed by two tangents, an angle formed by a tangent and a secant, and an angle formed by two secants.

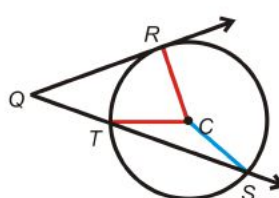
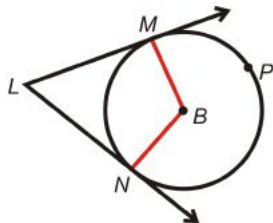
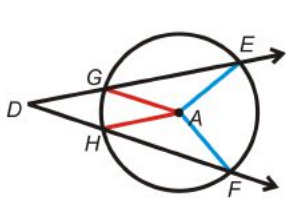
Investigation 9-8: Find the Measure of an Angle outside a Circle

Tools Needed: pencil, paper, ruler, compass, protractor, colored pencils (optional)

1. Draw three circles and label the centers A , B , and C . In $\odot A$ draw two secant rays with the same endpoint. In $\odot B$, draw two tangent rays with the same endpoint. In $\odot C$, draw a tangent ray and a secant ray with the same endpoint. Label the points like the pictures below.

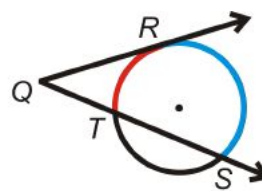
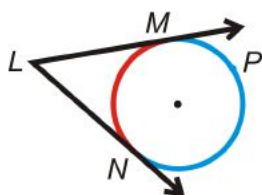
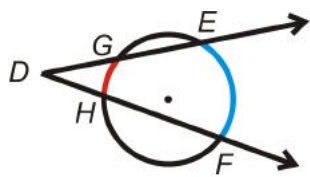


2. Draw in all the central angles. Using a protractor, measure the central angles and find the measures of each intercepted arc.



3. Find $m\angle EDF$, $m\angle MLN$, and $m\angle RQS$.
4. Find $\frac{m\widehat{EF} - m\widehat{GH}}{2}$, $\frac{m\widehat{MPN} - m\widehat{MN}}{2}$, and $\frac{m\widehat{RS} - m\widehat{RT}}{2}$. What do you notice?

Theorem 9-13: The measure of an angle formed by two secants, two tangents, or a secant and a tangent from a point outside the circle is half the difference of the measures of the intercepted arcs.



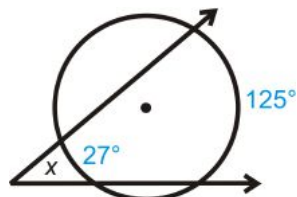
$$m\angle D = \frac{m\widehat{EF} - m\widehat{GH}}{2}$$

$$m\angle L = \frac{m\widehat{PN} - m\widehat{MN}}{2}$$

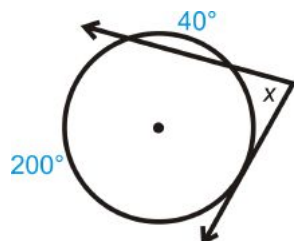
$$m\angle Q = \frac{m\widehat{RS} - m\widehat{RT}}{2}$$

Example 4: Find the measure of x .

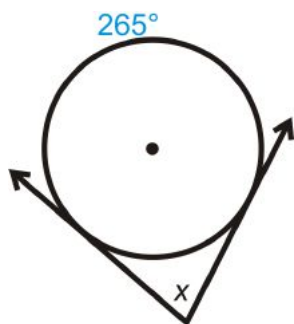
a)



b)



c)



Solution: For all of the above problems we can use Theorem 9-13.

a) $x = \frac{125^\circ - 27^\circ}{2} = \frac{98^\circ}{2} = 49^\circ$

b) 40° is not the intercepted arc. The intercepted arc is 120° , $(360^\circ - 200^\circ - 40^\circ)$. $x = \frac{200^\circ - 120^\circ}{2} = \frac{80^\circ}{2} = 40^\circ$

c) Find the other intercepted arc, $360^\circ - 265^\circ = 95^\circ$ $x = \frac{265^\circ - 95^\circ}{2} = \frac{170^\circ}{2} = 85^\circ$

Know What? Revisited From Theorem 9-13, we know $x = \frac{182^\circ - 178^\circ}{2} = \frac{4^\circ}{2} = 2^\circ$.

Review Questions

- Questions 1-3 use the definitions of tangent and secant lines.
- Questions 4-7 use the definition and theorems learned in this section.
- Questions 8-25 are similar to Examples 1-4.
- Questions 26 and 27 are similar to Example 4, but also a challenge.
- Questions 28 and 29 are fill-in-the-blank proofs of Theorems 9-12 and 9-13.

1. Draw two secants that intersect:

- inside a circle.
- on a circle.
- outside a circle.

2. Can two tangent lines intersect inside a circle? Why or why not?

3. Draw a tangent and a secant that intersect:

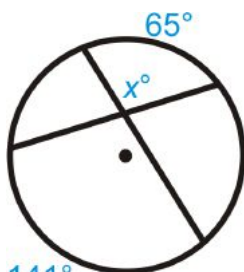
- on a circle.
- outside a circle.

Fill in the blanks.

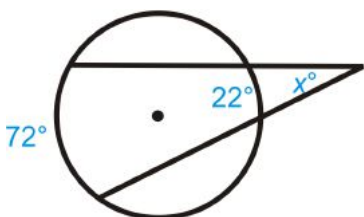
- If the vertex of an angle is on the _____ of a circle, then its measure is _____ - _____ to the intercepted arc.
- If the vertex of an angle is _____ a circle, then its measure is the average of the _____ arcs.
- If the vertex of an angle is _____ a circle, then its measure is _____ the intercepted arc.
- If the vertex of an angle is _____ a circle, then its measure is _____ the difference of the intercepted arcs.

For questions 8-25, find the value of the missing variable(s).

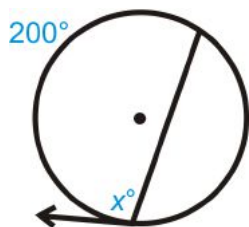
8.



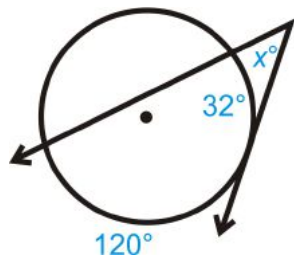
9.



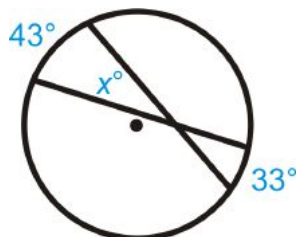
10.



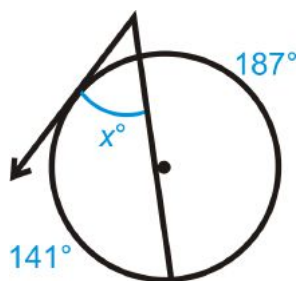
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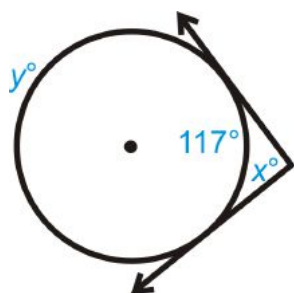
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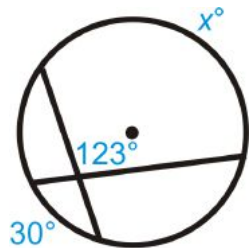
13.



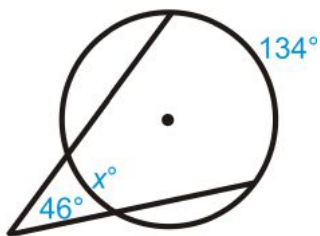
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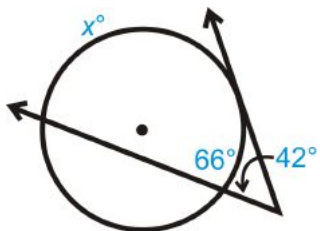
15.



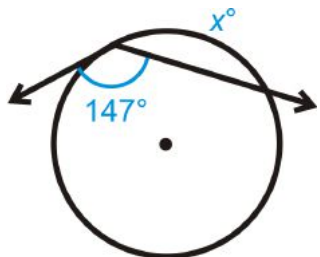
16.



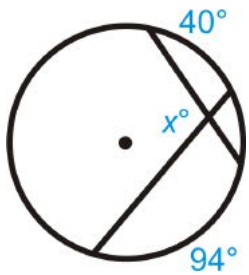
17.



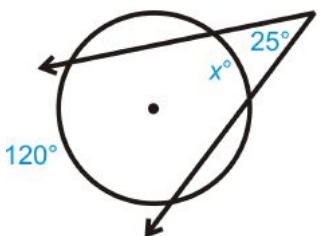
18.



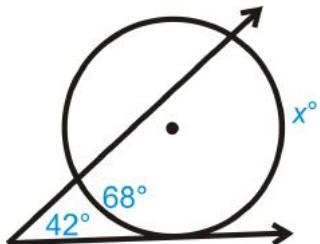
19.

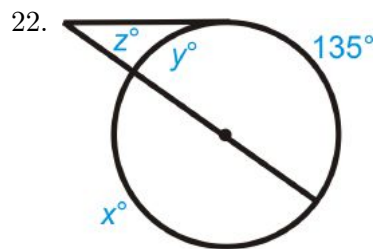


20.

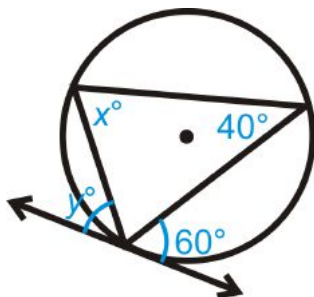


21.

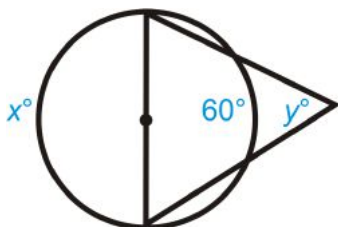




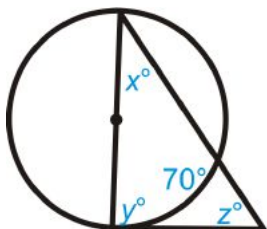
23. $y \neq 60^\circ$



24.

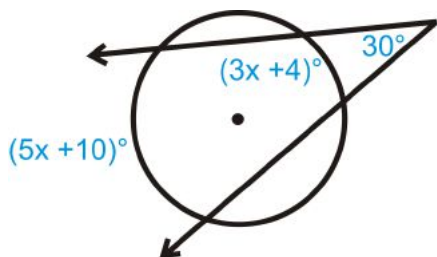


25.

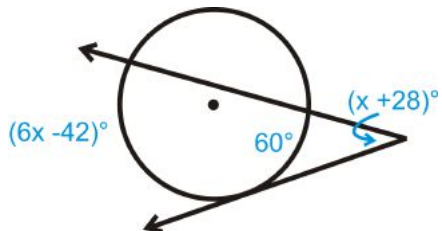


Challenge Solve for x .

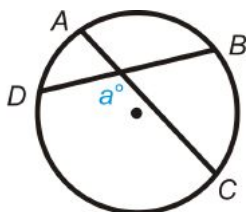
26.



27.



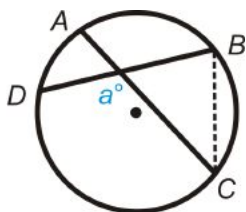
28. Fill in the blanks of the proof for Theorem 9-12.



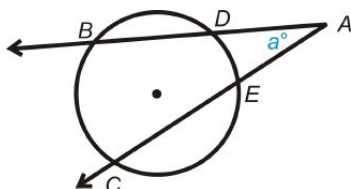
Given: Intersecting chords \overline{AC} and \overline{BD} .

Prove: $m\angle a = \frac{1}{2}(m\widehat{DC} + m\widehat{AB})$

Table 5.3:

<i>Statement</i>	<i>Reason</i>
1. Intersecting chords \overline{AC} and \overline{BD} .	
2. Draw \overline{BC}	Construction
	
3. $m\angle DBC = \frac{1}{2}m\widehat{DC}$ $m\angle ACB = \frac{1}{2}m\widehat{AB}$	
4. $m\angle a = m\angle DBC + m\angle ACB$	
5. $m\angle a = \frac{1}{2}m\widehat{DC} + \frac{1}{2}m\widehat{AB}$	

29. Fill in the blanks of the proof for Theorem 9-13.



Given: Secant rays \overrightarrow{AB} and \overrightarrow{AC}

Prove: $m\angle a = \frac{1}{2}(m\widehat{BC} - m\widehat{DE})$

Table 5.4:

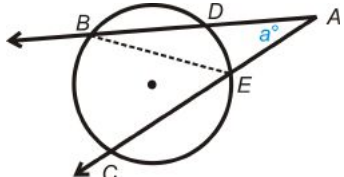
<i>Statement</i>	<i>Reason</i>
1. Intersecting secants \overrightarrow{AB} and \overrightarrow{AC} .	
2. Draw \overline{BE} .	Construction
	

Table 5.4: (continued)

<i>Statement</i>	<i>Reason</i>
3. $m\angle BEC = \frac{1}{2}m\widehat{BC}$ $m\angle DBE = \frac{1}{2}m\widehat{DE}$	
5. $m\angle a + m\angle DBE = m\angle BEC$	
6.	Subtraction PoE
7.	Substitution
8. $m\angle a = \frac{1}{2}(m\widehat{BC} - m\widehat{DE})$	

Review Queue Answers

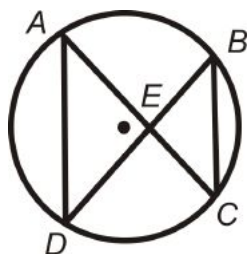
1. $m\angle OML = m\angle OPL = 90^\circ$ because a tangent line and a radius drawn to the point of tangency are perpendicular.
2. $165^\circ + m\angle OML + m\angle OPL + m\angle MLP = 360^\circ$
 $165^\circ + 90^\circ + 90^\circ + m\angle MLP = 360^\circ$
 $m\angle MLP = 15^\circ$
3. $m\widehat{MNP} = 360^\circ - 165^\circ = 195^\circ$

5.6 Segments of Chords, Secants, and Tangents

Learning Objectives

- Find the lengths of segments within circles.

Review Queue



1. What do you know about $m\angle DAC$ and $m\angle DBC$? Why?
2. What do you know about $m\angle AED$ and $m\angle BEC$? Why?
3. Is $\triangle AED \sim \triangle BEC$? How do you know?
4. If $AE = 8$, $ED = 7$, and $BE = 6$, find EC .

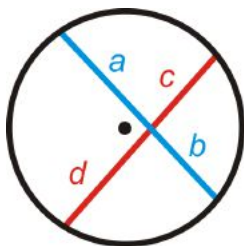
Know What? At a particular time during its orbit, the moon is 238,857 miles from Beijing, China. On the same line, Yukon is 12,451 miles from Beijing. Drawing another line from the moon to Cape Horn we see that Jakarta, Indonesia is collinear. If the distance from the moon to Jakarta is 240,128 miles, what is the distance from Cape Horn to Jakarta?



Segments from Chords

In the Review Queue above, we have two chords that intersect inside a circle. The two triangles are similar, making the sides in each triangle proportional.

Theorem 9-14: If two chords intersect inside a circle so that one is divided into segments of length a and b and the other into segments of length c and d then $ab = cd$.

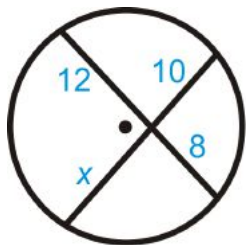


The product of the segments of one chord is equal to the product of segments of the second chord.

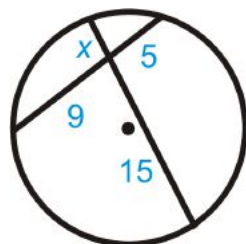
$$ab = cd$$

Example 1: Find x in each diagram below.

a)



b)



Solution: Use the ratio from Theorem 9-14.

a) $12 \cdot 8 = 10 \cdot x$

$$96 = 10x$$

$$9.6 = x$$

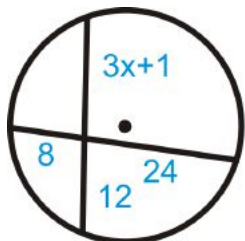
b) $x \cdot 15 = 5 \cdot 9$

$$15x = 45$$

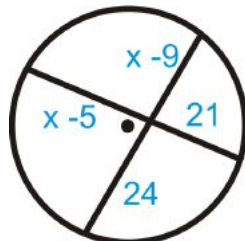
$$x = 3$$

Example 2: Algebra Connection Solve for x .

a)



b)



Solution: Use Theorem 9-13.

$$a) 8 \cdot 24 = (3x + 1) \cdot 12$$

$$192 = 36x + 12$$

$$180 = 36x$$

$$5 = x$$

$$b) (x - 5)21 = (x - 9)24$$

$$21x - 105 = 24x - 216$$

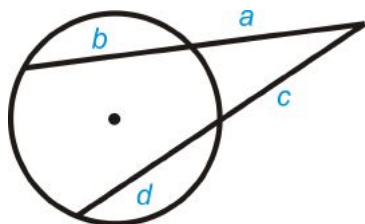
$$111 = 3x$$

$$37 = x$$

Segments from Secants

In addition to forming an angle outside of a circle, the circle can divide the secants into segments that are proportional with each other.

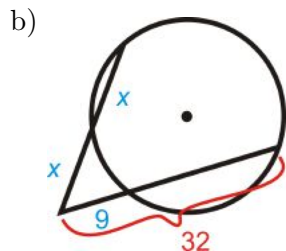
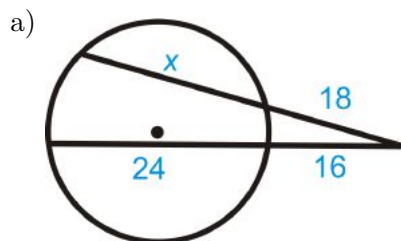
Theorem 9-15: If two secants are drawn from a common point outside a circle and the segments are labeled as below, then $a(a + b) = c(c + d)$.



The product of the outer segment and the whole of one secant equals the product of the outer segment and the whole of the other secant.

$$a(a + b) = c(c + d)$$

Example 3: Find the value of the missing variable.



Solution: Use Theorem 9-15 to set up an equation.

$$\text{a) } 18 \cdot (18 + x) = 16 \cdot (16 + 24)$$

$$324 + 18x = 256 + 384$$

$$18x = 316$$

$$x = 17\frac{5}{9}$$

$$\text{b) } x \cdot (x + x) = 9 \cdot 32$$

$$2x^2 = 288$$

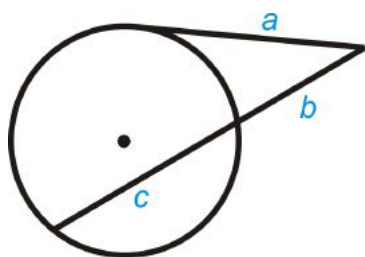
$$x^2 = 144$$

$$x = 12, x \neq -12 \text{ (length is not negative)}$$

Segments from Secants and Tangents

If a tangent and secant meet at a common point outside a circle, the segments created have a similar relationship to that of two secant rays in Example 3.

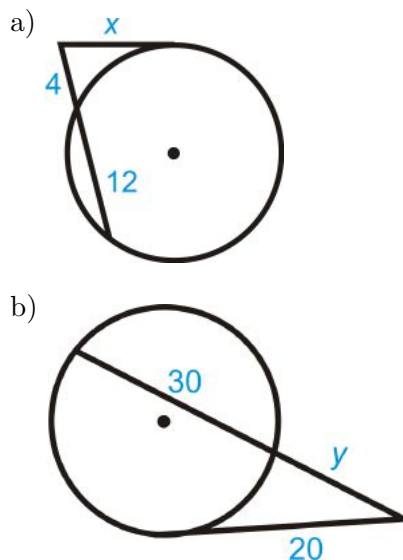
Theorem 9-16: If a tangent and a secant are drawn from a common point outside the circle (and the segments are labeled like the picture below), then $a^2 = b(b + c)$.



The product of the outside segment of the secant and the whole is equal to the square of the tangent.

$$a^2 = b(b + c)$$

Example 4: Find the value of the missing segment.



Solution: Use Theorem 9-16.

$$a) x^2 = 4(4 + 12)$$

$$x^2 = 4 \cdot 16 = 64$$

$$x = 8$$

$$b) 20^2 = y(y + 30)$$

$$400 = y^2 + 30y$$

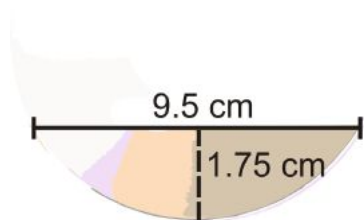
$$0 = y^2 + 30y - 400$$

$$0 = (y + 40)(y - 10)$$

$$y = -40, 10$$

When you have to factor a quadratic equation to find an answer, *always eliminate the negative answer* because length is never negative.

Example 5: Ishmael found a broken piece of a CD in his car. He places a ruler across two points on the rim, and the length of the chord is 9.5 cm. The distance from the midpoint of this chord to the nearest point on the rim is 1.75 cm. Find the diameter of the CD.

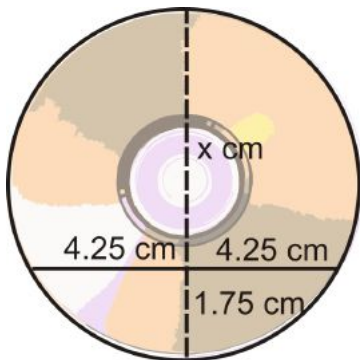


Solution: Think of this as two chords intersecting each other. If we were to extend the 1.75 cm segment, it would be a diameter. So, if we find x , in the diagram to the left, and add it to 1.75 cm, we would find the diameter.

$$4.25 \cdot 4.25 = 1.75 \cdot x$$

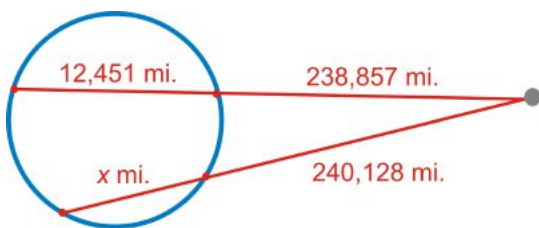
$$18.0625 = 1.75x$$

$x \approx 10.3$ cm, making the diameter 12 cm, which is the actual diameter of a CD.



Know What? Revisited The given information is to the left. Let's set up an equation using Theorem 9-15.

$$\begin{aligned} 238857 \cdot 251308 &= 240128(240128 + x) \\ 60026674956 &= 57661456380 + 240128x \\ 2365218572 &= 240128x \\ x &\approx 9849.8 \text{ miles} \end{aligned}$$

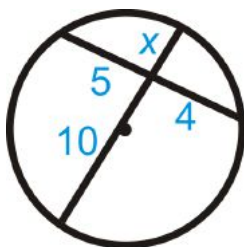


Review Questions

- Questions 1-25 are similar to Examples 1, 3, and 4.
- Questions 26-28 are similar to Example 2.
- Questions 29 is similar to Example 5.
- Questions 30 and 31 are proofs of Theorem 9-14 and 9-15.

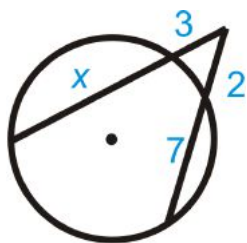
Fill in the blanks for each problem below. Then, solve for the missing segment.

1.



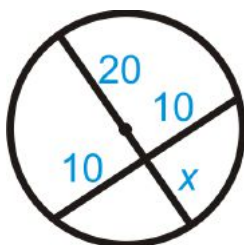
$$\underline{\hspace{1cm}} \cdot 4 = \underline{\hspace{1cm}} \cdot x$$

2.



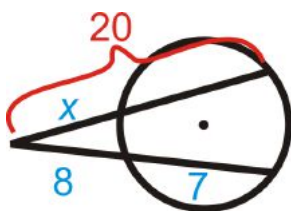
$$3(\underline{\quad} + \underline{\quad}) = 2(2 + 7)$$

3.



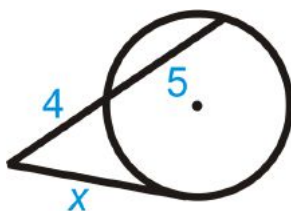
$$20x = \underline{\quad}$$

4.



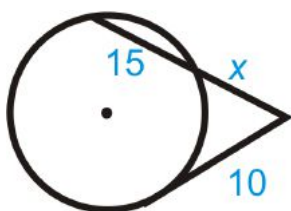
$$x \cdot \underline{\quad} = 8(\underline{\quad} + \underline{\quad})$$

5.



$$\underline{\quad} = \underline{\quad}(4 + 5)$$

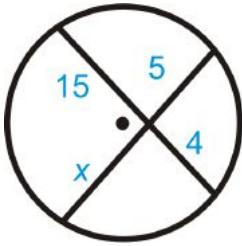
6.



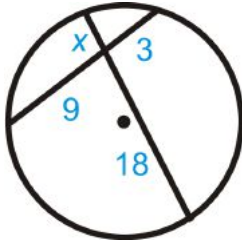
$$10^2 = x(\underline{\quad} + \underline{\quad})$$

Find x in each diagram below. Simplify any radicals.

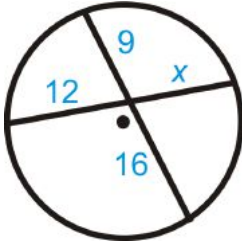
7.



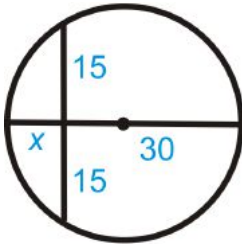
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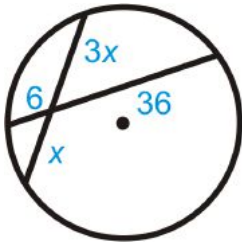
9.



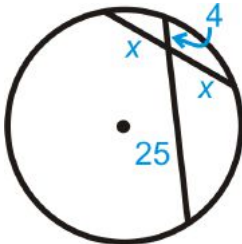
10.



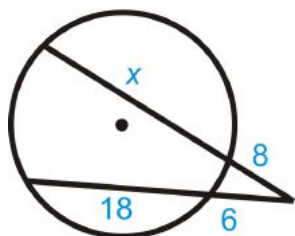
11.



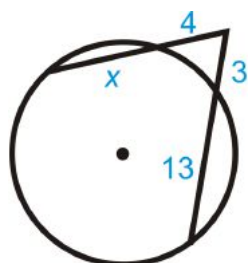
12.



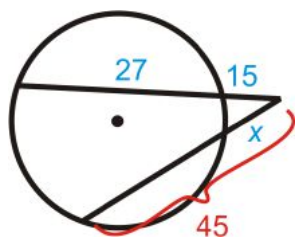
13.



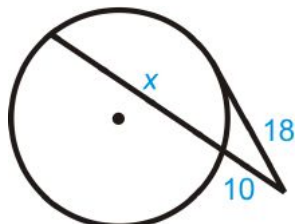
14.



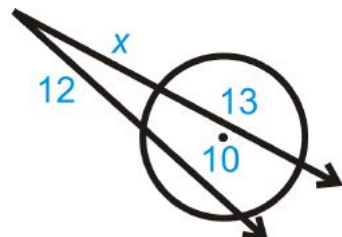
15.



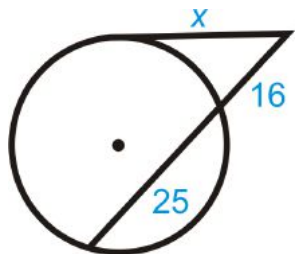
16.



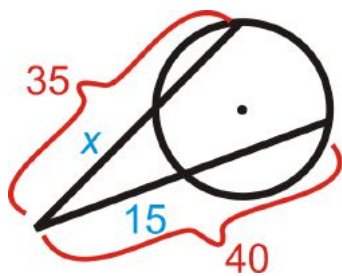
17.



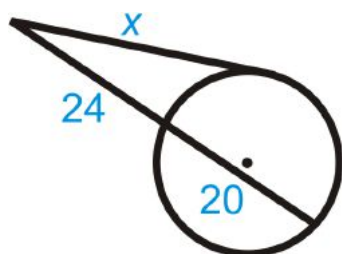
18.



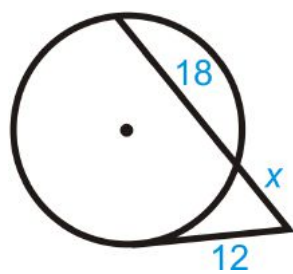
19.



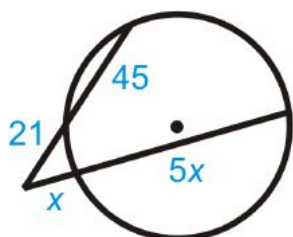
20.



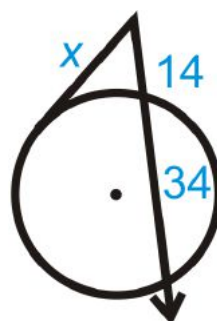
21.



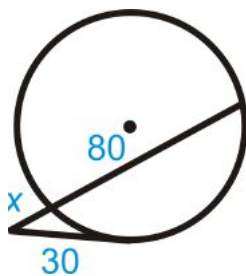
22.



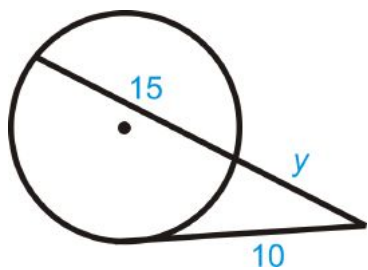
23.



24.



25. **Error Analysis** Describe and correct the error in finding y .



$$10 \cdot 10 = y \cdot 15y$$

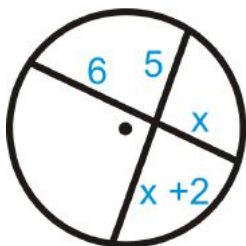
$$100 = 15y^2$$

$$\frac{20}{3} = y^2$$

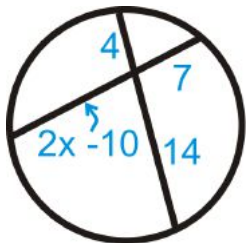
$$\frac{2\sqrt{15}}{3} = y \quad \leftarrow y \text{ in } \underline{\text{not}} \text{ correct}$$

Algebra Connection Find the value of x .

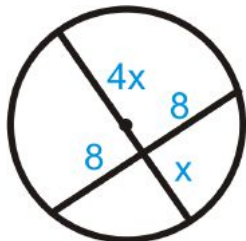
26.



27.

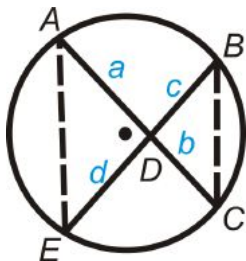


28.



29. Suzie found a piece of a broken plate. She places a ruler across two points on the rim, and the length of the chord is 6 inches. The distance from the midpoint of this chord to the nearest point on the rim is 1 inch. Find the diameter of the plate.

30. Fill in the blanks of the proof of Theorem 9-14.

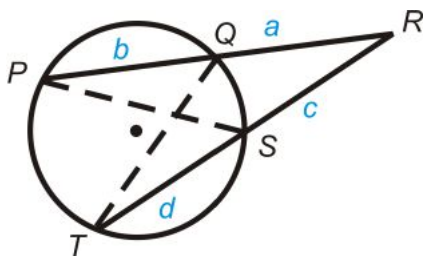


Given: Intersecting chords \overline{AC} and \overline{BE} .
Prove: $ab = cd$

Table 5.5:

<i>Statement</i>	<i>Reason</i>
1. Intersecting chords \overline{AC} and \overline{BE} with segments a , b , c , and d .	
2.	Theorem 9-8
3. $\triangle ADE \sim \triangle BDC$	
4.	Corresponding parts of similar triangles are proportional
5. $ab = cd$	

31. Fill in the blanks of the proof of Theorem 9-15.



Given: Secants \overline{PR} and \overline{RT}
Prove: $a(a + b) = c(c + d)$

Table 5.6:

<i>Statement</i>	<i>Reason</i>
1. Secants \overline{PR} and \overline{RT} with segments a , b , c , and d .	given
2. $\angle R \cong \angle R$	Reflexive PoC
3. $\angle QPS \cong \angle STQ$	Theorem 9-8
4. $\triangle RPS \sim \triangle RTQ$	AA Similarity Postulate
5. $\frac{a}{c+d} = \frac{c}{a+b}$	Corresponding parts of similar triangles are proportional
6. $a(a + b) = c(c + d)$	Cross multiplication

Review Queue Answers

- $m\angle DAC = m\angle DBC$ by Theorem 9-8, they are inscribed angles and intercept the same arc.
- $m\angle AED = m\angle BEC$ by the Vertical Angles Theorem.
- Yes, by AA Similarity Postulate.
- $$\frac{8}{6} = \frac{7}{EC}$$

$$8 \cdot EC = 42$$

$$EC = \frac{21}{4} = 5.25$$

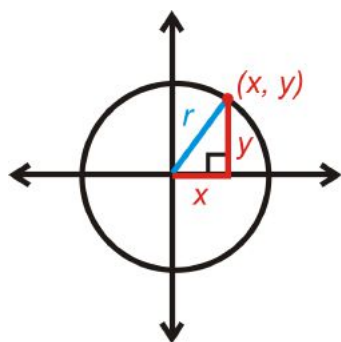
5.7 Extension: Writing and Graphing the Equations of Circles

Learning Objectives

- Graph a circle.
- Find the equation of a circle in the $x - y$ plane.
- Find the radius and center, given the equation of a circle and vice versa.
- Find the equation of a circle, given the center and a point on the circle.

Graphing a Circle in the Coordinate Plane

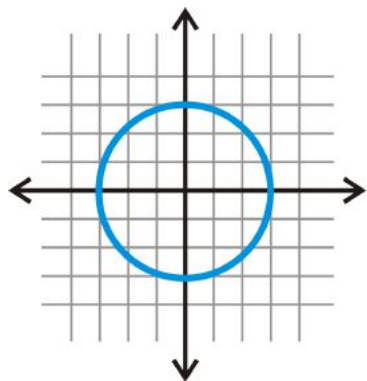
Recall that the definition of a circle is the set of all points that are the same distance from the center. This definition can be used to find an equation of a circle in the coordinate plane.



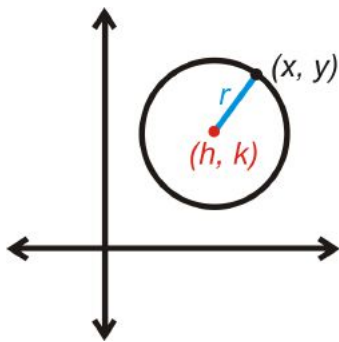
Let's start with the circle centered at $(0, 0)$. If (x, y) is a point on the circle, then the distance from the center to this point would be the radius, r . x is the horizontal distance y is the vertical distance. This forms a right triangle. From the Pythagorean Theorem, the equation of a circle, *centered at the origin* is $x^2 + y^2 = r^2$.

Example 1: Graph $x^2 + y^2 = 9$.

Solution: The center is $(0, 0)$. It's radius is the square root of 9, or 3. Plot the center, and then go out 3 units in every direction and connect them to form a circle.



The center does not always have to be on $(0, 0)$. If it is not, then we label the center (h, k) and would use the distance formula to find the length of the radius.



$$r = \sqrt{(x - h)^2 + (y - k)^2}$$

If you square both sides of this equation, then we would have the standard equation of a circle.

Standard Equation of a Circle: The standard equation of a circle with center (h, k) and radius r is $r^2 = (x - h)^2 + (y - k)^2$.

Example 2: Find the center and radius of the following circles.

a) $(x - 3)^2 + (y - 1)^2 = 25$

b) $(x + 2)^2 + (y - 5)^2 = 49$

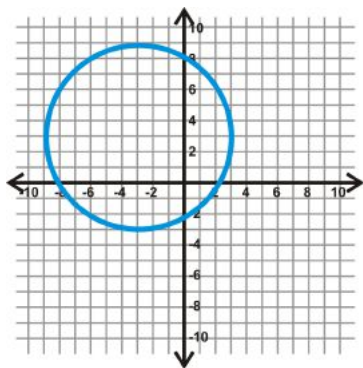
Solution:

a) Rewrite the equation as $(x - 3)^2 + (y - 1)^2 = 5^2$. The center is $(3, 1)$ and $r = 5$.

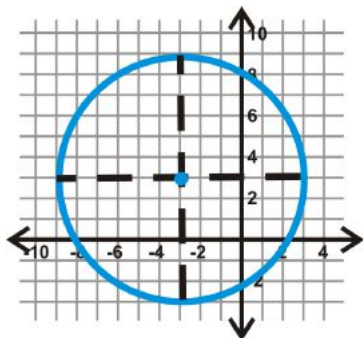
b) Rewrite the equation as $(x - (-2))^2 + (y - 5)^2 = 7^2$. The center is $(-2, 5)$ and $r = 7$.

When finding the center of a circle always take the **opposite sign** of what the value is in the equation.

Example 3: Find the equation of the circle below.



Solution: First locate the center. Draw in the horizontal and vertical diameters to see where they intersect.



From this, we see that the center is $(-3, 3)$. If we count the units from the center to the circle on either of these diameters, we find $r = 6$. Plugging this into the equation of a circle, we get: $(x - (-3))^2 + (y - 3)^2 = 6^2$ or $(x + 3)^2 + (y - 3)^2 = 36$.

Finding the Equation of a Circle

Example 4: Determine if the following points are on $(x + 1)^2 + (y - 5)^2 = 50$.

a) $(8, -3)$

b) $(-2, -2)$

Solution: Plug in the points for x and y in $(x + 1)^2 + (y - 5)^2 = 50$.

a) $(8 + 1)^2 + (-3 - 5)^2 = 50$

$$9^2 + (-8)^2 = 50$$

$$81 + 64 \neq 50$$

$(8, -3)$ is not on the circle

b) $(-2 + 1)^2 + (-2 - 5)^2 = 50$

$$(-1)^2 + (-7)^2 = 50$$

$$1 + 49 = 50$$

$(-2, -2)$ is on the circle

Example 5: Find the equation of the circle with center $(4, -1)$ and passes through $(-1, 2)$.

Solution: First plug in the center to the standard equation.

$$(x - 4)^2 + (y - (-1))^2 = r^2$$

$$(x - 4)^2 + (y + 1)^2 = r^2$$

Now, plug in $(-1, 2)$ for x and y and solve for r .

$$(-1 - 4)^2 + (2 + 1)^2 = r^2$$

$$(-5)^2 + (3)^2 = r^2$$

$$25 + 9 = r^2$$

$$34 = r^2$$

Substituting in 34 for r^2 , the equation is $(x - 4)^2 + (y + 1)^2 = 34$.

Review Questions

- Questions 1-4 are similar to Examples 1 and 2.
- Questions 5-8 are similar to Example 3.
- Questions 9-11 are similar to Example 4.
- Questions 12-15 are similar to Example 5.

Find the center and radius of each circle. Then, graph each circle.

1. $(x + 5)^2 + (y - 3)^2 = 16$

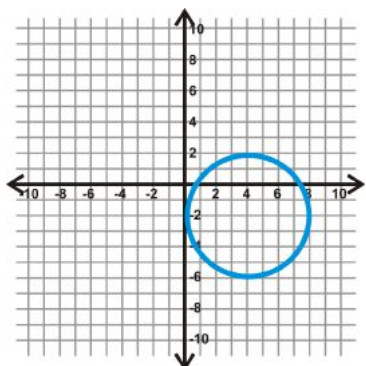
2. $x^2 + (y + 8)^2 = 4$

3. $(x - 7)^2 + (y - 10)^2 = 20$

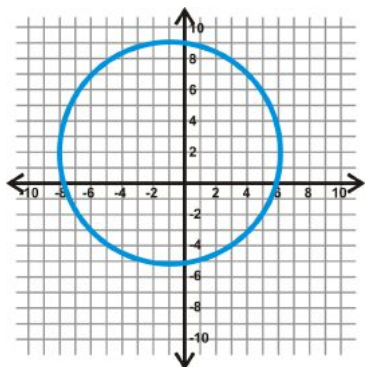
4. $(x + 2)^2 + y^2 = 8$

Find the equation of the circles below.

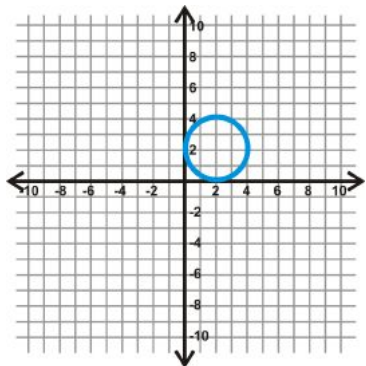
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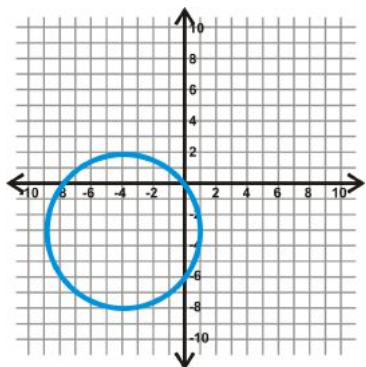
6.



7.



8.



9. Is $(-7, 3)$ on $(x + 1)^2 + (y - 6)^2 = 45$?

10. Is $(9, -1)$ on $(x - 2)^2 + (y - 2)^2 = 60$?

11. Is $(-4, -3)$ on $(x + 3)^2 + (y - 3)^2 = 37$?
12. Is $(5, -3)$ on $(x + 1)^2 + (y - 6)^2 = 45$?

Find the equation of the circle with the given center and point on the circle.

13. center: $(2, 3)$, point: $(-4, -1)$
14. center: $(10, 0)$, point: $(5, 2)$
15. center: $(-3, 8)$, point: $(7, -2)$
16. center: $(6, -6)$, point: $(-9, 4)$

5.8 Chapter 9 Review

Keywords & Theorems

Parts of Circles & Tangent Lines

- Circle
- Center
- Radius
- Chord
- Diameter
- Secant
- Tangent
- Point of Tangency
- Congruent Circles
- Concentric Circles
- Externally Tangent Circles
- Internally Tangent Circles
- Common Internal Tangent
- Common External Tangent
- Tangent to a Circle Theorem
- Theorem 9-2

Properties of Arcs

- Central Angle
- Arc
- Semicircle
- Minor Arc
- Major Arc
- Congruent Arcs
- Arc Addition Postulate

Properties of Chords

- Theorem 9-3
- Theorem 9-4

- Theorem 9-5
- Theorem 9-6

Inscribed Angles

- Inscribed Angle
- Intercepted Arc
- Inscribed Angle Theorem
- Theorem 9-8
- Theorem 9-9
- Inscribed Polygon
- Theorem 9-10

Angles from Chords, Secants and Tangents

- Theorem 9-11
- Theorem 9-12
- Theorem 9-13

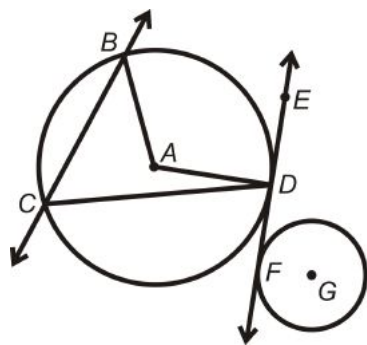
Segments from Secants and Tangents

- Theorem 9-14
- Theorem 9-15
- Theorem 9-16

Extension: Equations of Circles

- Standard Equation of a Circle

Vocabulary



Match the description with the correct label.

1. minor arc - A. \overline{CD}
2. chord - B. \overline{AD}
3. tangent line - C. \overleftrightarrow{CB}
4. central angle - D. \overleftrightarrow{EF}

5. secant - E. A
6. radius - F. D
7. inscribed angle - G. $\angle BAD$
8. center - H. $\angle BCD$
9. major arc - I. \widehat{BD}
10. point of tangency - J. \widehat{BCD}

Texas Instruments Resources

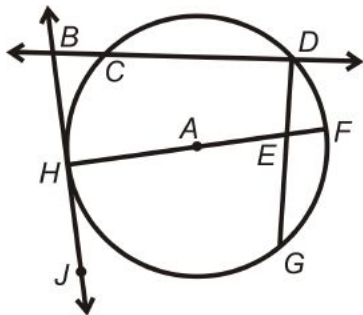
In the *CK-12 Texas Instruments Geometry FlexBook*, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9694>.

5.9 Study Guide

Keywords: Define, write theorems, and/or draw a diagram for each word below.

1st Section: Parts of Circles & Tangent Lines

Circle



Radius

Diameter

Tangent

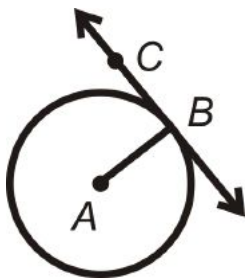
Congruent Circles

Concentric Circles

Externally Tangent Circles

Internally Tangent Circles

Common Internal Tangent



Common External Tangent
Tangent to a Circle Theorem
Theorem 9-2

Center

Chord

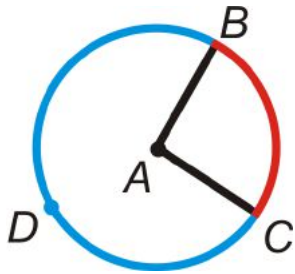
Secant

Point of Tangency

Homework:

2nd Section: Properties of Arcs

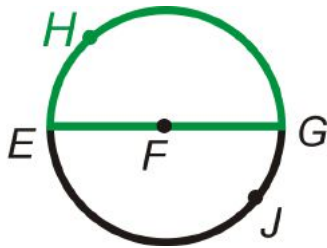
Central Angle



Arc

Semicircle

Minor Arc



Major Arc

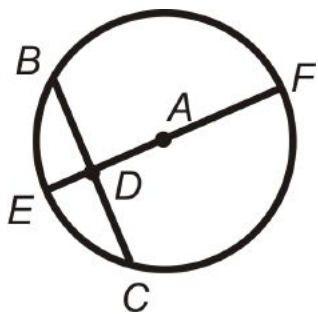
Congruent Arcs

Arc Addition Postulate

Homework:

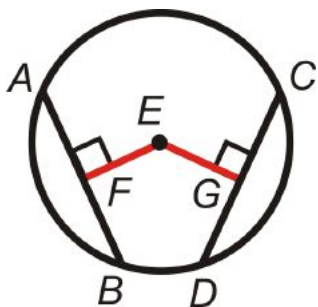
3rd Section: Properties of Chords

Theorem 9-3



Theorem 9-4

Theorem 9-5



Theorem 9-6

Homework:

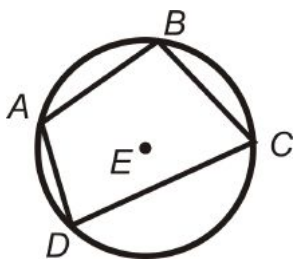
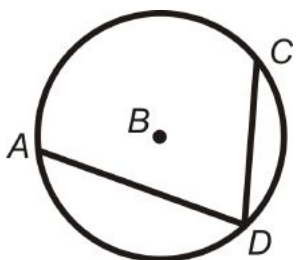
4th Section: Inscribed Angles

Inscribed Angle

Intercepted Arc

Inscribed Angle Theorem

Theorem 9-8



Theorem 9-9

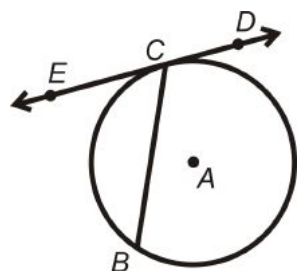
Inscribed Polygon

Theorem 9-10

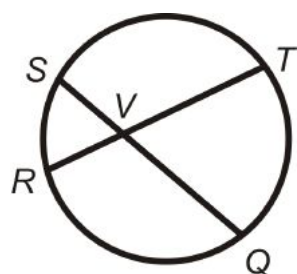
Homework:

5th Section: Angles from Chords, Secants and Tangents

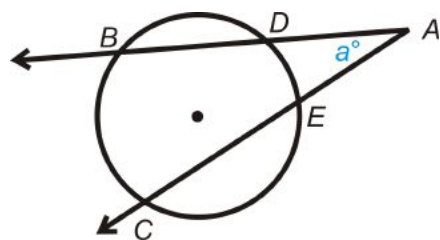
Theorem 9-11



Theorem 9-12



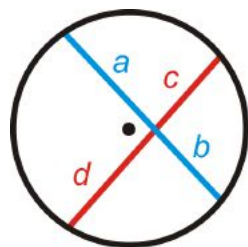
Theorem 9-13



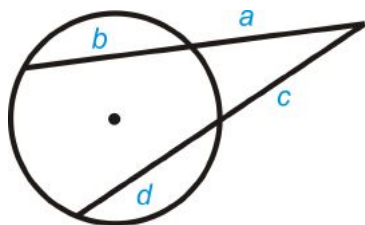
Homework:

6th Section: Segments from Secants and Tangents

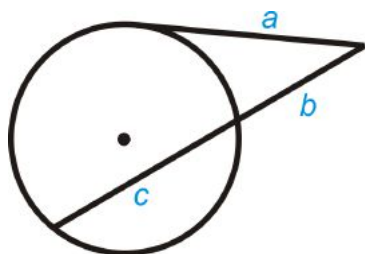
Theorem 9-14



Theorem 9-15



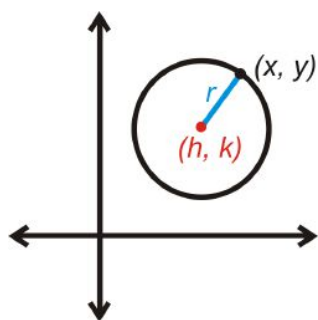
Theorem 9-16



Homework:

Extension: Equations of Circles

Standard Equation of a Circle



Homework:

Chapter 6

Perimeter and Area

Now that we have explored triangles, quadrilaterals, polygons, and circles, we are going to learn how to find the perimeter and area of each.

6.1 Triangles and Parallelograms

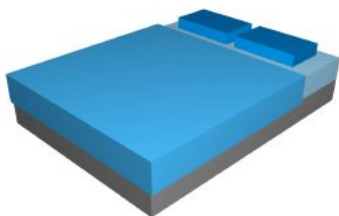
Learning Objectives

- Understand the basic concepts of area.
- Use formulas to find the area of triangles and parallelograms.

Review Queue

1. Define perimeter and area, in your own words.
2. Solve the equations below. Simplify any radicals.
 - (a) $x^2 = 121$
 - (b) $4x + 6 = 80$
 - (c) $x^2 - 6x + 8 = 0$
 - (d) $\frac{1}{2}x - 3 = 5$
 - (e) $x^2 + 2x - 15 = 0$
 - (f) $x^2 - x - 12 = 0$

Know What? Ed's parents are getting him a new king bed. Upon further research, Ed discovered there are two types of king beds, and Eastern (or standard) King and a California King. The Eastern King has $76'' \times 80''$ dimensions, while the California King is $72'' \times 84''$ (both dimensions are *width* \times *length*). Which bed has a larger area to lie on?



Areas and Perimeters of Squares and Rectangles

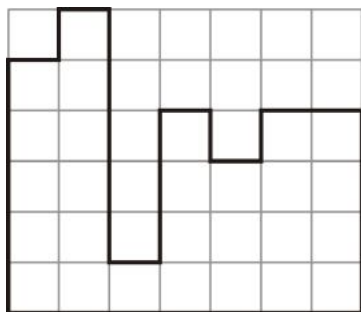
Perimeter: The distance around a shape.

The perimeter of any figure must have a unit of measurement attached to it. If no specific units are given (feet, inches, centimeters, etc), write “units.”

Example 1: Find the perimeter of the figure to the left.

Solution: Here, we can use the grid as our units. Count around the figure to find the perimeter.

$$5 + 1 + 1 + 1 + 5 + 1 + 3 + 1 + 1 + 1 + 1 + 2 + 4 + 7 = 34 \text{ units}$$

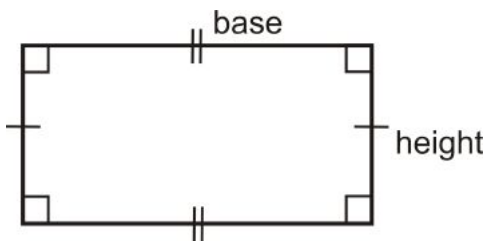


You are probably familiar with the area of squares and rectangles from a previous math class. Recall that you must always establish a unit of measure for area. Area is always measured in square units, square feet ($ft.^2$), square inches ($in.^2$), square centimeters ($cm.^2$), etc. If no specific units are given, write “ $units^2$.”

Example 2: Find the area of the figure from Example 1.

Solution: Count the number of squares within the figure. If we start on the left and count each column. $5 + 6 + 1 + 4 + 3 + 4 + 4 = 27 \text{ units}^2$

Area of a Rectangle: $A = bh$, where b is the base (width) and h is the height (length).



Example 3: Find the area and perimeter of a rectangle with sides 4 cm by 9 cm.

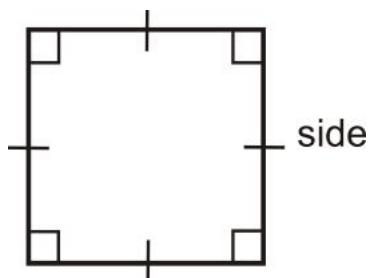


Solution: The perimeter is $4 + 9 + 4 + 9 = 36 \text{ cm}$. The area is $A = 9 \cdot 4 = 36 \text{ cm}^2$.

Perimeter of a Rectangle: $P = 2b + 2h$.

If a rectangle is a square, with sides of length s , the formulas are as follows:

Perimeter of a Square: $P_{\text{square}} = 2s + 2s = 4s$



Area of a Square: $A_{square} = s \cdot s = s^2$

Example 4: The area of a square is 75 in^2 . Find the perimeter.

Solution: To find the perimeter, we need to find the length of the sides.

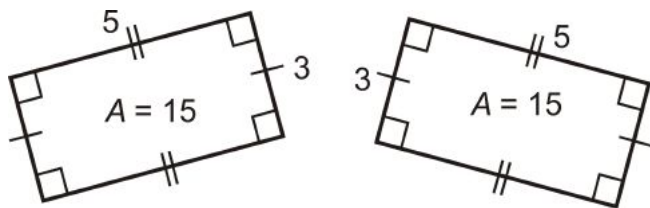
$$A = s^2 = 75 \text{ in}^2$$

$$s = \sqrt{75} = 5\sqrt{3} \text{ in}$$

From this, $P = 4(5\sqrt{3}) = 20\sqrt{3} \text{ in}$.

Area Postulates

Congruent Areas Postulate: If two figures are congruent, they have the same area.



Example 5: Draw two different rectangles with an area of 36 cm^2 .

Solution: Think of all the different factors of 36. These can all be dimensions of the different rectangles.

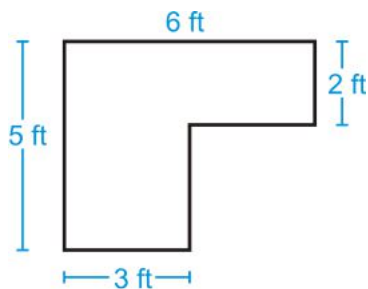


Other possibilities could be 6×6 , 2×18 , and 1×36 .

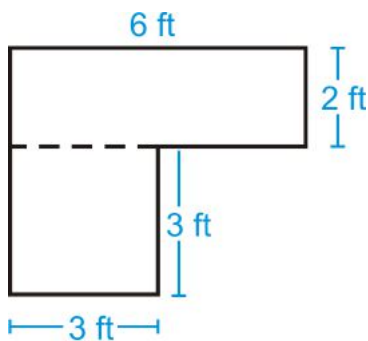
Example 5 shows two rectangles with the same area and are not congruent. This tells us that the converse of the Congruent Areas Postulate is not true.

Area Addition Postulate: If a figure is composed of two or more parts that do not overlap each other, then the area of the figure is the sum of the areas of the parts.

Example 6: Find the area of the figure below. You may assume all sides are perpendicular.



Solution: Split the shape into two rectangles and find the area of each.



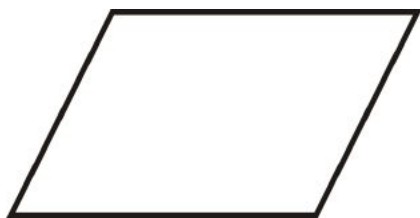
$$A_{\text{top rectangle}} = 6 \cdot 2 = 12 \text{ ft}^2$$

$$A_{\text{bottom square}} = 3 \cdot 3 = 9 \text{ ft}^2$$

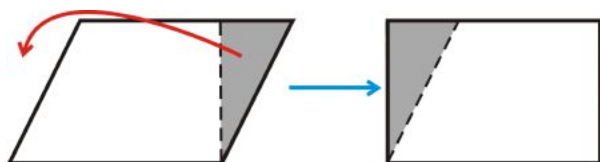
The total area is $12 + 9 = 21 \text{ ft}^2$.

Area of a Parallelogram

Recall that a parallelogram is a quadrilateral whose opposite sides are parallel.



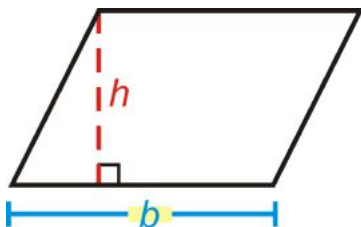
To find the area of a parallelogram, make it into a rectangle.



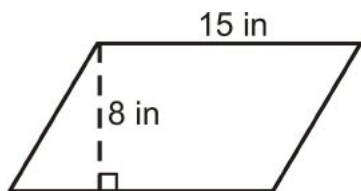
From this, we see that the area of a parallelogram is the same as the area of a rectangle.

Area of a Parallelogram: The area of a parallelogram is $A = bh$.

The height of a parallelogram is always perpendicular to the base. This means that the sides are *not* the height.



Example 7: Find the area of the parallelogram.



Solution: $A = 15 \cdot 8 = 120 \text{ in}^2$

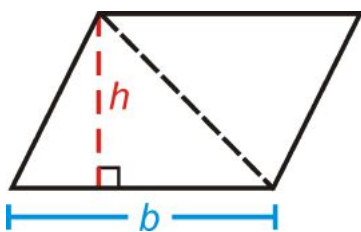
Example 8: If the area of a parallelogram is 56 units^2 and the base is 4 units, what is the height?

Solution: Solve for the height in $A = bh$.

$$56 = 4h$$

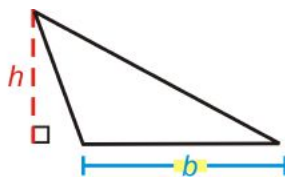
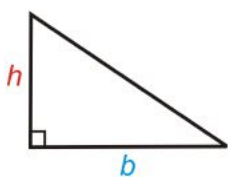
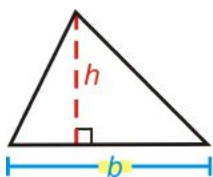
$$14 = h$$

Area of a Triangle

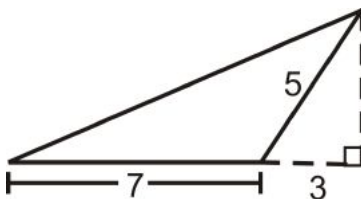


If we take parallelogram and cut it in half, along a diagonal, we would have two congruent triangles. The formula for the area of a triangle is half the area of a parallelogram.

Area of a Triangle: $A = \frac{1}{2} bh$ or $A = \frac{bh}{2}$.



Example 9: Find the area of the triangle.



Solution: To find the area, we need to find the height of the triangle. We are given the two sides of the

small right triangle, where the hypotenuse is also the short side of the obtuse triangle.

$$3^2 + h^2 = 5^2$$

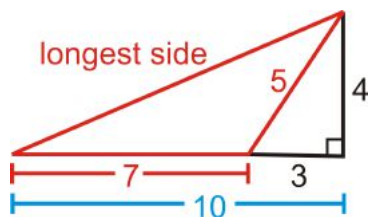
$$9 + h^2 = 25$$

$$h^2 = 16$$

$$h = 4$$

$$A = \frac{1}{2}(4)(7) = 14 \text{ units}^2$$

Example 10: Find the perimeter of the triangle in Example 9.



Solution: To find the perimeter, we need to find the longest side of the obtuse triangle. If we used the black lines in the picture, we would see that the longest side is also the hypotenuse of the right triangle with legs 4 and 10.

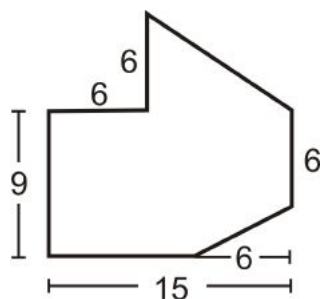
$$4^2 + 10^2 = c^2$$

$$16 + 100 = c^2$$

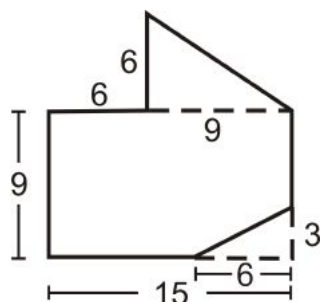
$$c = \sqrt{116} \approx 10.77$$

The perimeter is $7 + 5 + 10.77 = 22.77 \text{ units}$

Example 11: Find the area of the figure below.



Solution: Divide the figure into a triangle and a rectangle with a small rectangle cut out of the lower right-hand corner.



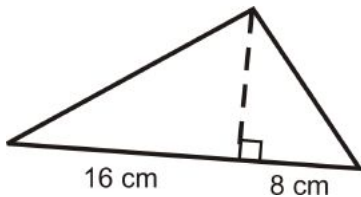
$$\begin{aligned}
 A &= A_{\text{top triangle}} + A_{\text{rectangle}} - A_{\text{small triangle}} \\
 A &= \left(\frac{1}{2} \cdot 6 \cdot 9\right) + (9 \cdot 15) + \left(\frac{1}{2} \cdot 3 \cdot 6\right) \\
 A &= 27 + 135 + 9 \\
 A &= 171 \text{ units}^2
 \end{aligned}$$

Know What? Revisited The area of an Eastern King is 6080 in^2 and the California King is 6048 in^2 .

Review Questions

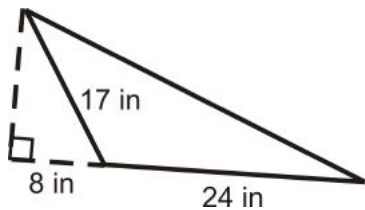
- Questions 1-12 are similar to Examples 3-5, 7-9.
 - Questions 13-18 are similar to Examples 9 and 10.
 - Questions 19-24 are similar to Examples 7 and 9.
 - Questions 25-30 are similar to Examples 6 and 11.
 - Questions 31-36 use the formula for the area of a triangle.
1. Find the area and perimeter of a square with sides of length 12 in.
 2. Find the area and perimeter of a rectangle with height of 9 cm and base of 16 cm.
 3. Find the area of a parallelogram with height of 20 m and base of 18 m.
 4. Find the area and perimeter of a rectangle if the height is 8 and the base is 14.
 5. Find the area and perimeter of a square if the sides are 18 ft.
 6. If the area of a square is 81 ft^2 , find the perimeter.
 7. If the perimeter of a square is 24 in, find the area.
 8. Find the area of a triangle with base of length 28 cm and height of 15 cm.
 9. What is the height of a triangle with area 144 m^2 and a base of 24 m?
 10. The perimeter of a rectangle is 32. Find two different dimensions that the rectangle could be.
 11. Draw two different rectangles that haven an area of 90 mm^2 .
 12. Write the converse of the Congruent Areas Postulate. Determine if it is a true statement. If not, write a counterexample. If it is true, explain why.

Use the triangle to answer the following questions.



13. Find the height of the triangle by using the geometric mean.
14. Find the perimeter.
15. Find the area.

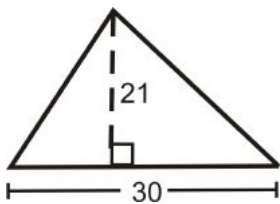
Use the triangle to answer the following questions.



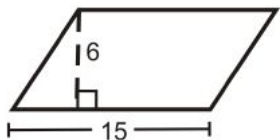
16. Find the height of the triangle.
17. Find the perimeter.
18. Find the area.

Find the area of the following shapes.

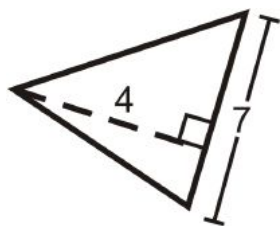
19.



20.



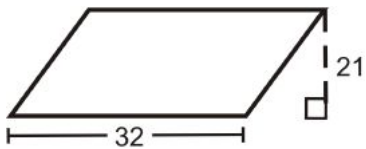
21.



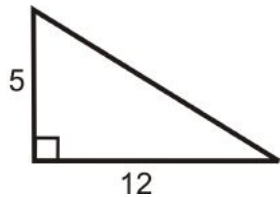
22.



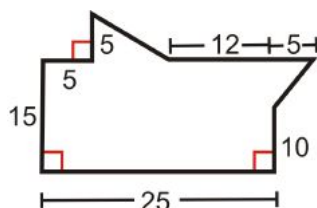
23.



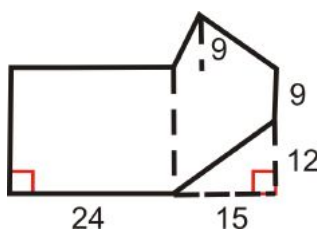
24.



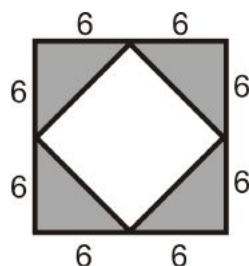
25. (a) Divide the shape into two triangles and one rectangle.
- (b) Find the area of the two triangles and rectangle.
- (c) Find the area of the entire shape.



26. (a) Divide the shape into two rectangles and one triangle.
 (b) Find the area of the two rectangles and triangle.
 (c) Find the area of the entire shape (you will need to subtract the area of the small triangle in the lower right-hand corner).

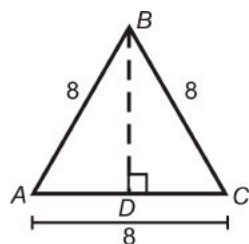


Use the picture below for questions 27-30. Both figures are squares.



27. Find the area of the outer square.
 28. Find the area of one grey triangle.
 29. Find the area of all four grey triangles.
 30. Find the area of the inner square.

In questions 31-36 we are going to derive a formula for the area of an equilateral triangle.



31. What kind of triangle is $\triangle ABD$? Find AD and BD .
 32. Find the area of $\triangle ABC$.
 33. If each side is x , what is AD and BD ?
 34. If each side is x , find the area of $\triangle ABC$.
 35. Using your formula from #34, find the area of an equilateral triangle with 12 inch sides.
 36. Using your formula from #34, find the area of an equilateral triangle with 5 inch sides.

Review Queue Answers

1. Possible Answers

Perimeter: The distance around a shape.

Area: The space inside a shape.

2. (a) $x = \pm 11$

(b) $x = 18.5$

(c) $x = 4, 2$

(d) $x = 16$

(e) $x = 3, -5$

(f) $x = 4, -3$

6.2 Trapezoids, Rhombi, and Kites

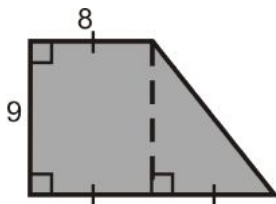
Learning Objectives

- Derive and use the area formulas for trapezoids, rhombi, and kites.

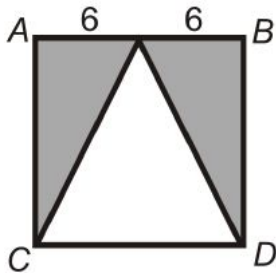
Review Queue

Find the area of the *shaded* regions in the figures below.

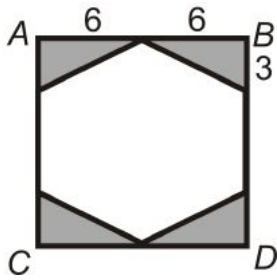
1.



2. $ABCD$ is a square.



3. $ABCD$ is a square.



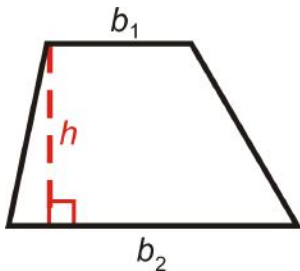
Know What? The Brazilian flag is to the right. The flag has dimensions of 20×14 (units vary depending on the size, so we will not use any here). The vertices of the yellow rhombus in the middle are 1.7 units from the midpoint of each side.



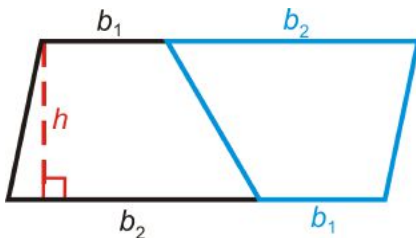
Find the area of the rhombus (including the circle). *Do not round your answer.*

Area of a Trapezoid

Recall that a trapezoid is a quadrilateral with one pair of parallel sides. The lengths of the parallel sides are the bases and the perpendicular distance between the parallel sides is the height of the trapezoid.



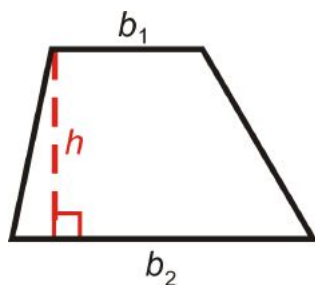
To find the area of the trapezoid, make a copy of the trapezoid and then rotate the copy 180° . Now, this is a parallelogram with height h and base $b_1 + b_2$. The area of this shape is $A = h(b_1 + b_2)$.



Because the area of this parallelogram is two congruent trapezoids, the area of one trapezoid would be $A = \frac{1}{2}h(b_1 + b_2)$.

Area of a Trapezoid: $A = \frac{1}{2}h(b_1 + b_2)$

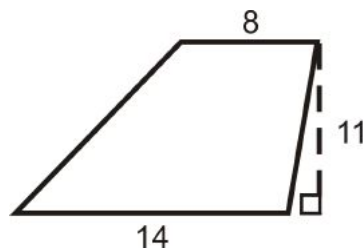
h is *always* perpendicular to the bases.



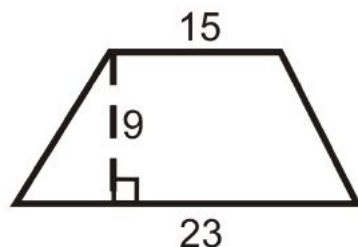
You could also say the area of a trapezoid is the average of the bases times the height.

Example 1: Find the area of the trapezoids below.

a)



b)



Solution:

$$\text{a) } A = \frac{1}{2}(11)(14 + 8)$$

$$A = \frac{1}{2}(11)(22)$$

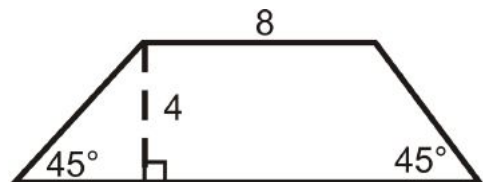
$$A = 121 \text{ units}^2$$

$$\text{b) } A = \frac{1}{2}(9)(15 + 23)$$

$$A = \frac{1}{2}(9)(38)$$

$$A = 171 \text{ units}^2$$

Example 2: Find the perimeter and area of the trapezoid.



Solution: Even though we are not told the length of the second base, we can find it using special right triangles. Both triangles at the ends of this trapezoid are isosceles right triangles, so the hypotenuses are $4\sqrt{2}$ and the other legs are of length 4.

$$P = 8 + 4\sqrt{2} + 16 + 4\sqrt{2}$$

$$P = 24 + 8\sqrt{2} \approx 35.3 \text{ units}$$

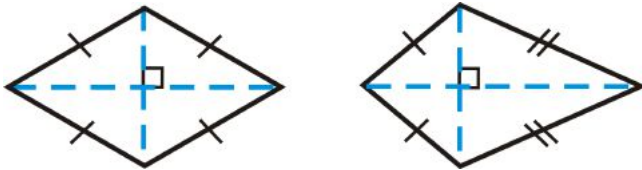
$$A = \frac{1}{2}(4)(8 + 16)$$

$$A = 48 \text{ units}^2$$

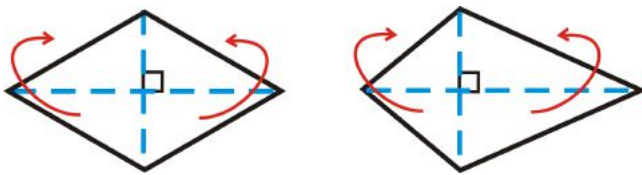
Area of a Rhombus and Kite

Recall that a rhombus is an equilateral quadrilateral and a kite has adjacent congruent sides.

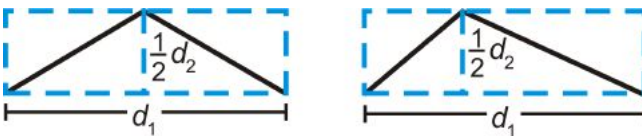
Both of these quadrilaterals have perpendicular diagonals, which is how we are going to find their areas.



Notice that the diagonals divide each quadrilateral into 4 triangles. If we move the two triangles on the bottom of each quadrilateral so that they match up with the triangles above the horizontal diagonal, we would have two rectangles.

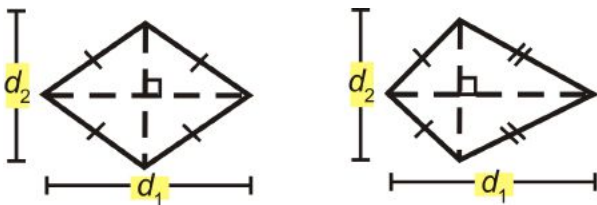


So, the height of these rectangles is half of one of the diagonals and the base is the length of the other diagonal.



Area of a Rhombus: $A = \frac{1}{2}d_1d_2$

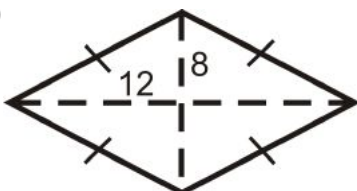
The area is half the product of the diagonals.



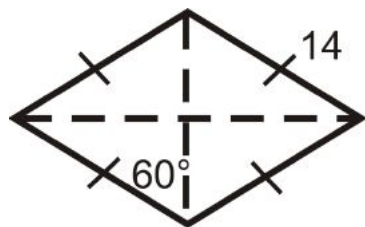
Area of a Kite: $A = \frac{1}{2}d_1d_2$

Example 3: Find the perimeter and area of the rhombi below.

a)



b)



Solution: In a rhombus, all four triangles created by the diagonals are congruent.

a) To find the perimeter, you must find the length of each side, which would be the hypotenuse of one of the four triangles. Use the Pythagorean Theorem.

$$12^2 + 8^2 = \text{side}^2$$

$$144 + 64 = \text{side}^2$$

$$\text{side} = \sqrt{208} = 4\sqrt{13}$$

$$P = 4(4\sqrt{13}) = 16\sqrt{13}$$

$$A = \frac{1}{2} \cdot 16 \cdot 24$$

$$A = 192$$

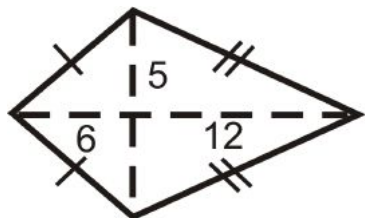
b) Here, each triangle is a 30-60-90 triangle with a hypotenuse of 14. From the special right triangle ratios the short leg is 7 and the long leg is $7\sqrt{3}$.

$$P = 4 \cdot 14 = 56$$

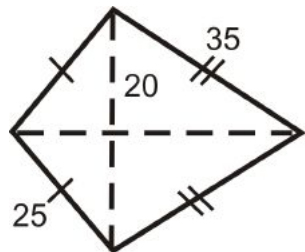
$$A = \frac{1}{2} \cdot 14 \cdot 14\sqrt{3} = 98\sqrt{3}$$

Example 4: Find the perimeter and area of the kites below.

a)



b)



Solution: In a kite, there are two pairs of congruent triangles. Use the Pythagorean Theorem in both problems to find the length of sides or diagonals.

a)

Shorter sides of kite

$$6^2 + 5^2 = s_1^2$$

$$36 + 25 = s_1^2$$

$$s_1 = \sqrt{61}$$

Longer sides of kite

$$12^2 + 5^2 = s_2^2$$

$$144 + 25 = s_2^2$$

$$s_2 = \sqrt{169} = 13$$

$$P = 2(\sqrt{61}) + 2(13) = 2\sqrt{61} + 26 \approx 41.6$$

$$A = \frac{1}{2}(10)(18) = 90$$

b)

Smaller diagonal portion

$$20^2 + d_s^2 = 25^2$$

$$d_s^2 = 225$$

$$d_s = 15$$

Larger diagonal portion

$$20^2 + d_l^2 = 35^2$$

$$d_l^2 = 825$$

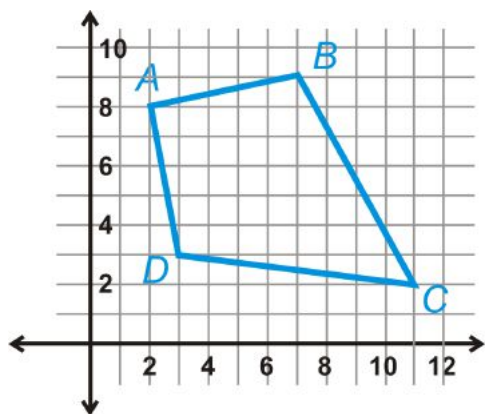
$$d_l = 5\sqrt{33}$$

$$A = \frac{1}{2}(15 + 5\sqrt{33})(40) \approx 874.5$$

$$P = 2(25) + 2(35) = 120$$

Example 5: The vertices of a quadrilateral are $A(2, 8)$, $B(7, 9)$, $C(11, 2)$, and $D(3, 3)$. Show $ABCD$ is a kite and find its area.

Solution: After plotting the points, it looks like a kite. $AB = AD$ and $BC = DC$. The diagonals are perpendicular if the slopes are opposite signs and flipped.



$$m_{AC} = \frac{2 - 8}{11 - 2} = -\frac{6}{9} = -\frac{2}{3}$$

$$m_{BD} = \frac{9 - 3}{7 - 3} = \frac{6}{4} = \frac{3}{2}$$

The diagonals are perpendicular, so $ABCD$ is a kite. To find the area, we need to find the length of the diagonals.

$$\begin{aligned} d_1 &= \sqrt{(2 - 11)^2 + (8 - 2)^2} \\ &= \sqrt{(-9)^2 + 6^2} \\ &= \sqrt{81 + 36} = \sqrt{117} = 3\sqrt{13} \end{aligned}$$

$$\begin{aligned} d_2 &= \sqrt{(7 - 3)^2 + (9 - 3)^2} \\ &= \sqrt{4^2 + 6^2} \\ &= \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13} \end{aligned}$$

Plug these lengths into the area formula for a kite. $A = \frac{1}{2}(3\sqrt{13})(2\sqrt{13}) = 39 \text{ units}^2$

Know What? Revisited To find the area of the rhombus, we need to find the length of the diagonals. One diagonal is $20 - 1.7 - 1.7 = 16.6$ and the other is $14 - 1.7 - 1.7 = 10.6$. The area is $A = \frac{1}{2}(16.6)(10.6) = 87.98 \text{ units}^2$.

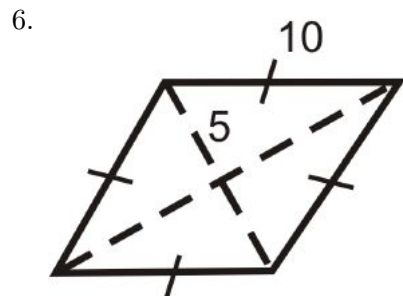
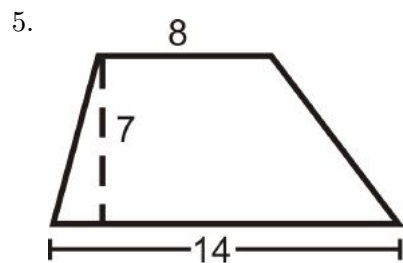
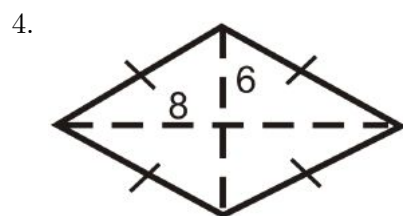
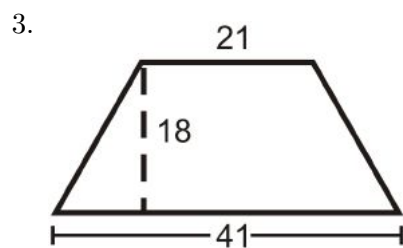
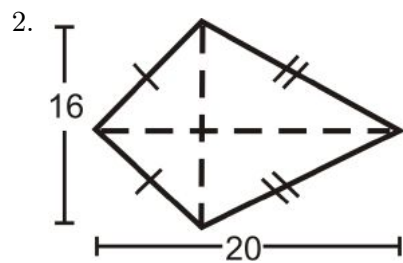
Review Questions

- Question 1 uses the formula of the area of a kite and rhombus.
- Questions 2-16 are similar to Examples 1-4.

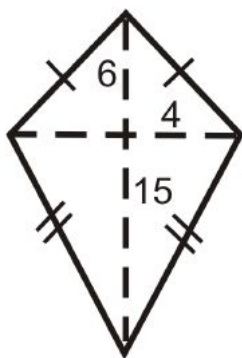
- Questions 17-23 are similar to Example 5.
- Questions 24-27 use the area formula for a kite and rhombus and factors.
- Questions 28-30 are similar to Example 4.

1. Do you think all rhombi and kites with the same diagonal lengths have the same area? *Explain* your answer.

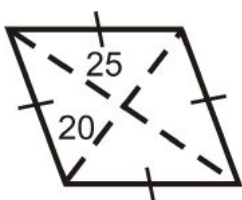
Find the area of the following shapes. *Round your answers to the nearest hundredth.*



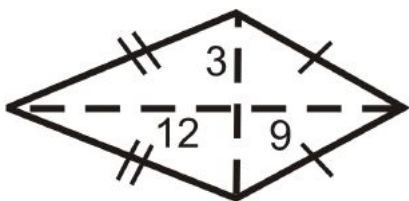
7.



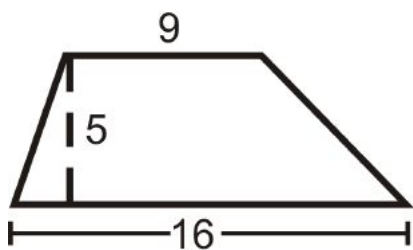
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9.

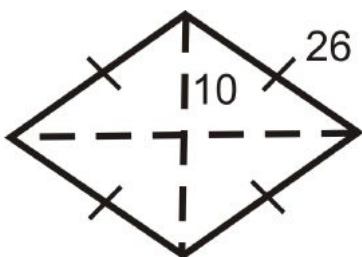


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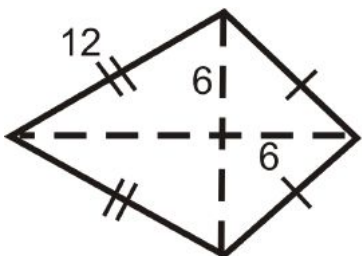


Find the area and perimeter of the following shapes. *Round your answers to the nearest hundredth.*

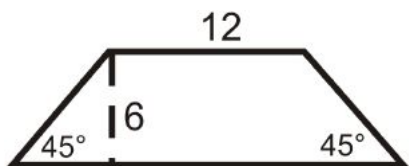
11.



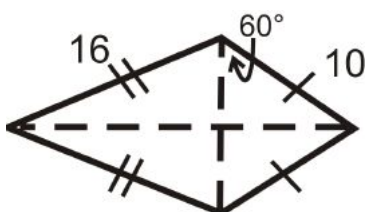
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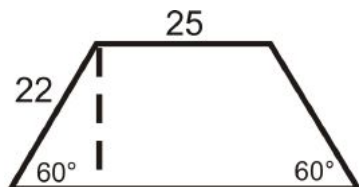
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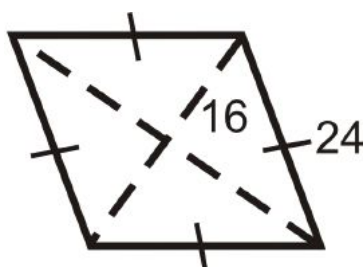
14.



15.



16.



Quadrilateral $ABCD$ has vertices $A(-2, 0)$, $B(0, 2)$, $C(4, 2)$, and $D(0, -2)$. Leave your answers in simplest radical form.

17. Find the slopes of \overline{AB} and \overline{DC} . What type of quadrilateral is this? *Plotting the points will help you find the answer.*
18. Find the slope of \overline{AD} . Is it perpendicular to \overline{AB} and \overline{DC} ?
19. Find AB , AD , and DC .
20. Use #19 to find the area of the shape.

Quadrilateral $EFGH$ has vertices $E(2, -1)$, $F(6, -4)$, $G(2, -7)$, and $H(-2, -4)$.

21. Find the slopes of all the sides and diagonals. What type of quadrilateral is this? *Plotting the points will help you find the answer.*
22. Find HF and EG .
23. Use #22 to find the area of the shape.

For Questions 24 and 25, the area of a rhombus is 32 units^2 .

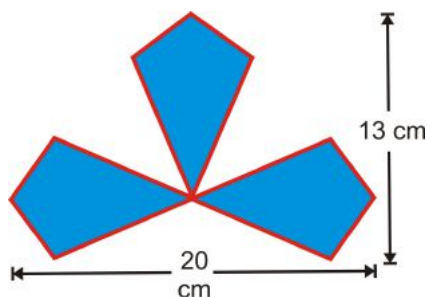
24. What would the product of the diagonals have to be for the area to be 32 units^2 ?
25. List two possibilities for the length of the diagonals, based on your answer from #24.

For Questions 26 and 27, the area of a kite is 54 units^2 .

26. What would the product of the diagonals have to be for the area to be 54 units^2 ?
27. List two possibilities for the length of the diagonals, based on your answer from #26.

Sherry designed the logo for a new company, made up of 3 congruent kites.

28. What are the lengths of the diagonals for one kite?
29. Find the area of one kite.
30. Find the area of the entire logo.



Review Queue Answers

1. $A = 9(8) + \left[\frac{1}{2}(9)(8)\right] = 72 + 36 = 108 \text{ units}^2$
2. $A = \frac{1}{2}(6)(12)2 = 72 \text{ units}^2$
3. $A = 4\left[\frac{1}{2}(6)(3)\right] = 36 \text{ units}^2$

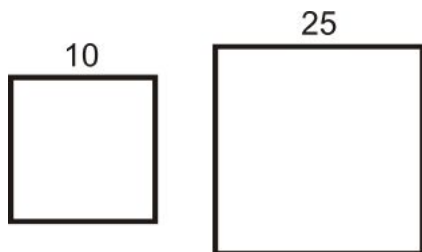
6.3 Areas of Similar Polygons

Learning Objectives

- Understand the relationship between the scale factor of similar polygons and their areas.
- Apply scale factors to solve problems about areas of similar polygons.

Review Queue

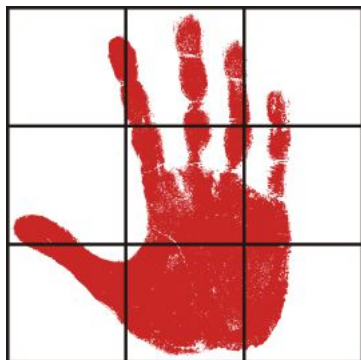
1. Are two squares similar? Are two rectangles?



2. Find the scale factor of the sides of the similar shapes. Both figures are squares.
3. Find the area of each square.
4. Find the ratio of the smaller square's area to the larger square's area. Reduce it.

Know What? One use of scale factors and areas is scale drawings. This technique takes a small object, like the handprint to the right, divides it up into smaller squares and then blows up the individual squares.

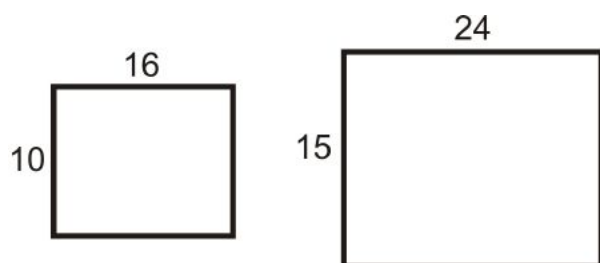
In this Know What? you are going to make a scale drawing of your own hand. Trace your hand on a piece of paper. Then, divide your hand into 9 squares, like the one to the right, $2\text{ in} \times 2\text{ in}$. Take a larger piece of paper and blow up each square to be $6\text{ in} \times 6\text{ in}$ (you will need at least an 18 in square piece of paper). Once you have your $6\text{ in} \times 6\text{ in}$ squares drawn, use the proportions and area to draw in your enlarged handprint.



Areas of Similar Polygons

In Chapter 7, we learned about similar polygons. Polygons are similar when the corresponding angles are equal and the corresponding sides are in the same proportion.

Example 1: The two rectangles below are similar. Find the scale factor and the ratio of the perimeters.



Solution: The scale factor is $\frac{16}{24} = \frac{2}{3}$.

$$P_{small} = 2(10) + 2(16) = 52 \text{ units}$$

$$P_{large} = 2(15) + 2(24) = 78 \text{ units}$$

The ratio of the perimeters is $\frac{52}{78} = \frac{2}{3}$.

The ratio of the perimeters is the same as the scale factor. In fact, the ratio of any part of two similar shapes (diagonals, medians, midsegments, altitudes, etc.) is the same as the scale factor.

Example 2: Find the area of each rectangle from Example 1. Then, find the ratio of the areas.

Solution:

$$A_{small} = 10 \cdot 16 = 160 \text{ units}^2$$

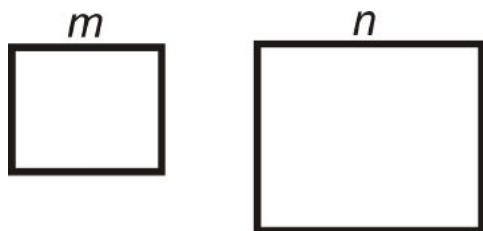
$$A_{large} = 15 \cdot 24 = 360 \text{ units}^2$$

The ratio of the areas would be $\frac{160}{360} = \frac{4}{9}$.

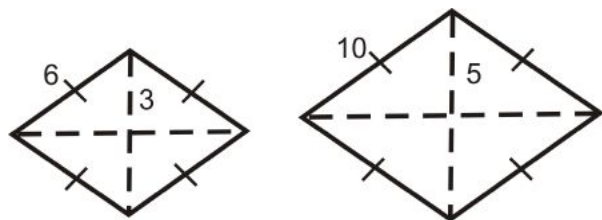
The ratio of the sides, or scale factor was $\frac{2}{3}$ and the ratio of the areas is $\frac{4}{9}$. Notice that the ratio of the areas is the **square** of the scale factor.

Area of Similar Polygons Theorem: If the scale factor of the sides of two similar polygons is $\frac{m}{n}$, then the ratio of the areas would be $\left(\frac{m}{n}\right)^2$.

If the scale factor is $\frac{m}{n}$, then the ratio of the areas is $\left(\frac{m}{n}\right)^2$.



Example 3: Find the ratio of the areas of the rhombi below. The rhombi are similar.



Solution: Find the ratio of the sides and square it.

$$\left(\frac{3}{5}\right)^2 = \frac{9}{25}$$

Example 4: Two trapezoids are similar. If the scale factor is $\frac{3}{4}$ and the area of the smaller trapezoid is 81 cm^2 , what is the area of the larger trapezoid?

Solution: First, the ratio of the areas would be $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$. Now, we need the area of the larger trapezoid. To find this, set up a proportion using the area ratio.

$$\begin{aligned} \frac{9}{16} &= \frac{81}{A} \rightarrow 9A = 1296 \\ A &= 144 \text{ cm}^2 \end{aligned}$$

Example 5: Two triangles are similar. The ratio of the areas is $\frac{25}{64}$. What is the scale factor?

Solution: The scale factor is $\sqrt{\frac{25}{64}} = \frac{5}{8}$.

Example 6: Using the ratios from Example 5, find the length of the base of the smaller triangle if the length of the base of the larger triangle is 24 units.

Solution: Set up a proportion using the scale factor.

$$\begin{aligned} \frac{5}{8} &= \frac{b}{24} \rightarrow 8b = 120 \\ b &= 15 \text{ units} \end{aligned}$$

Know What? Revisited You should end up with an $18 \text{ in} \times 18 \text{ in}$ drawing of your handprint.

Review Questions

- Questions 1-4 are similar to Example 3.

- Questions 5-8 are similar to Example 5.
- Questions 9-18 are similar to Examples 1-3, and 5.
- Questions 19-22 are similar to Examples 4 and 6.
- Questions 23-26 are similar to Examples 5 and 6.

Determine the ratio of the areas, given the ratio of the sides of a polygon.

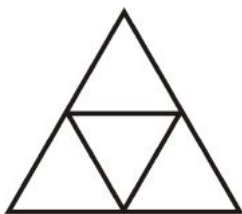
1. $\frac{3}{5}$
2. $\frac{1}{4}$
3. $\frac{7}{2}$
4. $\frac{6}{11}$

Determine the ratio of the sides of a polygon, given the ratio of the areas.

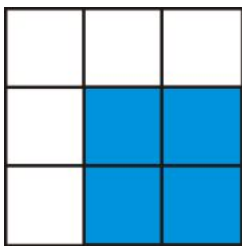
5. $\frac{1}{36}$
6. $\frac{4}{81}$
7. $\frac{49}{9}$
8. $\frac{25}{144}$

This is an equilateral triangle made up of 4 congruent equilateral triangles.

9. What is the ratio of the areas of the large triangle to one of the small triangles?



10. What is the scale factor of large to small triangle?
11. If the area of the large triangle is 20 units^2 , what is the area of a small triangle?
12. If the length of the altitude of a small triangle is $2\sqrt{3}$, find the perimeter of the large triangle.



13. Find the perimeter of the large square and the blue square.
14. Find the scale factor of the blue square and large square.
15. Find the ratio of their perimeters.
16. Find the area of the blue and large squares.
17. Find the ratio of their areas.
18. Find the length of the diagonals of the blue and large squares. Put them into a ratio. Which ratio is this the same as?
19. Two rectangles are similar with a scale factor of $\frac{4}{7}$. If the area of the larger rectangle is 294 in^2 , find the area of the smaller rectangle.
20. Two triangles are similar with a scale factor of $\frac{1}{3}$. If the area of the smaller triangle is 22 ft^2 , find the area of the larger triangle.

21. The ratio of the areas of two similar squares is $\frac{16}{81}$. If the length of a side of the smaller square is 24 units, find the length of a side in the larger square.
22. The ratio of the areas of two right triangles is $\frac{4}{9}$. If the length of the hypotenuse of the larger triangle is 48 units, find the length of the smaller triangle's hypotenuse.

Questions 23-26 build off of each other. You may assume the problems are connected.

23. Two similar rhombi have areas of 72 units^2 and 162 units^2 . Find the ratio of the areas.
24. Find the scale factor.
25. The diagonals in these rhombi are congruent. Find the length of the diagonals and the sides.
26. What type of rhombi are these quadrilaterals?

Review Queue Answers

1. Two squares are always similar. Two rectangles can be similar as long as the sides are in the same proportion.
2. $\frac{10}{25} = \frac{2}{5}$
3. $A_{small} = 100, A_{large} = 625$

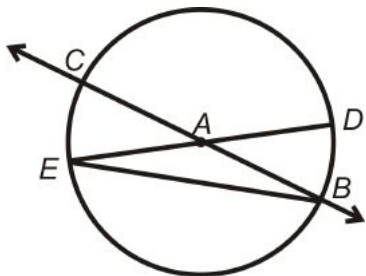
6.4 Circumference and Arc Length

Learning Objectives

- Find the circumference of a circle.
- Define the length of an arc and find arc length.

Review Queue

1. Find a central angle in that intercepts \widehat{CE}



2. Find an inscribed angle that intercepts \widehat{CE} .
3. How many degrees are in a circle? Find $m\widehat{ECD}$.
4. If $m\widehat{CE} = 26^\circ$, find $m\widehat{CD}$ and $m\angle CBE$.

Know What? A typical large pizza has a diameter of 14 inches and is cut into 8 pieces. Think of the crust as the circumference of the pizza. Find the *length* of the crust for the entire pizza. Then, find the length of the crust for one piece of pizza if the entire pizza is cut into 8 pieces.



Circumference of a Circle

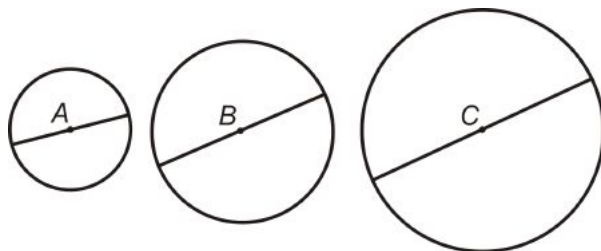
Circumference: The distance around a circle.

The circumference can also be called the perimeter of a circle. However, we use the term circumference for circles because they are round. In order to find the circumference of a circle, we need to explore π (pi).

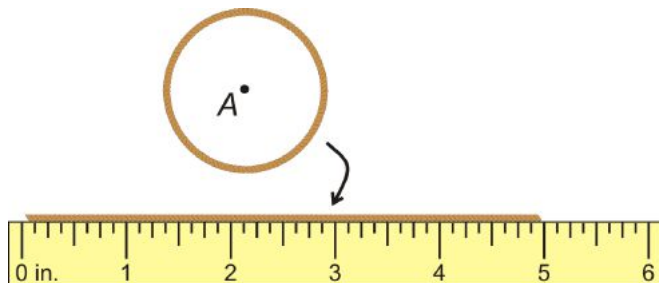
Investigation 10-1: Finding π (pi)

Tools Needed: paper, pencil, compass, ruler, string, and scissors

1. Draw three circles with radii of 2 in, 3 in, and 4 in. Label the centers of each A , B , and C .
2. Draw in the diameters and determine their lengths.



3. Take the string and outline each circle with it. Cut the string so that it perfectly outlines the circle. Then, lay it out straight and measure it in inches. Round your answer to the nearest $\frac{1}{8}$ -inch. Repeat this for the other two circles.



4. Find $\frac{\text{circumference}}{\text{diameter}}$ for each circle. Record your answers to the nearest thousandth.

You should see that $\frac{\text{circumference}}{\text{diameter}}$ approaches 3.14159... We call this number π , the Greek letter “pi.” When finding the circumference and area of circles, we must use π .

π , or “pi”: The ratio of the circumference of a circle to its diameter. It is approximately equal to 3.14159265358979323846...

To see more digits of π , go to <http://www.eveandersson.com/pi/digits/>.

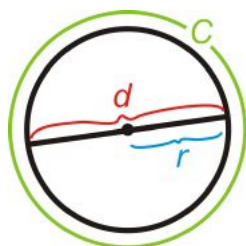
From Investigation 10-1, we found that $\frac{\text{circumference}}{\text{diameter}} = \pi$. Let’s solve for the circumference, C .

$$\frac{C}{d} = \pi$$

$$C = \pi d$$

We can also say $C = 2\pi r$ because $d = 2r$.

Circumference Formula: $C = \pi d$ or $C = 2\pi r$



$$d = 2r$$

Example 1: Find the circumference of a circle with a radius of 7 cm.

Solution: Plug the radius into the formula.

$$C = 2\pi(7) = 14\pi \approx 44 \text{ cm}$$

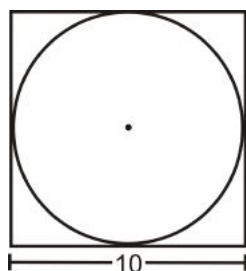
Example 2: The circumference of a circle is 64π . Find the diameter.

Solution: Again, you can plug in what you know into the circumference formula and solve for d .

$$64\pi = \pi d$$

$$64 = d$$

Example 3: A circle is inscribed in a square with 10 in. sides. What is the circumference of the circle? Leave your answer in terms of π .



Solution: From the picture, we can see that the diameter of the circle is equal to the length of a side. $C = 10\pi \text{ in.}$

Example 4: Find the perimeter of the square. Is it more or less than the circumference of the circle? Why?

Solution: The perimeter is $P = 4(10) = 40 \text{ in.}$ In order to compare the perimeter with the circumference we should change the circumference into a decimal.

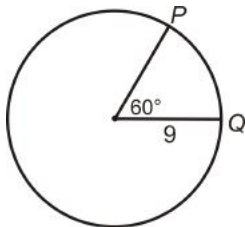
$C = 10\pi \approx 31.42 \text{ in.}$ This is less than the perimeter of the square, which makes sense because the circle is inside the square.

Arc Length

In Chapter 9, we measured arcs in degrees. This was called the “arc measure” or “degree measure.” Arcs can also be measured in length, as a portion of the circumference.

Arc Length: The length of an arc or a portion of a circle’s circumference.

The arc length is directly related to the degree arc measure.

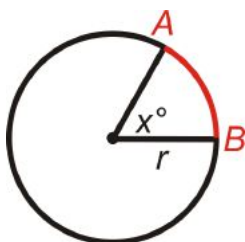


Example 5: Find the length of \widehat{PQ} . Leave your answer in terms of π .

Solution: In the picture, the central angle that corresponds with \widehat{PQ} is 60° . This means that $m\widehat{PQ} = 60^\circ$. Think of the arc length as a portion of the circumference. There are 360° in a circle, so 60° would be $\frac{1}{6}$ of that ($\frac{60^\circ}{360^\circ} = \frac{1}{6}$). Therefore, the length of \widehat{PQ} is $\frac{1}{6}$ of the circumference. $length\ of\ \widehat{PQ} = \frac{1}{6} \cdot 2\pi(9) = 3\pi$

Arc Length Formula: The length of $\widehat{AB} = \frac{m\widehat{AB}}{360^\circ} \cdot \pi d$ or $\frac{m\widehat{AB}}{360^\circ} \cdot 2\pi r$.

Another way to write this could be $\frac{x^\circ}{360^\circ} \cdot 2\pi r$, where x is the central angle.



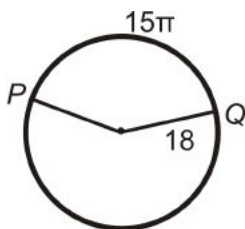
Example 6: The arc length of $\widehat{AB} = 6\pi$ and is $\frac{1}{4}$ the circumference. Find the radius of the circle.

Solution: If 6π is $\frac{1}{4}$ the circumference, then the total circumference is $4(6\pi) = 24\pi$. To find the radius, plug this into the circumference formula and solve for r .

$$24\pi = 2\pi r$$

$$12 = r$$

Example 7: Find the measure of the central angle or \widehat{PQ} .



Solution: Let’s plug in what we know to the Arc Length Formula.

$$15\pi = \frac{m\widehat{PQ}}{360^\circ} \cdot 2\pi(18)$$

$$15 = \frac{m\widehat{PQ}}{10^\circ}$$

$$150^\circ = m\widehat{PQ}$$

Example 8: The tires on a compact car are 18 inches in diameter. How far does the car travel after the tires turn once? How far does the car travel after 2500 rotations of the tires?



Solution: One turn of the tire is the circumference. This would be $C = 18\pi \approx 56.55$ in. 2500 rotations would be $2500 \cdot 56.55$ in = 141371.67 in, 11781 ft, or 2.23 miles.

Know What? Revisited The entire length of the crust, or the circumference of the pizza is $14\pi \approx 44$ in. In $\frac{1}{8}$ of the pizza, one piece would have $\frac{44}{8} \approx 5.5$ inches of crust.

Review Questions

- Questions 1-10 are similar to Examples 1 and 2.
- Questions 11-14 are similar to Examples 3 and 4.
- Questions 15-20 are similar to Example 5.
- Questions 21-23 are similar to Example 6.
- Questions 24-26 are similar to Example 7.
- Questions 27-30 are similar to Example 8.

Fill in the following table. Leave all answers in terms of π .

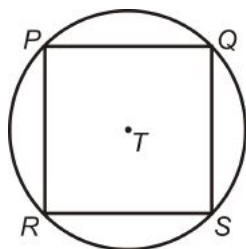
Table 6.1:

	<i>diameter</i>	<i>radius</i>	<i>circumference</i>
1.	15		
2.		4	
3.	6		
4.			84π
5.		9	
6.			25π
7.			2π
8.	36		

9. Find the radius of circle with circumference 88 in.

10. Find the circumference of a circle with $d = \frac{20}{\pi}$ cm.

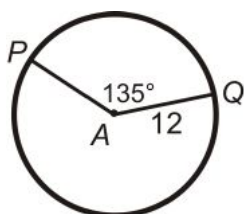
Square $PQSR$ is inscribed in $\odot T$. $RS = 8\sqrt{2}$.



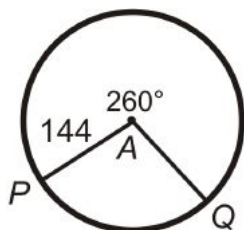
11. Find the length of the diameter of $\odot T$.
12. How does the diameter relate to $PQSR$?
13. Find the perimeter of $PQSR$.
14. Find the circumference of $\odot T$.

Find the arc length of \widehat{PQ} in $\odot A$. Leave your answers in terms of π .

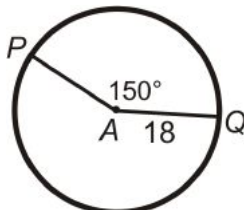
15.



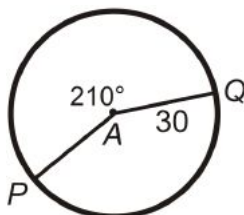
16.



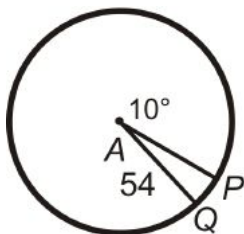
17.



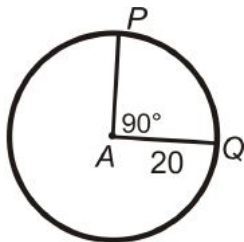
18.



19.

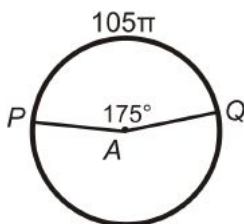


20.

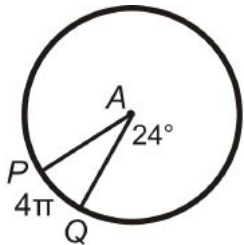


Find PA (the radius) in $\odot A$. Leave your answer in terms of π .

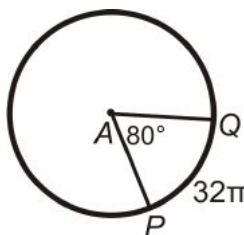
21.



22.

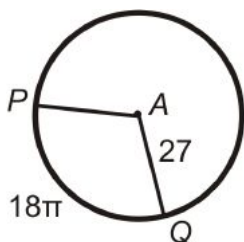


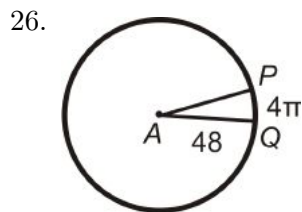
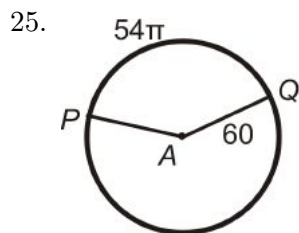
23.



Find the central angle or $m\widehat{PQ}$ in $\odot A$. Round any decimal answers to the nearest tenth.

24.





For questions 27-30, a truck has tires with a 26 in diameter.

27. How far does the truck travel every time a tire turns exactly once? What is this the same as?
28. How many times will the tire turn after the truck travels 1 mile? (1 mile = 5280 feet)
29. The truck has travelled 4072 tire rotations. How many miles is this?
30. The average recommendation for the life of a tire is 30,000 miles. How many rotations is this?

Review Queue Answers

1. $\angle CAE$
2. $\angle CBE$
3. $360^\circ, 180^\circ$
4. $m\widehat{CD} = 180^\circ - 26^\circ = 154^\circ, m\angle CBE = 13^\circ$

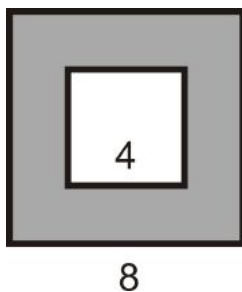
6.5 Areas of Circles and Sectors

Learning Objectives

- Find the area of circles, sectors, and segments.

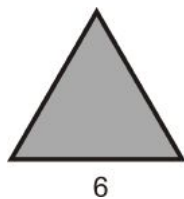
Review Queue

1. Find the area of both squares.
2. Find the area of the shaded region.



3. The triangle to the right is an equilateral triangle.

- (a) Find the height of the triangle.
- (b) Find the area of the triangle.



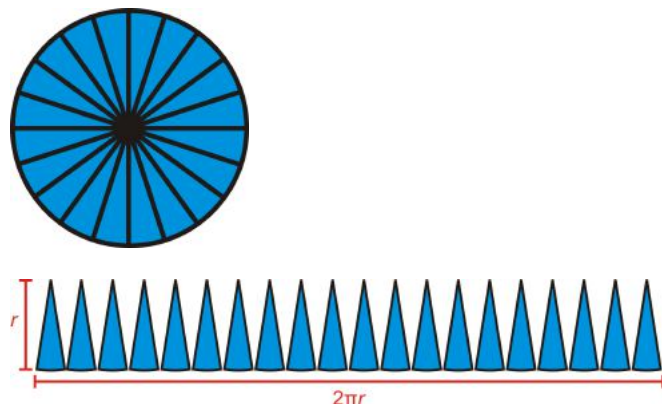
Know What? Back to the pizza. In the previous section, we found the length of the crust for a 14 in pizza. However, crust typically takes up some area on a pizza. Round your answers to the nearest hundredth.



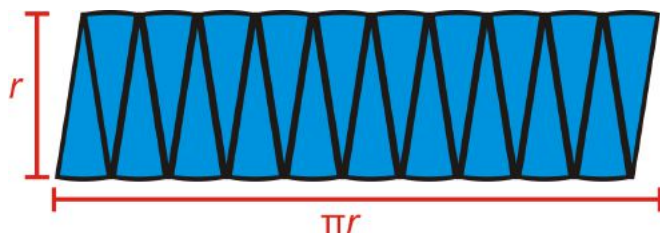
- a) Find the area of the crust of a deep-dish 16 in pizza. A typical deep-dish pizza has 1 in of crust around the toppings.
- b) A thin crust pizza has $\frac{1}{2}$ -in of crust around the edge of the pizza. Find the area of a thin crust 16 in pizza.

Area of a Circle

Take a circle and divide it into several wedges. Then, unfold the wedges so they are in a line, with the points at the top.



The height of the wedges is the radius and the length is the circumference of the circle. Now, take half of these wedges and flip them upside-down and place them so they all fit together.



Now our circle looks like a parallelogram. The area of this parallelogram is $A = bh = \pi r \cdot r = \pi r^2$.

To see an animation of this derivation, see <http://www.rkm.com.au/ANIMATIONS/animation-Circle-Area-Derivation.html>, by Russell Knightley.

Area of a Circle: If r is the radius of a circle, then $A = \pi r^2$.

Example 1: Find the area of a circle with a diameter of 12 cm.

Solution: If $d = 12$ cm, then $r = 6$ cm. The area is $A = \pi(6^2) = 36\pi$ cm².

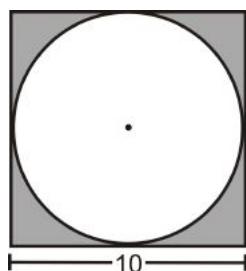
Example 2: If the area of a circle is 20π , what is the radius?

Solution: Plug in the area and solve for the radius.

$$\begin{aligned} 20\pi &= \pi r^2 \\ 20 &= r^2 \\ r &= \sqrt{20} = 2\sqrt{5} \end{aligned}$$

Just like the circumference, we will leave our answers in terms of π , unless otherwise specified.

Example 3: A circle is inscribed in a square. Each side of the square is 10 cm long. What is the area of the circle?



Solution: The diameter of the circle is the same as the length of a side of the square. Therefore, the radius is 5 cm.

$$A = \pi 5^2 = 25\pi$$

Example 4: Find the area of the shaded region.

Solution: The area of the shaded region would be the area of the square minus the area of the circle.

$$A = 10^2 - 25\pi = 100 - 25\pi \approx 21.46$$

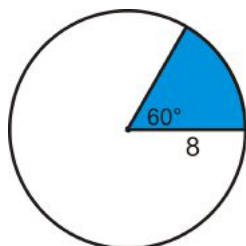
Area of a Sector

Sector of a Circle: The area bounded by two radii and the arc between the endpoints of the radii.



Area of a Sector: If r is the radius and \widehat{AB} is the arc bounding a sector, then $A = \frac{m\widehat{AB}}{360^\circ} \cdot \pi r^2$.

Example 5: Find the area of the blue sector. Leave your answer in terms of π .

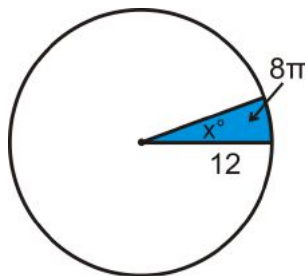


Solution: In the picture, the central angle that corresponds with the sector is 60° . 60° would be $\frac{1}{6}$ of 360° , so this sector is $\frac{1}{6}$ of the total area. *area of blue sector* $= \frac{1}{6} \cdot \pi 8^2 = \frac{32}{3}\pi$

Another way to write the sector formula is $A = \frac{\text{central angle}}{360^\circ} \cdot \pi r^2$.

Example 6: The area of a sector is 8π and the radius of the circle is 12. What is the central angle?

Solution: Plug in what you know to the sector area formula and then solve for the central angle, we will call it x .



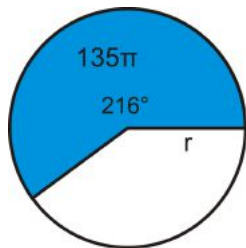
$$8\pi = \frac{x}{360^\circ} \cdot \pi 12^2$$

$$8\pi = \frac{x}{360^\circ} \cdot 144\pi$$

$$8 = \frac{2x}{5^\circ}$$

$$x = 8 \cdot \frac{5^\circ}{2} = 20^\circ$$

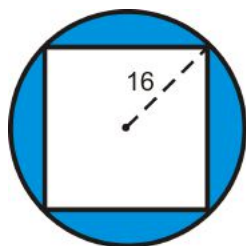
Example 7: The area of a sector is 135π and the arc measure is 216° . What is the radius of the circle?



Solution: Plug in what you know to the sector area formula and solve for r .

$$\begin{aligned}
 135\pi &= \frac{216^\circ}{360^\circ} \cdot \pi r^2 \\
 135 &= \frac{3}{5} \cdot r^2 \\
 \frac{5}{3} \cdot 135 &= r^2 \\
 225 &= r^2 \rightarrow r = \sqrt{225} = 15
 \end{aligned}$$

Example 8: Find the area of the shaded region. The quadrilateral is a square.



Solution: The radius of the circle is 16, which is also half of the diagonal of the square. So, the diagonal is 32 and the sides would be $\frac{32}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = 16\sqrt{2}$ because each half of a square is a 45-45-90 triangle.

$$A_{\text{circle}} = 16^2\pi = 256\pi$$

$$A_{\text{square}} = (16\sqrt{2})^2 = 256 \cdot 2 = 512$$

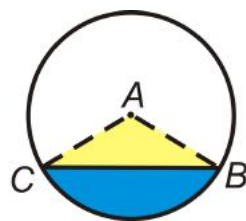
The area of the shaded region is $256\pi - 512 \approx 292.25$

Segments of a Circle

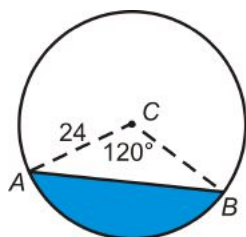
The last part of a circle that we can find the area of is called a segment, not to be confused with a line segment.

Segment of a Circle: The area of a circle that is bounded by a chord and the arc with the same endpoints as the chord.

$$A_{\text{segment}} = A_{\text{sector}} - A_{\triangle ABC}$$



Example 9: Find the area of the blue segment below.



Solution: The area of the segment is the area of the sector minus the area of the isosceles triangle made by the radii. If we split the isosceles triangle in half, each half is a 30-60-90 triangle, where the radius is the hypotenuse. The height of $\triangle ABC$ is 12 and the base would be $2(12\sqrt{3}) = 24\sqrt{3}$.

$$\begin{aligned} A_{\text{sector}} &= \frac{120}{360}\pi \cdot 24^2 \\ &= 192\pi \end{aligned}$$

$$\begin{aligned} A_{\triangle} &= \frac{1}{2}(24\sqrt{3})(12) \\ &= 144\sqrt{3} \end{aligned}$$

The area of the segment is $A = 192\pi - 144\sqrt{3} \approx 353.8$.

Know What? Revisited The area of the crust for a deep-dish pizza is $8^2\pi - 7^2\pi = 15\pi$. The area of the crust of the thin crust pizza is $8^2\pi - 7.5^2\pi = \frac{31}{4}\pi$.

Review Questions

- Questions 1-10 are similar to Examples 1 and 2.
- Questions 11-16 are similar to Example 5.
- Questions 17-19 are similar to Example 7.
- Questions 20-22 are similar to Example 6.
- Questions 23-25 are similar to Examples 3, 4, and 8.
- Questions 26-31 are similar to Example 9.

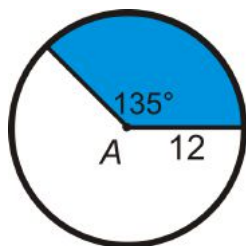
Fill in the following table. Leave all answers in terms of π .

Table 6.2:

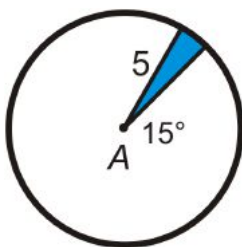
	<i>radius</i>	<i>Area</i>	<i>circumference</i>
1.	2		
2.		16π	
3.			10π
4.			24π
5.	9		
6.		90π	
7.			35π
8.	$\frac{7}{\pi}$		
9.			60
10.		36	

Find the area of the blue sector or segment in $\odot A$. Leave your answers in terms of π . Round any decimal answers to the nearest hundredth.

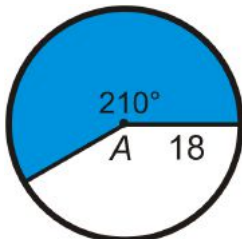
11.



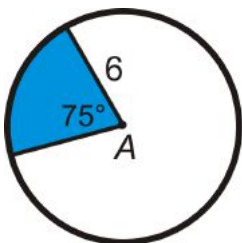
12.



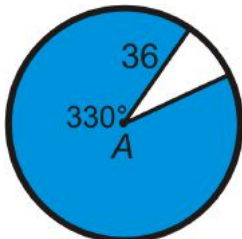
13.



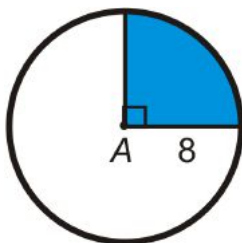
14.



15.

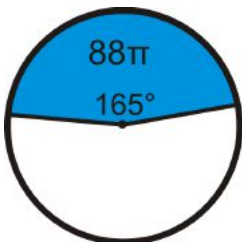


16.

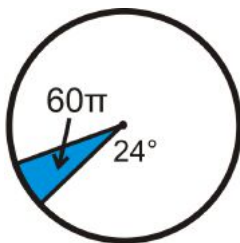


Find the radius of the circle. Leave your answer in terms of π .

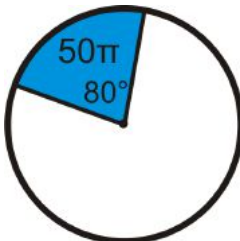
17.



18.

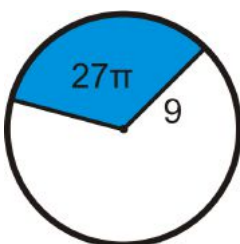


19.

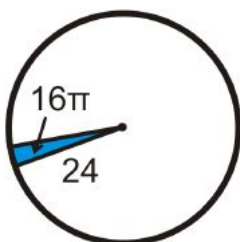


Find the central angle of each blue sector. Round any decimal answers to the nearest tenth.

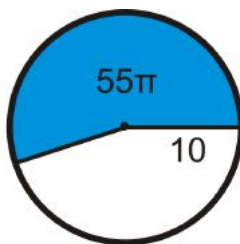
20.



21.

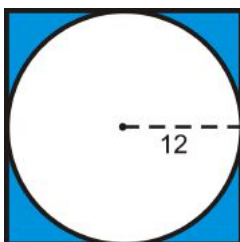


22.

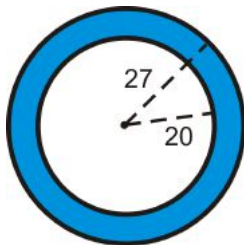


Find the area of the shaded region. Round your answer to the nearest hundredth.

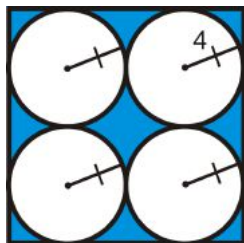
23.



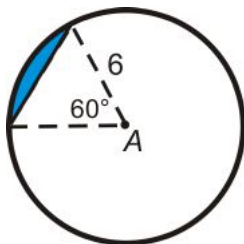
24.



25.



26. Find the area of the sector in $\odot A$. Leave your answer in terms of π .

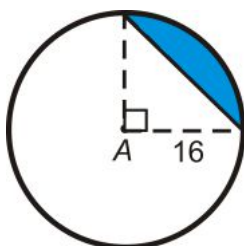


27. Find the area of the equilateral triangle.

28. Find the area of the segment. Round your answer to the nearest hundredth.

29. Find the area of the sector in $\odot A$. Leave your answer in terms of π .

30. Find the area of the right triangle.



31. Find the area of the segment. Round your answer to the nearest hundredth.

Review Queue Answers

- $8^2 - 4^2 = 64 - 16 = 48$
- $6(10) - \frac{1}{2}(7)(3) = 60 - 10.5 = 49.5$
- $\frac{1}{2}(6)(3\sqrt{3}) = 9\sqrt{3}$
- $\frac{1}{2}(s)(\frac{1}{2}s\sqrt{3}) = \frac{1}{4}s^2\sqrt{3}$

6.6 Chapter 10 Review

Keywords, Theorems and Formulas

Triangles and Parallelograms

- Perimeter
- Area of a Rectangle: $A = bh$
- Perimeter of a Rectangle $P = 2b + 2h$
- Perimeter of a Square: $P = 4s$
- Area of a Square: $A = s^2$
- Congruent Areas Postulate
- Area Addition Postulate
- Area of a Parallelogram: $A = bh$
- Area of a Triangle: $A = \frac{1}{2}bh$ or $A = \frac{bh}{2}$

Trapezoids, Rhombi, and Kites

- Area of a Trapezoid: $A = \frac{1}{2}h(b_1 + b_2)$
- Area of a Rhombus: $A = \frac{1}{2}d_1d_2$
- Area of a Kite: $A = \frac{1}{2}d_1d_2$

Area of Similar Polygons

- Area of Similar Polygons Theorem

Circumference and Arc Length

- π
- Circumference: $C = \pi d$ or $C = 2\pi r$
- Arc Length
- Arc Length Formula: length of $\widehat{AB} = \frac{m\widehat{AB}}{360^\circ} \cdot \pi d$ or $\frac{m\widehat{AB}}{360^\circ} \cdot 2\pi r$

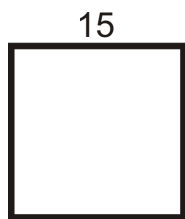
Area of Circles and Sectors

- Area of a Circle: $A = \pi r^2$
- Sector
- Area of a Sector: $A = \frac{m\widehat{AB}}{360^\circ} \cdot \pi r^2$
- Segment of a Circle

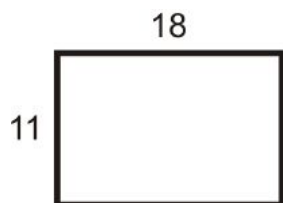
Review Questions

Find the area and perimeter of the following figures. Round your answers to the nearest hundredth.

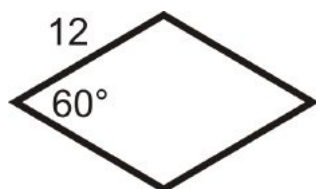
1. square



2. rectangle



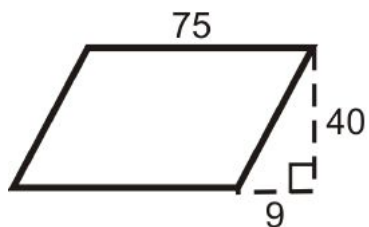
3. rhombus



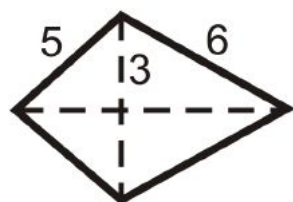
4. equilateral triangle



5. parallelogram

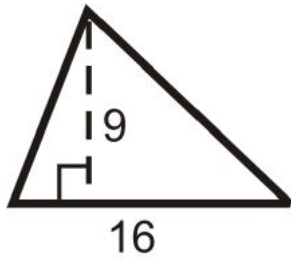


6. kite

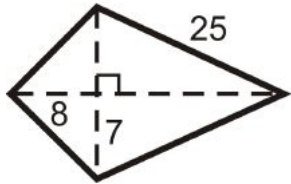


Find the area of the following figures. Leave your answers in simplest radical form.

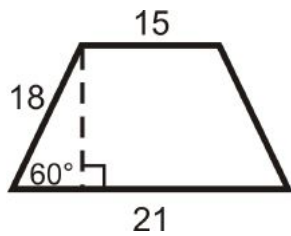
7. triangle



8. kite



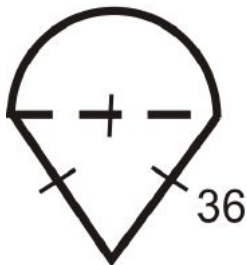
9. isosceles trapezoid



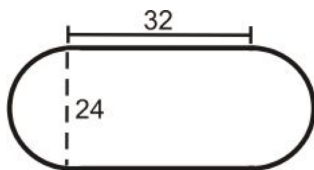
10. Find the area and circumference of a circle with radius 17.
11. Find the area and circumference of a circle with diameter 30.
12. Two similar rectangles have a scale factor $\frac{4}{3}$. If the area of the larger rectangle is 96 units^2 , find the area of the smaller rectangle.

Find the area of the following figures. Round your answers to the nearest hundredth.

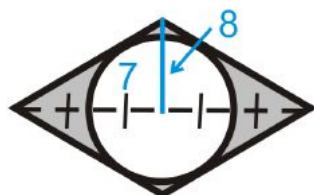
13.



14.



15. find the shaded area



(figure is a rhombus)

Texas Instruments Resources

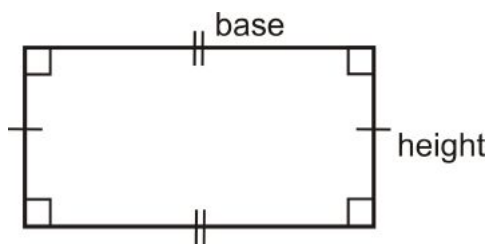
In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9695>.

6.7 Study Guide

Keywords: Define, write theorems, and/or draw a diagram for each word below.

1st Section: Triangles and Parallelograms

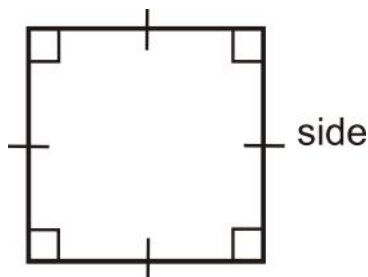
Perimeter



Area of a Rectangle: $A = bh$

Perimeter of a Rectangle $P = 2b + 2h$

Perimeter of a Square: $P = 4s$

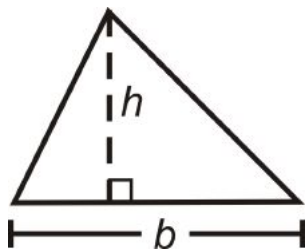


Area of a Square: $A = s^2$

Congruent Areas Postulate

Area Addition Postulate

Area of a Parallelogram: $A = bh$

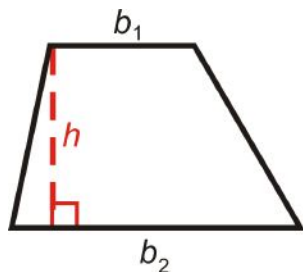


Area of a Triangle: $A = \frac{1}{2}bh$ or $A = \frac{bh}{2}$

Homework:

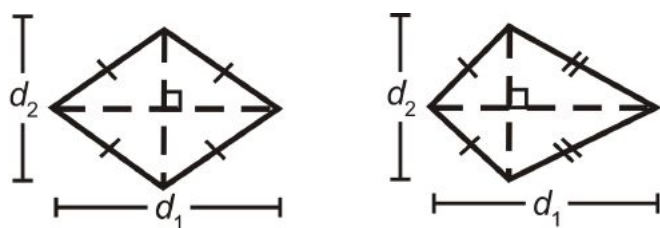
2nd Section: Trapezoids, Rhombi, and Kites

Area of a Trapezoid: $A = \frac{1}{2}h(b_1 + b_2)$



Area of a Rhombus: $A = \frac{1}{2}d_1d_2$

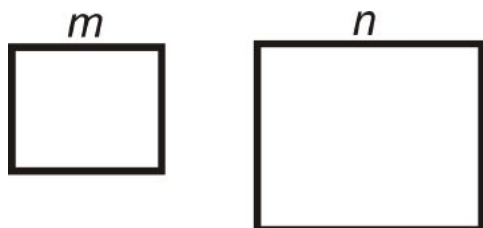
Area of a Kite: $A = \frac{1}{2}d_1d_2$



Homework:

3rd Section: Area of Similar Polygons

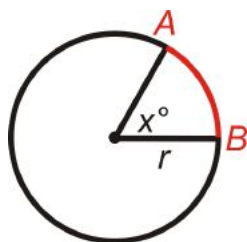
Area of Similar Polygons Theorem



Homework:

4th Section: Circumference and Arc Length

π



Circumference: $C = \pi d$ or $C = 2\pi r$

Arc Length

Arc Length Formula: length of $\widehat{AB} = \frac{m\widehat{AB}}{360^\circ} \cdot \pi d$ or $\frac{m\widehat{AB}}{360^\circ} \cdot 2\pi r$

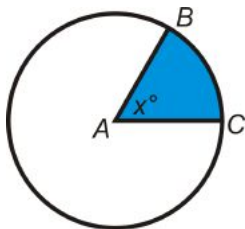
Homework:

5th Section: Area of Circles and Sectors

Area of a Circle: $A = \pi r^2$

Sector

Area of a Sector: $A = \frac{\widehat{mAB}}{360^\circ} \cdot \pi r^2$



Segment of a Circle

Homework:

Chapter 7

Surface Area and Volume

In this chapter we extend what we know about two-dimensional figures to three-dimensional shapes. First, we will define the different types of 3D shapes and their parts. Then, we will find the surface area and volume of prisms, cylinders, pyramids, cones, and spheres.

7.1 Exploring Solids

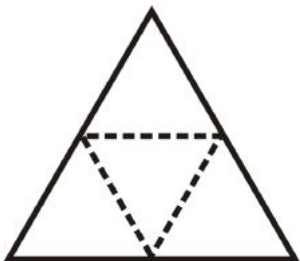
Learning Objectives

- Identify different types of solids and their parts.
- Use Euler's formula and nets.

Review Queue

1. Draw an octagon and identify the edges and vertices of the octagon. How many of each are there?
2. Find the area of a square with 5 cm sides.
3. Draw the following polygons.
 - (a) A convex pentagon.
 - (b) A concave nonagon.

Know What? Until now, we have only talked about two-dimensional, or flat, shapes. Copy the equilateral triangle to the right onto a piece of paper and cut it out. Fold on the dotted lines. What shape do these four equilateral triangles make?



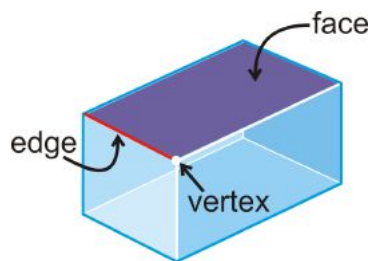
Polyhedrons

Polyhedron: A 3-dimensional figure that is formed by polygons that enclose a region in space.

Each polygon in a polyhedron is a **face**.

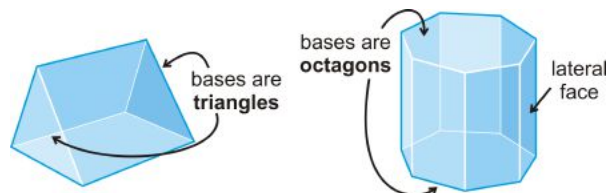
The line segment where two faces intersect is an **edge**.

The point of intersection of two edges is a **vertex**.

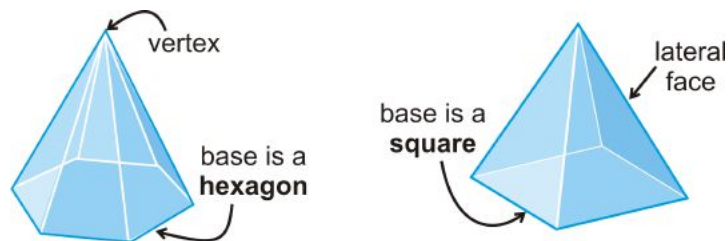


Examples of polyhedrons include a cube, prism, or pyramid. Non-polyhedrons are cones, spheres, and cylinders because they have sides that are not polygons.

Prism: A polyhedron with two congruent bases, in parallel planes, and the lateral sides are rectangles.



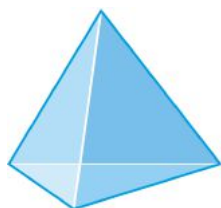
Pyramid: A polyhedron with one base and all the lateral sides meet at a common vertex.

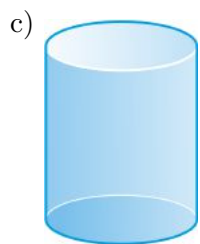
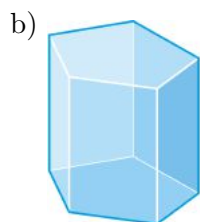


All prisms and pyramids are named by their bases. So, the first prism would be a triangular prism and the first pyramid would be a hexagonal pyramid.

Example 1: Determine if the following solids are polyhedrons. If the solid is a polyhedron, name it and find the number of faces, edges and vertices each has.

a)





Solution:

- a) The base is a triangle and all the sides are triangles, so this is a triangular pyramid. There are 4 faces, 6 edges and 4 vertices.
- b) This solid is also a polyhedron. The bases are both pentagons, so it is a pentagonal prism. There are 7 faces, 15 edges, and 10 vertices.
- c) The bases that are circles. Circles are not polygons, so it is not a polyhedron.

Euler's Theorem

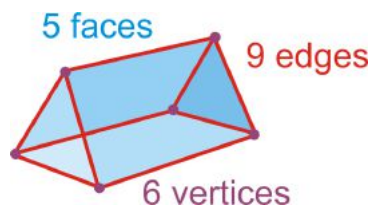
Let's put our results from Example 1 into a table.

Table 7.1:

	<i>Faces</i>	<i>Vertices</i>	<i>Edges</i>
<i>Triangular Pyramid</i>	4	4	6
<i>Pentagonal Prism</i>	7	10	15

Notice that faces + vertices is two more than the number of edges. This is called Euler's Theorem, after the Swiss mathematician Leonhard Euler.

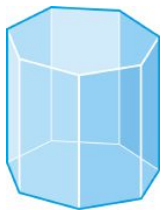
Euler's Theorem: $F + V = E + 2$.



$$Faces + Vertices = Edges + 2$$

$$5 + 6 = 9 + 2$$

Example 2: Find the number of faces, vertices, and edges in the octagonal prism.



Solution: There are 10 faces and 16 vertices. Use Euler's Theorem, to solve for E .

$$\begin{aligned} F + V &= E + 2 \\ 10 + 16 &= E + 2 \\ 24 &= E \end{aligned}$$

Example 3: In a six-faced polyhedron, there are 10 edges. How many vertices does the polyhedron have?

Solution: Solve for V in Euler's Theorem.

$$\begin{aligned} F + V &= E + 2 \\ 6 + V &= 10 + 2 \\ V &= 6 \end{aligned}$$

Example 4: A three-dimensional figure has 10 vertices, 5 faces, and 12 edges. Is it a polyhedron?

Solution: Plug in all three numbers into Euler's Theorem.

$$\begin{aligned} F + V &= E + 2 \\ 5 + 10 &= 12 + 2 \\ 15 &\neq 14 \end{aligned}$$

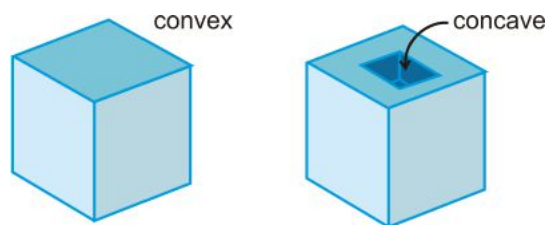
Because the two sides are not equal, this figure is not a polyhedron.

Regular Polyhedra

Regular Polyhedron: A polyhedron where all the faces are congruent regular polygons.

All regular polyhedron are *convex*.

A *concave* polyhedron “caves in.”



There are only *five regular polyhedra, called the Platonic solids*.

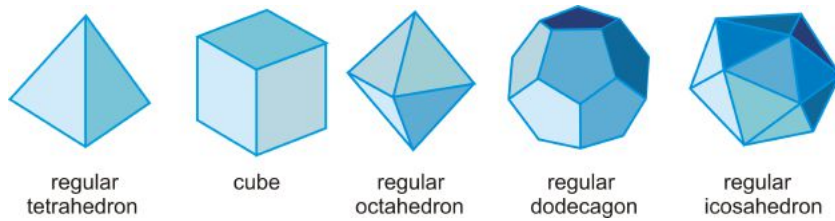
Regular Tetrahedron: A 4-faced polyhedron and all the faces are equilateral triangles.

Cube: A 6-faced polyhedron and all the faces are squares.

Regular Octahedron: An 8-faced polyhedron and all the faces are equilateral triangles.

Regular Dodecahedron: A 12-faced polyhedron and all the faces are regular pentagons.

Regular Icosahedron: A 20-faced polyhedron and all the faces are equilateral triangles.

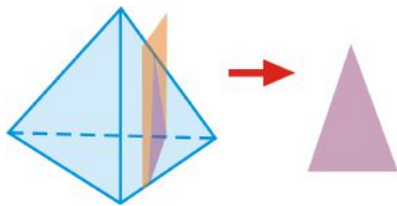


Cross-Sections

One way to “view” a three-dimensional figure in a two-dimensional plane, like in this text, is to use cross-sections.

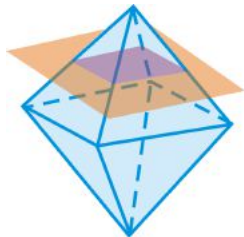
Cross-Section: The intersection of a plane with a solid.

The cross-section of the peach plane and the tetrahedron is a *triangle*.

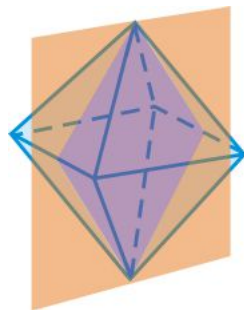


Example 5: What is the shape formed by the intersection of the plane and the regular octahedron?

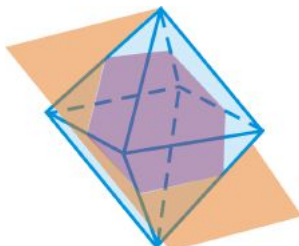
a)



b)



c)

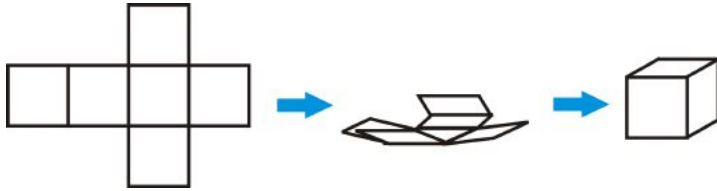


Solution:

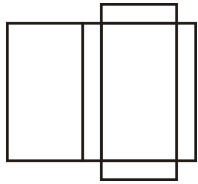
- a) Square
- b) Rhombus
- c) Hexagon

Nets

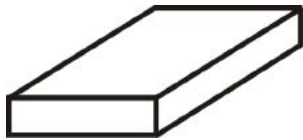
Net: An unfolded, flat representation of the sides of a three-dimensional shape.



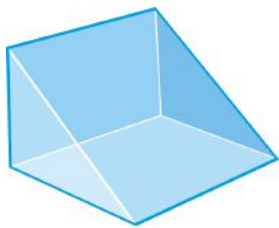
Example 6: What kind of figure does this net create?



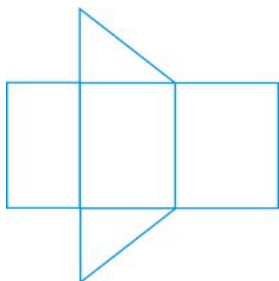
Solution: The net creates a rectangular prism.



Example 7: Draw a net of the right triangular prism below.



Solution: The net will have two triangles and three rectangles. The rectangles are different sizes and the two triangles are the same.



There are several different nets of any polyhedron. For example, this net could have the triangles anywhere along the top or bottom of the three rectangles. Click the site <http://www.cs.mcgill.ca/~sqrt/unfold/unfolding.html> to see a few animations of other nets.

Know What? Revisited The net of the shape is a regular tetrahedron.

Review Questions

- Questions 1-8 are similar to Examples 2-4.
- Questions 9-14 are similar to Example 1.
- Questions 15-17 are similar to Example 5.
- Questions 18-23 are similar to Example 7.
- Questions 24-29 are similar to Example 6.
- Question 30 uses Euler's Theorem.

Complete the table using Euler's Theorem.

Table 7.2:

	<i>Name</i>	<i>Faces</i>	<i>Edges</i>	<i>Vertices</i>
1.	Rectangular Prism	6	12	
2.	Octagonal Pyramid		16	9
3.	Regular Icosahedron	20		12
4.	Cube		12	8
5.	Triangular Pyramid	4		4
6.	Octahedron	8	12	
7.	Heptagonal Prism		21	14
8.	Triangular Prism	5	9	

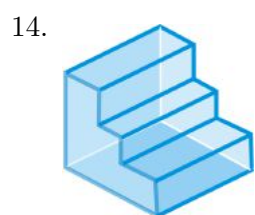
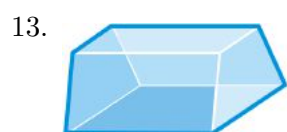
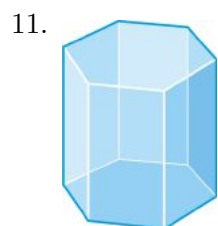
Determine if the following figures are polyhedra. If so, name the figure and find the number of faces, edges, and vertices.

9.

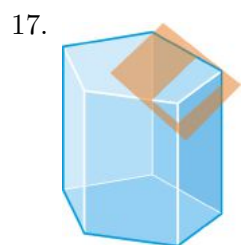
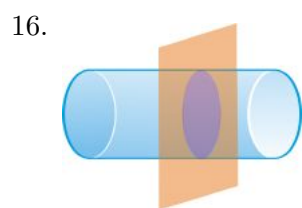
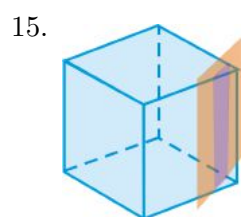


10.



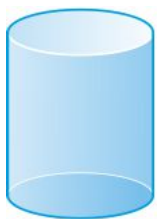


Describe the cross section formed by the intersection of the plane and the solid.



Draw the net for the following solids.

18.



19.



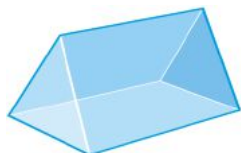
20.



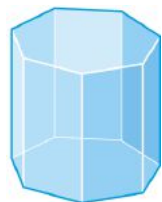
21.



22.

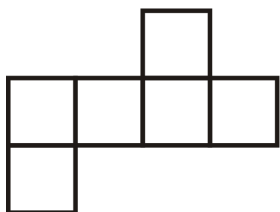


23.

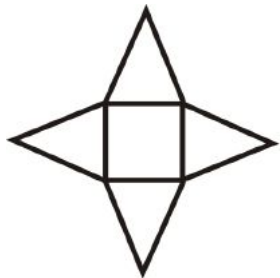


Determine what shape is formed by the following nets.

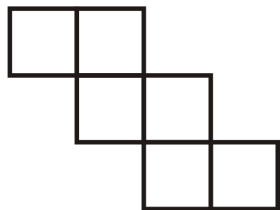
24.



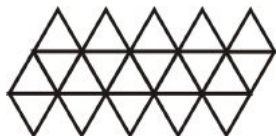
25.



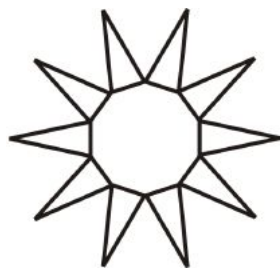
26.



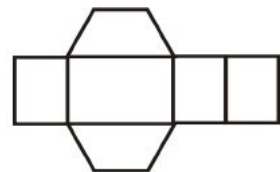
27.



28.



29.

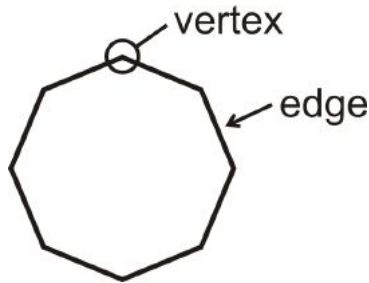


30. A **truncated icosahedron** is a polyhedron with 12 regular pentagonal faces and 20 regular hexagonal faces and 90 edges. This icosahedron closely resembles a soccer ball. How many vertices does it have? Explain your reasoning.



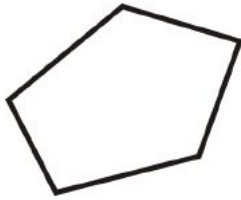
Review Queue Answers

1. There are 8 vertices and 8 edges in an octagon.

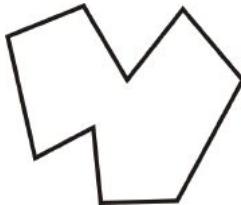


2. $5^2 = 25 \text{ cm}^2$

3. (a)



- (b)



7.2 Surface Area of Prisms and Cylinders

Learning Objectives

- Find the surface area of a prism and cylinder.

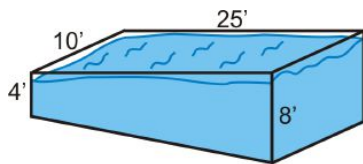
Review Queue

1. Find the area of a rectangle with sides:

- (a) 6 and 9
- (b) 11 and 4
- (c) $5\sqrt{2}$ and $8\sqrt{6}$

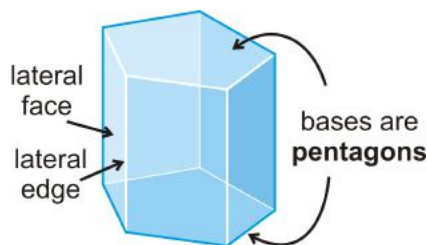
2. If the area of a square is 36 units^2 , what are the lengths of the sides?
3. If the area of a square is 45 units^2 , what are the lengths of the sides?

Know What? Your parents decide they want to put a pool in the backyard. The shallow end will be 4 ft. and the deep end will be 8 ft. The pool will be 10 ft. by 25 ft. How much siding do they need to cover the sides and bottom of the pool?



Parts of a Prism

Prism: A 3-dimensional figure with 2 congruent bases, in parallel planes, and the other faces are rectangles.



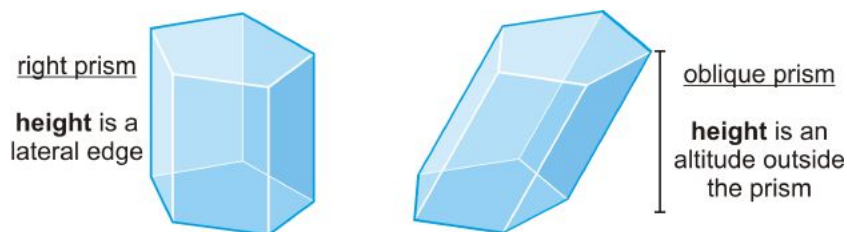
The non-base faces are *lateral faces*.

The edges between the lateral faces are *lateral edges*.

This is a *pentagonal prism*.

Right Prism: A prism where all the lateral faces are perpendicular to the bases.

Oblique Prism: A prism that leans to one side and the height is outside the prism.

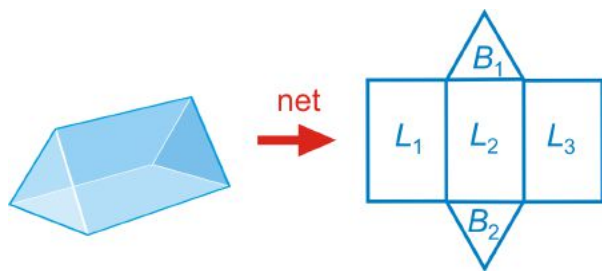


Surface Area of a Prism

Surface Area: The sum of the areas of the faces.

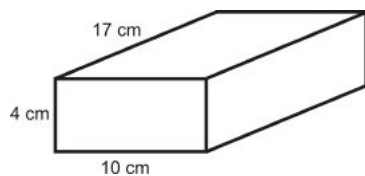
$$\text{Surface Area} = B_1 + B_2 + L_1 + L_2 + L_3$$

$$\text{Lateral Area} = L_1 + L_2 + L_3$$



Lateral Area: The sum of the areas of the *lateral* faces.

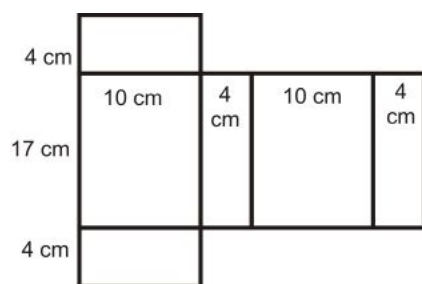
Example 1: Find the surface area of the prism below.



Solution: Draw the net of the prism.

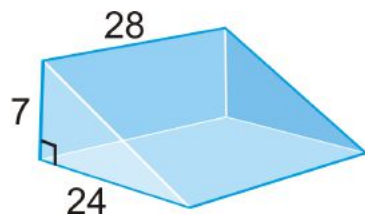
Using the net, we have:

$$\begin{aligned} SA_{prism} &= 2(4)(10) + 2(10)(17) + 2(17)(4) \\ &= 80 + 340 + 136 \\ &= 556 \text{ cm}^2 \end{aligned}$$

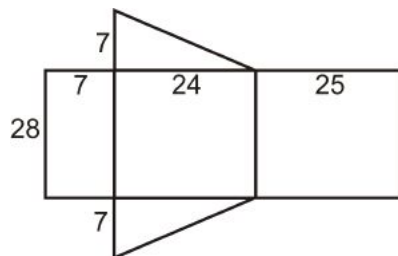


Surface Area of a Right Prism: The surface area of a right prism is the sum of the area of the bases and the area of each rectangular lateral face.

Example 2: Find the surface area of the prism below.



Solution: This is a right triangular prism. To find the surface area, we need to find the length of the hypotenuse of the base because it is the width of one of the lateral faces.



$$\begin{aligned} 7^2 + 24^2 &= c^2 \\ 49 + 576 &= c^2 \\ 625 &= c^2 \quad c = 25 \end{aligned}$$

Looking at the net, the surface area is:

$$SA = 28(7) + 28(24) + 28(25) + 2\left(\frac{1}{2} \cdot 7 \cdot 24\right)$$

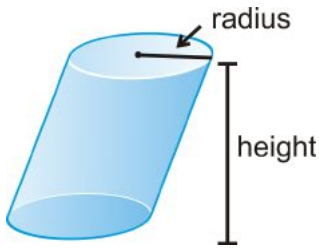
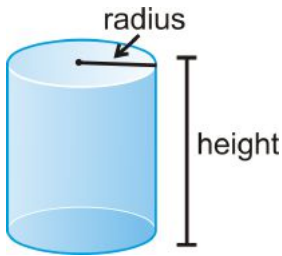
$$SA = 196 + 672 + 700 + 168 = 1736 \text{ units}^2$$

Cylinders

Cylinder: A solid with congruent circular bases that are in parallel planes. The space between the circles is enclosed.

A cylinder has a *radius* and a *height*.

A cylinder can also be *oblique*, like the one on the far right.

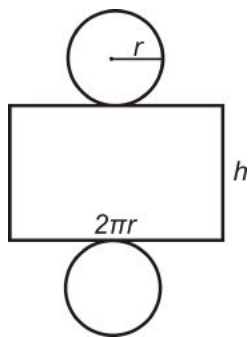


Surface Area of a Right Cylinder

Let's find the net of a right cylinder. One way to do this is to take the label off of a soup can. The label is a rectangle where the height is the height of the cylinder and the base is the circumference of the circle.



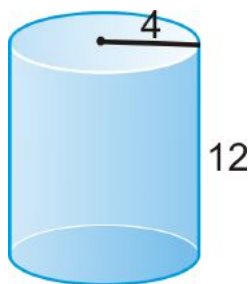
Surface Area of a Right Cylinder: $SA = 2\pi r^2 + 2\pi rh$.



$$\underbrace{2\pi r^2}_{\substack{\text{area of} \\ \text{both} \\ \text{circles}}} + \underbrace{2\pi r}_{\substack{\text{length} \\ \text{of} \\ \text{rectangle}}} h$$

To see an animation of the surface area, click <http://www.rkm.com.au/ANIMATIONS/animation-Cylinder-Surface-A.html>, by Russell Knightley.

Example 3: Find the surface area of the cylinder.



Solution: $r = 4$ and $h = 12$.

$$\begin{aligned} SA &= 2\pi(4)^2 + 2\pi(4)(12) \\ &= 32\pi + 96\pi \\ &= 128\pi \text{ units}^2 \end{aligned}$$

Example 4: The circumference of the base of a cylinder is 16π and the height is 21. Find the surface area of the cylinder.

Solution: We need to solve for the radius, using the circumference.

$$\begin{aligned} 2\pi r &= 16\pi \\ r &= 8 \end{aligned}$$

Now, we can find the surface area.

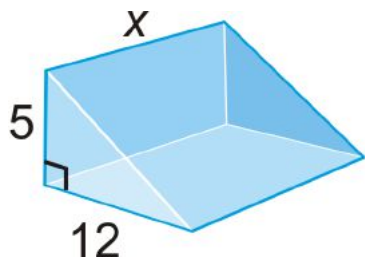
$$\begin{aligned} SA &= 2\pi(8)^2 + (16\pi)(21) \\ &= 128\pi + 336\pi \\ &= 464\pi \text{ units}^2 \end{aligned}$$

Example 5: Algebra Connection The total surface area of the triangular prism is 540 units^2 . What is x ?

Solution: The total surface area is equal to:

$$A_{2 \text{ triangles}} + A_{3 \text{ rectangles}} = 540$$

The hypotenuse of the triangle bases is 13, $\sqrt{5^2 + 12^2}$. Let's fill in what we know.



$$A_{2 \text{ triangles}} = 2 \left(\frac{1}{2} \cdot 5 \cdot 12 \right) = 60$$

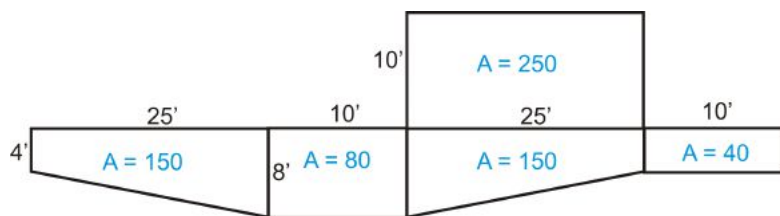
$$A_{3 \text{ rectangles}} = 5x + 12x + 13x = 30x$$

$$60 + 30x = 540$$

$$30x = 480$$

$$x = 16 \text{ units} \quad \text{The height is 16 units.}$$

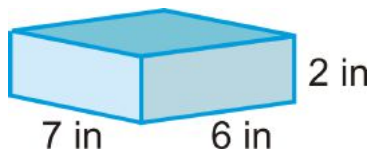
Know What? Revisited To the right is the net of the pool (minus the top). From this, we can see that your parents would need 670 square feet of siding.



Review Questions

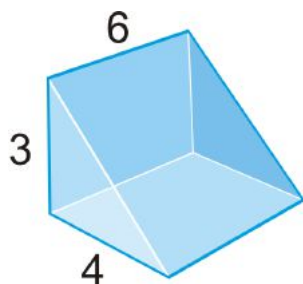
- Questions 1-9 are similar to Examples 1 and 2.
- Question 10 uses the definition of lateral and total surface area.
- Questions 11-18 are similar to Examples 1-3.
- Questions 19-21 are similar to Example 5.
- Questions 22-24 are similar to Example 4.
- Questions 25-30 use the Pythagorean Theorem and are similar to Examples 1-3.

1. What type of prism is this?



- Draw the net of this prism.
- Find the area of the bases.
- Find the area of lateral faces, or the lateral surface area.
- Find the total surface area of the prism.

Use the right triangular prism to answer questions 6-9.



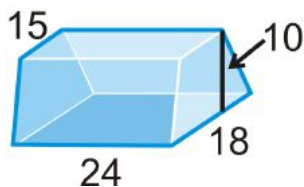
6. What shape are the bases of this prism? What are their areas?
7. What are the dimensions of each of the lateral faces? What are their areas?
8. Find the lateral surface area of the prism.
9. Find the total surface area of the prism.
10. **Writing** Describe the difference between *lateral* surface area and *total* surface area.
11. Fuzzy dice are cubes with 4 inch sides.



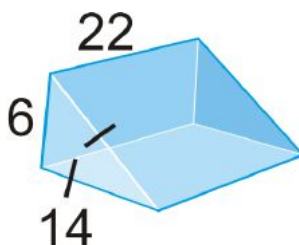
- (a) What is the surface area of one die?
- (b) Typically, the dice are sold in pairs. What is the surface area of two dice?
12. A right cylinder has a 7 cm radius and a height of 18 cm. Find the surface area.

Find the surface area of the following solids. Round your answer to the nearest hundredth.

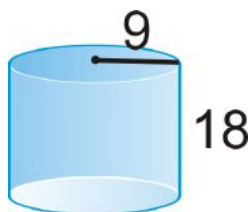
13. bases are isosceles trapezoids



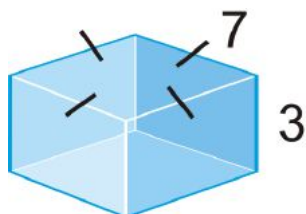
- 14.



15.



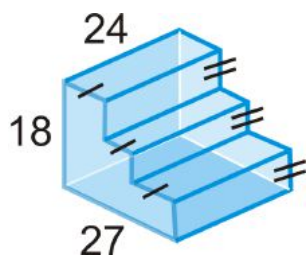
16.



17.

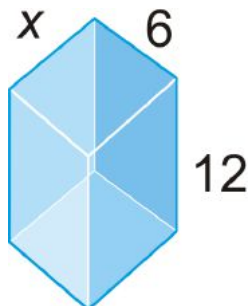


18.

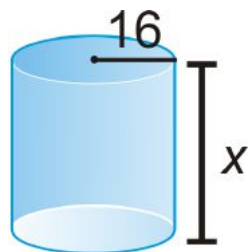


Algebra Connection Find the value of x , given the surface area.

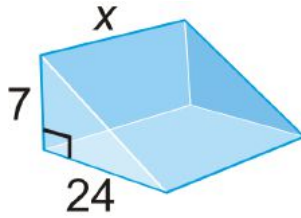
19. $SA = 432 \text{ units}^2$



20. $SA = 1536\pi \text{ units}^2$

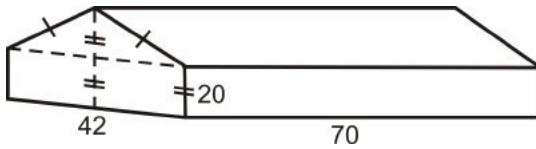


21. $SA = 1568 \text{ units}^2$



22. The area of the base of a cylinder is $25\pi \text{ in}^2$ and the height is 6 in. Find the *lateral* surface area.
23. The circumference of the base of a cylinder is $80\pi \text{ cm}$ and the height is 36 cm. Find the total surface area.
24. The lateral surface area of a cylinder is $30\pi \text{ m}^2$ and the height is 5m. What is the radius?

Use the diagram below for questions 25-30. The barn is shaped like a pentagonal prism with dimensions shown in feet.



25. What is the width of the roof? (HINT: Use the Pythagorean Theorem)
26. What is the area of the roof? (Both sides)
27. What is the floor area of the barn?
28. What is the area of the rectangular sides of the barn?
29. What is the area of the two pentagon sides of the barn? (HINT: Find the area of two congruent trapezoids for each side)
30. Find the total surface area of the barn (Roof and sides).

Review Queue Answers

1. (a) 54
(b) 44
(c) $80\sqrt{3}$
2. $s = 6$
3. $s = 3\sqrt{5}$

7.3 Surface Area of Pyramids and Cones

Learning Objectives

- Find the surface area of pyramids and cones.

Review Queue

- A rectangular prism has sides of 5 cm, 6 cm, and 7 cm. What is the surface area?
- A cylinder has a diameter of 10 in and a height of 25 in. What is the surface area?
- A cylinder has a circumference of $72\pi \text{ ft.}$ and a height of 24 ft. What is the surface area?
- Draw the net of a square pyramid.

Know What? A typical waffle cone is 6 inches tall and has a diameter of 2 inches. What is the surface area of the waffle cone? (You may assume that the cone is straight across at the top)

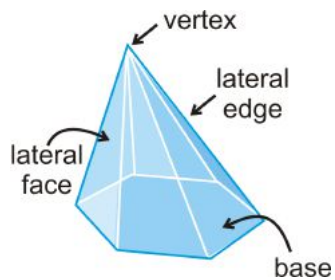


Parts of a Pyramid

Pyramid: A solid with one *base* and the *lateral faces* meet at a common *vertex*.

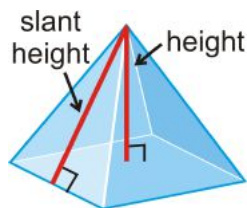
The edges between the lateral faces are *lateral edges*.

The edges between the base and the lateral faces are *base edges*.

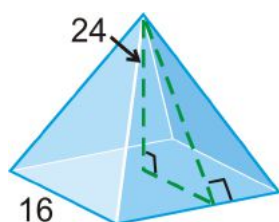


Regular Pyramid: A pyramid where the base is a regular polygon.

All regular pyramids also have a *slant height* which is the height of a lateral face. A non-regular pyramid does not have a slant height.



Example 1: Find the slant height of the square pyramid.



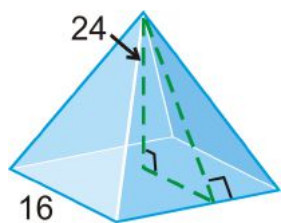
Solution: The slant height is the hypotenuse of the right triangle formed by the height and half the base length. Use the Pythagorean Theorem.

$$\begin{aligned}8^2 + 24^2 &= l^2 \\640 &= l^2 \\l &= \sqrt{640} = 8\sqrt{10}\end{aligned}$$

Surface Area of a Regular Pyramid

Using the slant height, which is labeled l , the area of each triangular face is $A = \frac{1}{2}bl$.

Example 2: Find the surface area of the pyramid from Example 1.



Solution: The four triangular faces are $4\left(\frac{1}{2}bl\right) = 2(16)(8\sqrt{10}) = 256\sqrt{10}$. To find the total surface area, we also need the area of the base, which is $16^2 = 256$. The total surface area is $256\sqrt{10} + 256 \approx 1065.54 \text{ units}^2$.

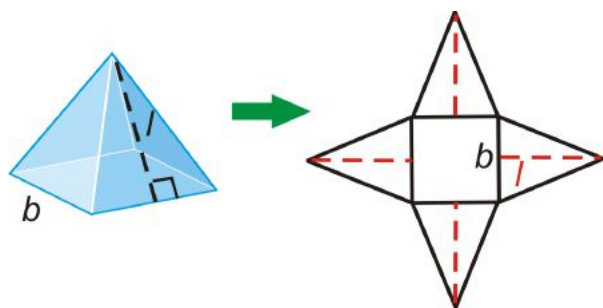
From this example, we see that the formula for a square pyramid is:

$$\begin{aligned}SA &= (\text{area of the base}) + 4(\text{area of triangular faces}) \\SA &= B + n\left(\frac{1}{2}bl\right)\end{aligned}$$

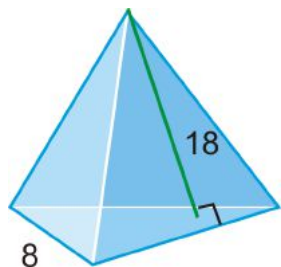
B is the area of the base and n is the number of triangles.

Surface Area of a Regular Pyramid: If B is the area of the base, then $SA = B + \frac{1}{2}nbl$.

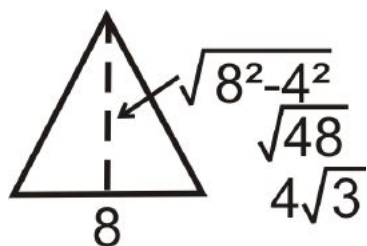
The net shows the surface area of a pyramid. If you ever forget the formula, use the net.



Example 3: Find the area of the *regular* triangular pyramid.



Solution: “Regular” tells us the base is an equilateral triangle. Let’s draw it and find its area.



$$B = \frac{1}{2} \cdot 8 \cdot 4\sqrt{3} = 16\sqrt{3}$$

The surface area is:

$$SA = 16\sqrt{3} + \frac{1}{2} \cdot 3 \cdot 8 \cdot 18 = 16\sqrt{3} + 216 \approx 243.71$$

Example 4: If the lateral surface area of a square pyramid is 72 ft^2 and the base edge is equal to the slant height. What is the length of the base edge?

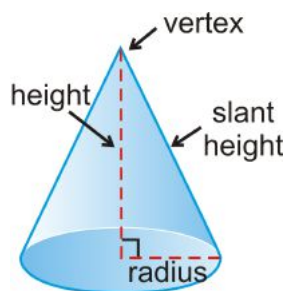
Solution: In the formula for surface area, the lateral surface area is $\frac{1}{2}nbl$. We know that $n = 4$ and $b = l$. Let’s solve for b .

$$\begin{aligned}\frac{1}{2}nbl &= 72 \text{ ft}^2 \\ \frac{1}{2}(4)b^2 &= 72 \\ 2b^2 &= 72 \\ b^2 &= 36 \\ b &= 6\end{aligned}$$

Surface Area of a Cone

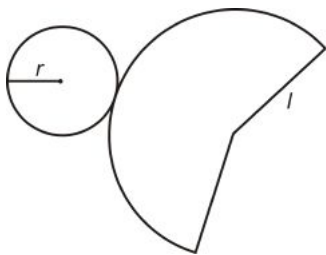
Cone: A solid with a circular base and sides taper up towards a vertex.

A cone has a slant height, just like a pyramid.



A cone is generated from rotating a right triangle, around one leg, in a circle.

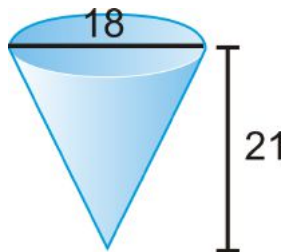
Surface Area of a Right Cone: $SA = \pi r^2 + \pi rl$.



Area of the base: πr^2

Area of the sides: $\pi r l$

Example 5: What is the surface area of the cone?



Solution: First, we need to find the slant height. Use the Pythagorean Theorem.

$$\begin{aligned} l^2 &= 9^2 + 21^2 \\ &= 81 + 441 \\ l &= \sqrt{522} \approx 22.85 \end{aligned}$$

The surface area would be $SA = \pi 9^2 + \pi(9)(22.85) \approx 900.54 \text{ units}^2$.

Example 6: The surface area of a cone is 36π and the radius is 4 units. What is the slant height?

Solution: Plug in what you know into the formula for the surface area of a cone and solve for l .

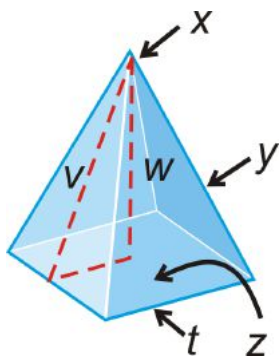
$$\begin{aligned} 36\pi &= \pi 4^2 + \pi 4l \\ 36 &= 16 + 4l && \text{When each term has a } \pi, \text{ they cancel out.} \\ 20 &= 4l \\ 5 &= l \end{aligned}$$

Know What? Revisited The standard cone has a surface area of $\pi + \sqrt{35}\pi \approx 21.73 \text{ in}^2$.

Review Questions

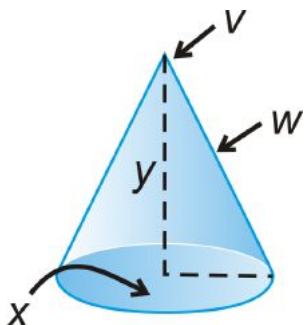
- Questions 1-10 use the definitions of pyramids and cones.
- Questions 11-19 are similar to Example 1.
- Questions 20-26 are similar to Examples 2, 3, and 5.
- Questions 27-31 are similar to Examples 4 and 6.
- Questions 32-25 are similar to Example 5.

Fill in the blanks about the diagram to the left.



1. x is the _____.
2. The slant height is _____.
3. y is the _____.
4. The height is _____.
5. The base is _____.
6. The base edge is _____.

Use the cone to fill in the blanks.



7. v is the _____.
8. The height of the cone is _____.
9. x is a _____ and it is the _____ of the cone.
10. w is the _____.

For questions 11-13, sketch each of the following solids and answer the question. Your drawings should be to scale, but not one-to-one. Leave your answer in simplest radical form.

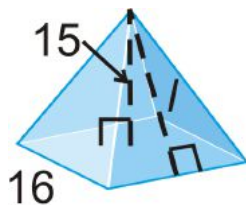
11. Draw a right cone with a radius of 5 cm and a height of 15 cm. What is the slant height?
12. Draw a square pyramid with an edge length of 9 in and a 12 in height. Find the slant height.
13. Draw an equilateral triangle pyramid with an edge length of 6 cm and a height of 6 cm. What is the height of the base?

Find the slant height, l , of one lateral face in each pyramid or of the cone. Round your answer to the nearest hundredth.

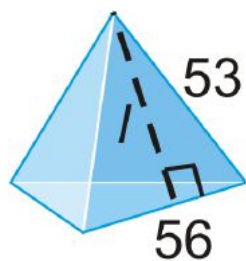
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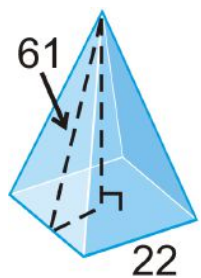


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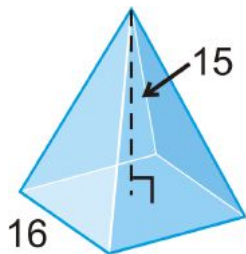


Find the area of a lateral face of the regular pyramid. Round your answers to the nearest hundredth.

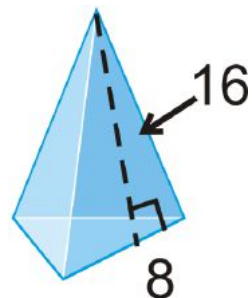
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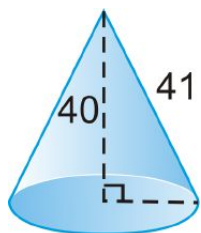


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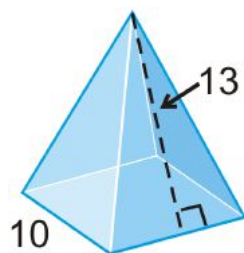


Find the surface area of the regular pyramids and right cones. Round your answers to 2 decimal places.

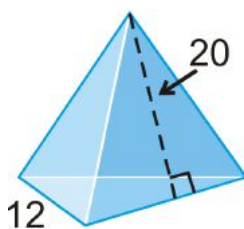
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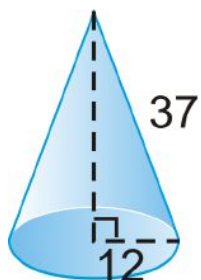
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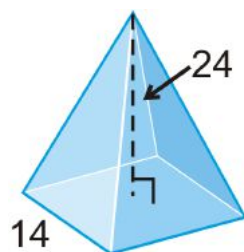
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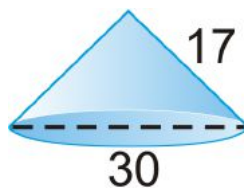
23.



24.



25.

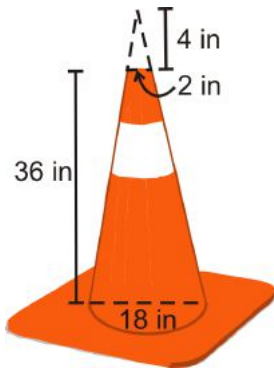


26. A **regular tetrahedron** has four equilateral triangles as its faces.

- Find the height of one of the faces if the edge length is 6 units.
- Find the area of one face.
- Find the total surface area of the regular tetrahedron.

27. If the lateral surface area of a cone is $30\pi \text{ cm}^2$ and the radius is 5 cm, what is the slant height?
28. If the surface area of a cone is $105\pi \text{ cm}^2$ and the slant height is 8 cm, what is the radius?
29. If the surface area of a square pyramid is 40 ft^2 and the base edge is 4 ft, what is the slant height?
30. If the lateral area of a square pyramid is 800 in^2 and the slant height is 16 in, what is the length of the base edge?
31. If the lateral area of a regular triangle pyramid is 252 in^2 and the base edge is 8 in, what is the slant height?

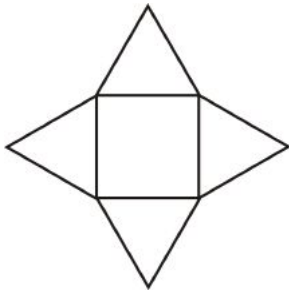
The traffic cone is cut off at the top and the base is a square with 24 in sides. Round answers to the nearest hundredth.



32. Find the area of the entire square. Then, subtract the area of the base of the cone.
33. Find the lateral area of the cone portion (include the 4 inch cut off top of the cone).
34. Subtract the cut-off top of the cone, to only have the lateral area of the cone portion of the traffic cone.
35. Combine your answers from #27 and #30 to find the entire surface area of the traffic cone.

Review Queue Answers

1. $2(5 \cdot 6) + 2(5 \cdot 7) + 2(6 \cdot 7) = 214 \text{ cm}^2$
2. $2(15 \cdot 18) + 2(15 \cdot 21) + 2(18 \cdot 21) = 1926 \text{ cm}^2$
3. $2 \cdot 25\pi + 250\pi = 300\pi \text{ in}^2$
4. $36^2(2\pi) + 72\pi(24) = 4320\pi \text{ ft}^2$
- 5.



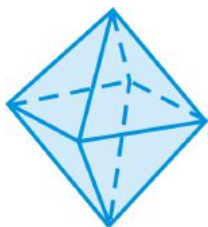
7.4 Volume of Prisms and Cylinders

Learning Objectives

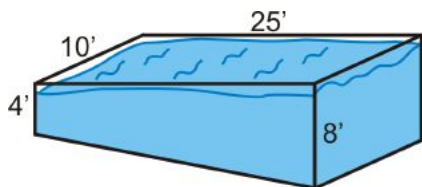
- Find the volume of prisms and cylinders.

Review Queue

1. Define volume in your own words.
2. What is the surface area of a cube with 3 inch sides?
3. A regular octahedron has 8 congruent equilateral triangles as the faces.
 - (a) If each edge is 4 cm, what is the slant height for one face?
 - (b) What is the surface area of one face?
 - (c) What is the total surface area?



Know What? Let's fill the pool it with water. The shallow end is 4 ft. and the deep end is 8 ft. The pool is 10 ft. wide by 25 ft. long. How many cubic feet of water is needed to fill the pool?



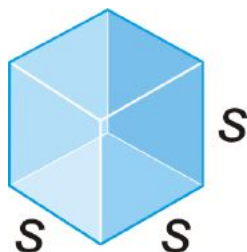
Volume of a Rectangular Prism

Volume: The measure of how much space a three-dimensional figure occupies.

Another way to define volume would be how much a three-dimensional figure can hold. The basic unit of volume is the cubic unit: cubic centimeter (cm^3), cubic inch (in^3), cubic meter (m^3), cubic foot (ft^3).

Volume of a Cube Postulate: $V = s^3$.

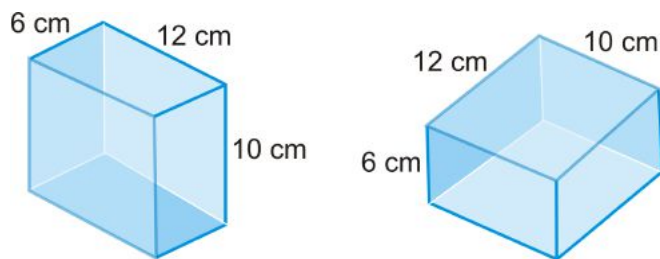
$$V = s \cdot s \cdot s = s^3$$



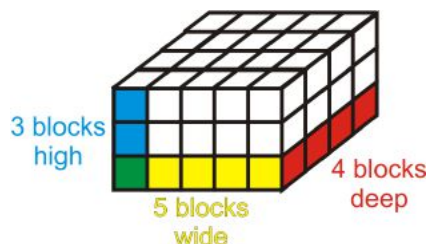
What this postulate tells us is that every solid can be broken down into cubes. For example, if we wanted to find the volume of a cube with 9 inch sides, it would be $9^3 = 729 in^3$.

Volume Congruence Postulate: If two solids are congruent, then their volumes are congruent.

These prisms are congruent, so their volumes are congruent.



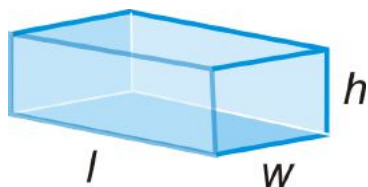
Example 1: Find the volume of the right rectangular prism below.



Solution: Count the cubes. The bottom layer has 20 cubes, or 4×5 , and there are 3 layers. There are 60 cubes. The volume is also 60 units^3 .

Each layer in Example 1 is the same as the area of the base and the number of layers is the same as the height. This is the formula for volume.

Volume of a Rectangular Prism: $V = l \cdot w \cdot h$.



Example 2: A typical shoe box is 8 in by 14 in by 6 in. What is the volume of the box?

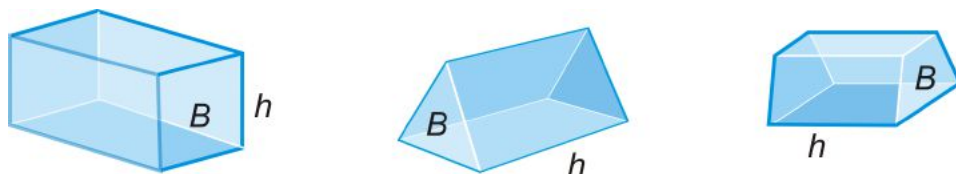
Solution: We can assume that a shoe box is a rectangular prism.

$$V = (8)(14)(6) = 672 \text{ in}^3$$

Volume of any Prism

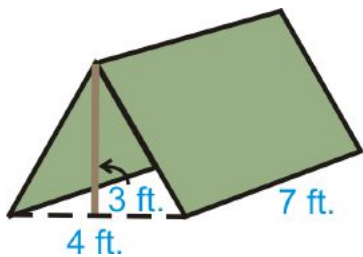
Notice that $l \cdot w$ is equal to the area of the base of the prism, which we will re-label B .

Volume of a Prism: $V = B \cdot h$.



“ B ” is not always going to be the same. So, to find the volume of a prism, you would first find the area of the base and then multiply it by the height.

Example 3: You have a small, triangular prism shaped tent. How much volume does it have, once it is set up?



Solution: First, we need to find the area of the base.

$$B = \frac{1}{2}(3)(4) = 6 \text{ ft}^2.$$

$$V = Bh = 6(7) = 42 \text{ ft}^3$$

Even though the height in this problem does not look like a “height,” it is, according to the formula. Usually, the height of a prism is going to be the last length you need to use.

Oblique Prisms

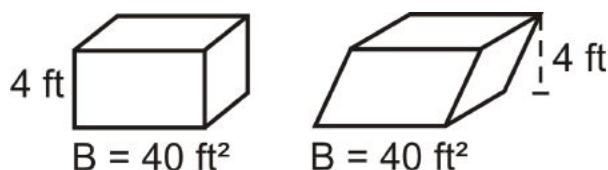
Recall that oblique prisms are prisms that lean to one side and the height is outside the prism. What would be the volume of an oblique prism? Consider to piles of books below.



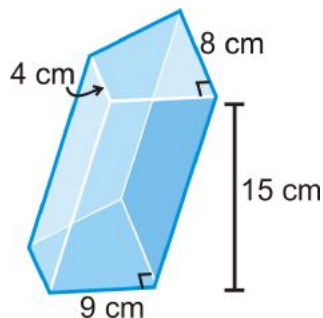
Both piles have 15 books, which means they will have the same volume. Cavalieri’s Principle says that leaning does not matter, the volumes are the same.

Cavalieri’s Principle: If two solids have the same height and the same cross-sectional area at every level, then they will have the same volume.

If an oblique prism and a right prism have the same base area and height, then they will have the same volume.



Example 4: Find the area of the oblique prism below.



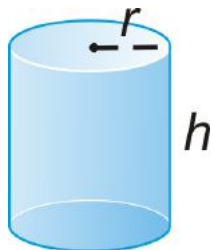
Solution: This is an oblique right trapezoidal prism. Find the area of the trapezoid.

$$B = \frac{1}{2}(9)(8 + 4) = 9(6) = 54 \text{ cm}^2$$
$$V = 54(15) = 810 \text{ cm}^3$$

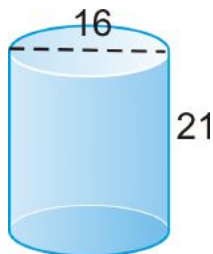
Volume of a Cylinder

If we use the formula for the volume of a prism, $V = Bh$, we can find the volume of a cylinder. In the case of a cylinder, the base is the area of a circle. Like a prism, Cavalieri's Principle holds.

Volume of a Cylinder: $V = \pi r^2 h$.



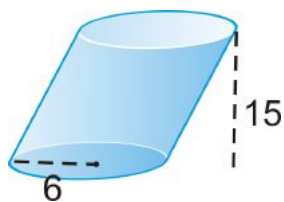
Example 5: Find the volume of the cylinder.



Solution: If the diameter is 16, then the radius is 8.

$$V = \pi 8^2(21) = 1344\pi \text{ units}^3$$

Example 6: Find the volume of the cylinder.



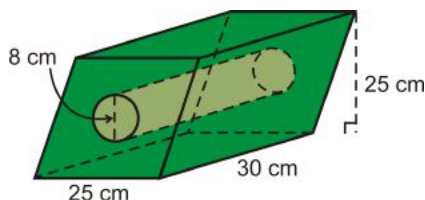
Solution: $V = \pi 6^2(15) = 540\pi \text{ units}^3$

Example 7: If the volume of a cylinder is $484\pi \text{ in}^3$ and the height is 4 in, what is the radius?

Solution: Solve for r .

$$484\pi = \pi r^2(4)$$
$$121 = r^2$$
$$11 = r$$

Example 8: Find the volume of the solid below.



Solution: This solid is a parallelogram-based prism with a cylinder cut out of the middle.

$$V_{prism} = (25 \cdot 25)30 = 18750 \text{ cm}^3$$

$$V_{cylinder} = \pi(4)^2(30) = 480\pi \text{ cm}^3$$

The total volume is $18750 - 480\pi \approx 17242.04 \text{ cm}^3$.

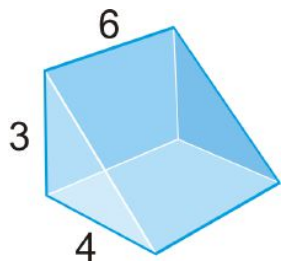
Know What? Revisited Even though it doesn't look like it, the trapezoid is the base of this prism. The area of the trapezoids are $\frac{1}{2}(4 + 8)25 = 150 \text{ ft}^2$. $V = 150(10) = 1500 \text{ ft}^3$

Review Questions

- Question 1 uses the volume formula for a cylinder.
- Questions 2-4 are similar to Example 1.
- Questions 5-18 are similar to Examples 2-6.
- Questions 19-24 are similar to Example 7.
- Questions 25-30 are similar to Example 8.

1. Two cylinders have the same surface area. Do they have the same volume? How do you know?
2. How many one-inch cubes can fit into a box that is 8 inches wide, 10 inches long, and 12 inches tall?
Is this the same as the volume of the box?
3. A cereal box is 2 inches wide, 10 inches long and 14 inches tall. How much cereal does the box hold?
4. A can of soda is 4 inches tall and has a diameter of 2 inches. How much soda does the can hold?
Round your answer to the nearest hundredth.
5. A cube holds 216 in^3 . What is the length of each edge?
6. A cube has sides that are 8 inches. What is the volume?
7. A cylinder has $r = h$ and the radius is 4 cm. What is the volume?
8. A cylinder has a volume of $486\pi \text{ ft}^3$. If the height is 6 ft., what is the diameter?

Use the right triangular prism to answer questions 9 and 10.



9. What is the length of the third base edge?
10. Find the volume of the prism.

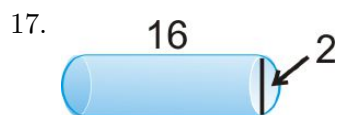
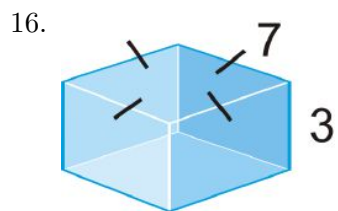
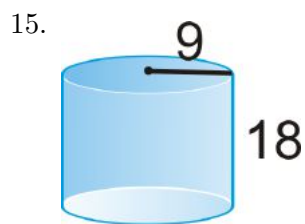
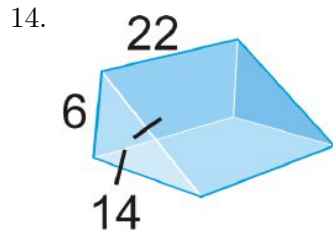
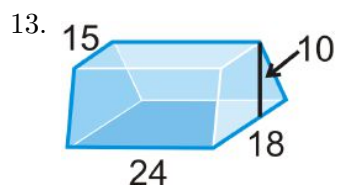
11. Fuzzy dice are cubes with 4 inch sides.



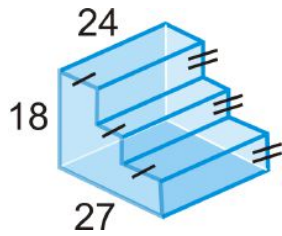
- (a) What is the volume of one die?
(b) What is the volume of both dice?

12. A right cylinder has a 7 cm radius and a height of 18 cm. Find the volume.

Find the volume of the following solids. Round your answers to the nearest hundredth.

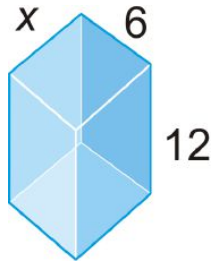


18.

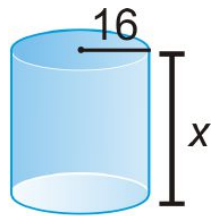


Algebra Connection Find the value of x , given the surface area.

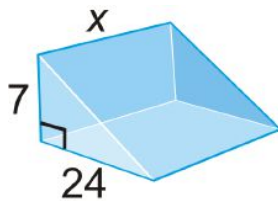
19. $V = 504 \text{ units}^3$



20. $V = 6144\pi \text{ units}^3$



21. $V = 2688 \text{ units}^3$

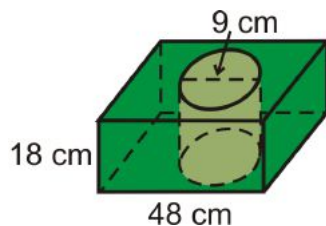


22. The area of the base of a cylinder is $49\pi \text{ in}^2$ and the height is 6 in. Find the volume.

23. The circumference of the base of a cylinder is $80\pi \text{ cm}$ and the height is 15 cm. Find the volume.

24. The lateral surface area of a cylinder is $30\pi \text{ m}^2$ and the circumference is $10\pi \text{ m}$. What is the volume of the cylinder?

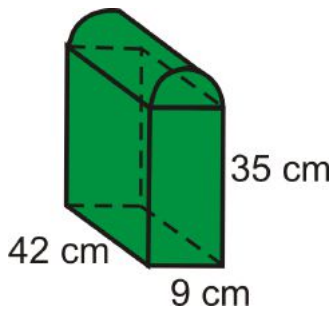
The bases of the prism are squares and a cylinder is cut out of the center.



25. Find the volume of the prism.

26. Find the volume of the cylinder in the center.
27. Find the volume of the figure.

This is a prism with half a cylinder on the top.



28. Find the volume of the prism.
29. Find the volume of the half-cylinder.
30. Find the volume of the entire figure.

Review Queue Answers

1. The amount a three-dimensional figure can hold.
2. 54 in^2
3. (a) $2\sqrt{3}$
 (b) $\frac{1}{2} \cdot 4 \cdot 2\sqrt{3} = 4\sqrt{3}$
 (c) $8 \cdot 4\sqrt{3} = 32\sqrt{3}$

7.5 Volume of Pyramids and Cones

Learning Objectives

- Find the volume of pyramids and cones.

Review Queue

1. Find the volume of a square prism with 8 inch base edges and a 12 inch height.
2. Find the volume of a cylinder with a diameter of 8 inches and a height of 12 inches.
3. Find the surface area of a square pyramid with 10 inch base edges and a height of 12 inches.

Know What? The Khafre Pyramid is a pyramid in Giza, Egypt. It is a square pyramid with a base edge of 706 feet and an original height of 407.5 feet. What was the original volume of the Khafre Pyramid?



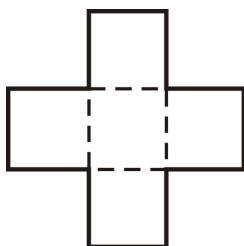
Volume of a Pyramid

The volume of a pyramid is closely related to the volume of a prism with the same sized base.

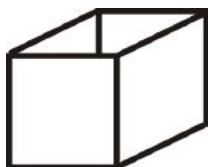
Investigation 11-1: Finding the Volume of a Pyramid

Tools needed: pencil, paper, scissors, tape, ruler, dry rice.

1. Make an open net (omit one base) of a cube, with 2 inch sides.



2. Cut out the net and tape up the sides to form an open cube.



3. Make an open net (no base) of a square pyramid, with lateral edges of 2.5 inches and base edges of 2 inches.

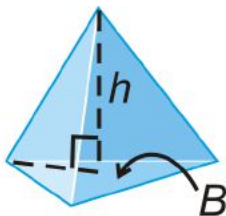
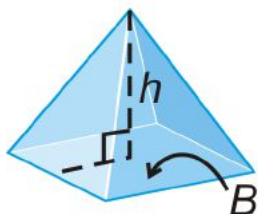


4. Cut out the net and tape up the sides to form an open pyramid.

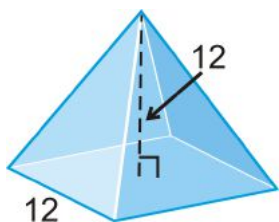


5. Fill the pyramid with dry rice and dump the rice into the open cube. Repeat this until you have filled the cube?

Volume of a Pyramid: $V = \frac{1}{3}Bh$.

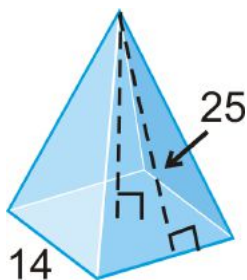


Example 1: Find the volume of the pyramid.



Solution: $V = \frac{1}{3}(12^2)12 = 576 \text{ units}^3$

Example 2a: Find the height of the pyramid.



Solution: In this example, we are given the *slant* height. Use the Pythagorean Theorem.

$$7^2 + h^2 = 25^2$$

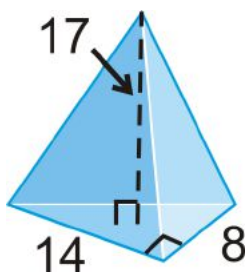
$$h^2 = 576$$

$$h = 24$$

Example 2b: Find the volume of the pyramid in Example 2a.

Solution: $V = \frac{1}{3}(14^2)(24) = 1568 \text{ units}^3$.

Example 3: Find the volume of the pyramid.



Solution: The base of the pyramid is a right triangle. The area of the base is $\frac{1}{2}(14)(8) = 56 \text{ units}^2$.

$$V = \frac{1}{3}(56)(17) \approx 317.33 \text{ units}^3$$

Example 4: A rectangular pyramid has a base area of 56 cm^2 and a volume of 224 cm^3 . What is the height of the pyramid?

Solution:

$$\begin{aligned} V &= \frac{1}{3}Bh \\ 224 &= \frac{1}{3} \cdot 56h \\ 12 &= h \end{aligned}$$

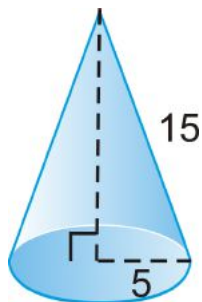
Volume of a Cone

Volume of a Cone: $V = \frac{1}{3}\pi r^2 h$.

This is the the same relationship as a pyramid's volume with a prism's volume.



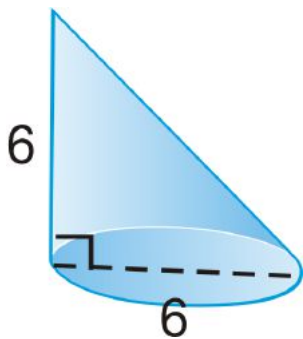
Example 5: Find the volume of the cone.



Solution: First, we need the height. Use the Pythagorean Theorem.

$$\begin{aligned} 5^2 + h^2 &= 15^2 \\ h &= \sqrt{200} = 10\sqrt{2} \\ V &= \frac{1}{3}(5^2)(10\sqrt{2})\pi \approx 370.24 \end{aligned}$$

Example 6: Find the volume of the cone.



Solution: We can use the same volume formula. Find the *radius*.

$$V = \frac{1}{3}\pi(3^2)(6) = 18\pi \approx 56.55$$

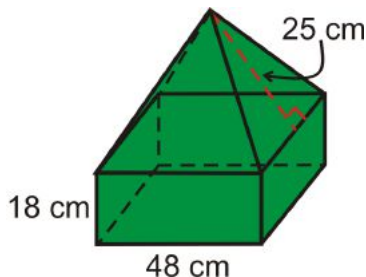
Example 7: The volume of a cone is $484\pi \text{ cm}^3$ and the height is 12 cm. What is the radius?

Solution: Plug in what you know to the volume formula.

$$\begin{aligned} 484\pi &= \frac{1}{3}\pi r^2(12) \\ 121 &= r^2 \\ 11 &= r \end{aligned}$$

Composite Solids

Example 8: Find the volume of the composite solid. All bases are squares.



Solution: This is a square prism with a square pyramid on top. First, we need the height of the pyramid portion. Using the Pythagorean Theorem, we have, $h = \sqrt{25^2 - 24^2} = 7$.

$$\begin{aligned} V_{prism} &= (48)(48)(18) = 41472 \text{ cm}^3 \\ V_{pyramid} &= \frac{1}{3}(48^2)(7) = 5376 \text{ cm}^3 \end{aligned}$$

The total volume is $41472 + 5376 = 46,848 \text{ cm}^3$.

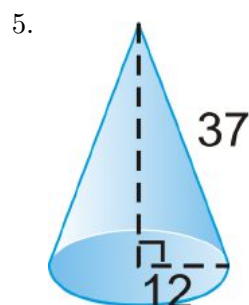
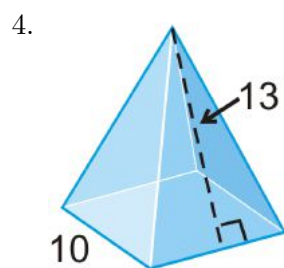
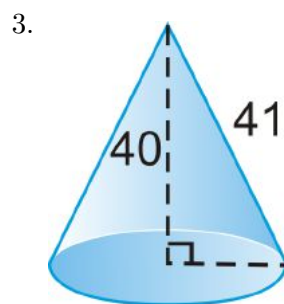
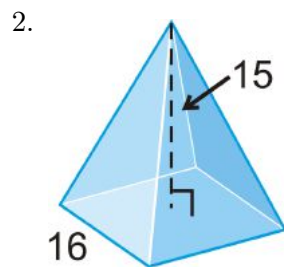
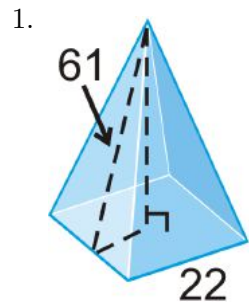
Know What? Revisited The original volume of the pyramid is $\frac{1}{3}(706^2)(407.5) \approx 67,704,223.33 \text{ ft}^3$.

Review Questions

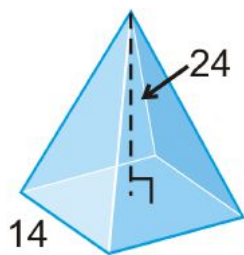
- Questions 1-13 are similar to Examples 1-3, 5 and 6.
- Questions 14-22 are similar to Examples 4 and 7.

- Questions 23-31 are similar to Example 8.

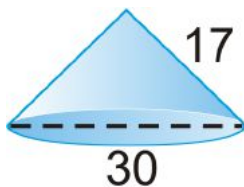
Find the volume of each regular pyramid and right cone. Round any decimal answers to the nearest hundredth. The bases of these pyramids are either squares or equilateral triangles.



6.

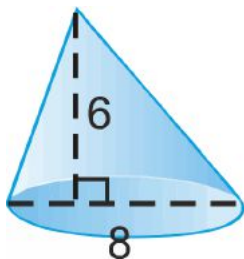


7.

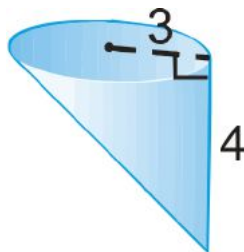


Find the volume of the following non-regular pyramids and cones. Round any decimal answers to the nearest hundredth.

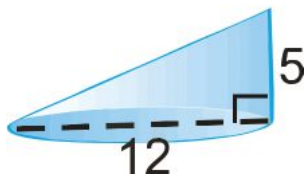
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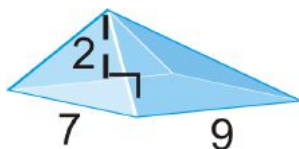
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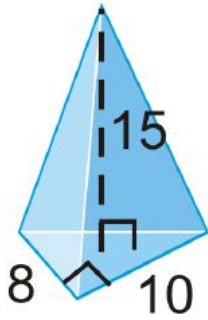
10.



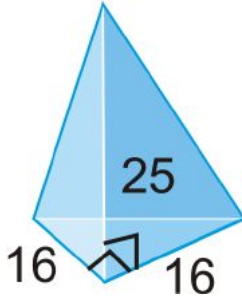
11. base is a rectangle



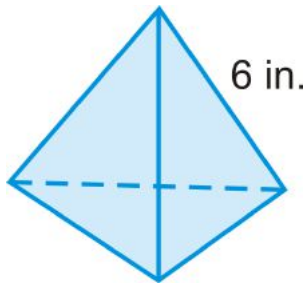
12.



13.

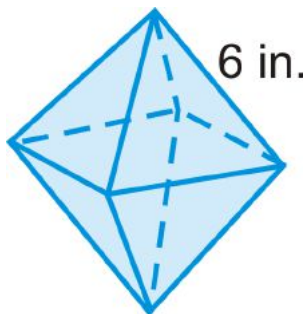


A **regular tetrahedron** has four equilateral triangles as its faces. Use the diagram to answer questions 14-16. Round your answers to the nearest hundredth.



14. What is the area of the base of this regular tetrahedron?
15. What is the height of this figure? Be careful!
16. Find the volume.

A **regular octahedron** has eight equilateral triangles as its faces. Use the diagram to answer questions 17-21. Round your answers to the nearest hundredth.

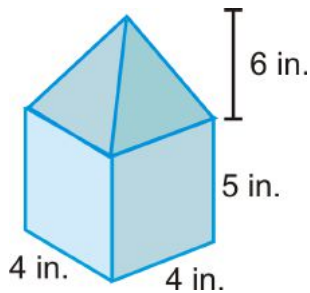


17. *Describe* how you would find the volume of this figure.
18. Find the volume.

19. The volume of a square pyramid is 72 square inches and the base edge is 4 inches. What is the height?
20. If the volume of a cone is $30\pi \text{ cm}^3$ and the radius is 5 cm, what is the height?
21. If the volume of a cone is $105\pi \text{ cm}^3$ and the height is 35 cm, what is the radius?
22. The volume of a triangle pyramid is 170 in^3 and the base area is 34 in^2 . What is the height of the pyramid?

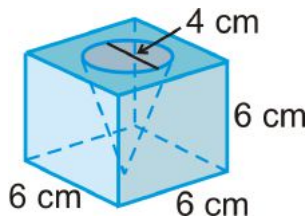
For questions 23-31, round your answer to the nearest hundredth.

23. Find the volume of the base prism.



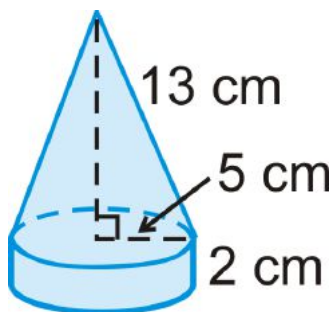
24. Find the volume of the pyramid.
25. Find the volume of the entire solid.

The solid to the right is a cube with a cone cut out.



26. Find the volume of the cube.
27. Find the volume of the cone.
28. Find the volume of the entire solid.

The solid to the left is a cylinder with a cone on top.



29. Find the volume of the cylinder.
30. Find the volume of the cone.
31. Find the volume of the entire solid.

Review Queue Answers

1. $(8^2)(12) = 768 \text{ in}^3$
2. $(4^2)(12)\pi = 192\pi \approx 603.19$
3. Find slant height, $l = 13$. $SA = 100 + \frac{1}{2}(40)(13) = 360 \text{ in}^2$

7.6 Surface Area and Volume of Spheres

Learning Objectives

- Find the surface area of a sphere.
- Find the volume of a sphere.

Review Queue

1. List three spheres you would see in real life.
2. Find the area of a circle with a 6 cm radius.
3. Find the volume of a cylinder with the circle from #2 as the base and a height of 5 cm.

Know What? A regulation bowling ball is a sphere with a circumference of 27 inches. Find the radius of a bowling ball, its surface area and volume. You may assume the bowling ball does not have any finger holes. Round your answers to the nearest hundredth.

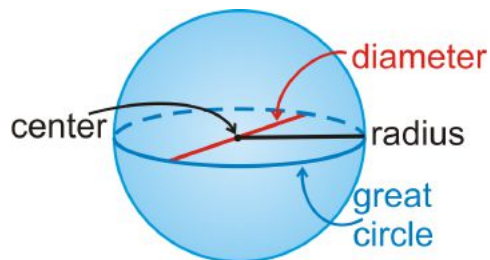


Defining a Sphere

A sphere is the last of the three-dimensional shapes that we will find the surface area and volume of. Think of a sphere as a three-dimensional circle.

Sphere: The set of all points, in three-dimensional space, which are equidistant from a point.

The **radius** has an endpoint on the sphere and the other endpoint is the center.

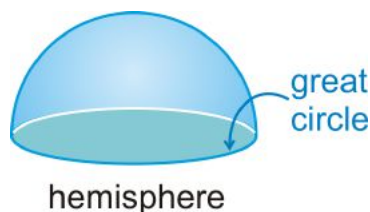


The **diameter** must contain the center.

Great Circle: A cross section of a sphere that contains the diameter.

A great circle is the largest circle cross section in a sphere. *The circumference of a sphere is the circumference of a great circle.*

Every great circle divides a sphere into two congruent *hemispheres*.



Example 1: The circumference of a sphere is 26π feet. What is the radius of the sphere?

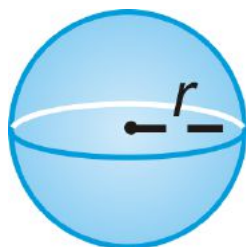
Solution: The circumference is referring to the circumference of a great circle. Use $C = 2\pi r$.

$$\begin{aligned}2\pi r &= 26\pi \\ r &= 13 \text{ ft.}\end{aligned}$$

Surface Area of a Sphere

The best way to understand the surface area of a sphere is to watch the link by Russell Knightley, <http://www.rkm.com.au/ANIMATIONS/animation-Sphere-Surface-Area-Derivation.html>.

Surface Area of a Sphere: $SA = 4\pi r^2$.

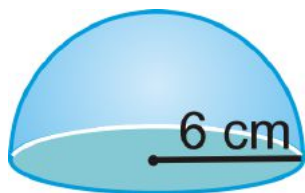


Example 2: Find the surface area of a sphere with a radius of 14 feet.

Solution:

$$\begin{aligned}SA &= 4\pi(14)^2 \\ &= 784\pi \text{ ft}^2\end{aligned}$$

Example 3: Find the surface area of the figure below.



Solution: Be careful when finding the surface area of a hemisphere because you need to include the area of the base.

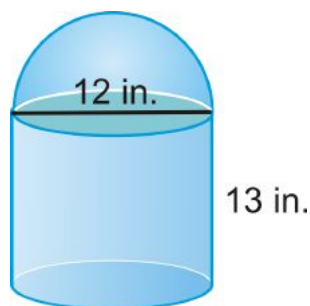
$$\begin{aligned}
 SA &= \pi r^2 + \frac{1}{2}4\pi r^2 \\
 &= \pi(6^2) + 2\pi(6^2) \\
 &= 36\pi + 72\pi = 108\pi \text{ cm}^2
 \end{aligned}$$

Example 4: The surface area of a sphere is $100\pi \text{ in}^2$. What is the radius?

Solution:

$$\begin{aligned}
 SA &= 4\pi r^2 \\
 100\pi &= 4\pi r^2 \\
 25 &= r^2 \\
 5 &= r
 \end{aligned}$$

Example 5: Find the surface area of the following solid.



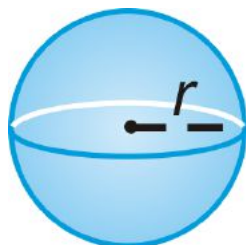
Solution: This solid is a cylinder with a hemisphere on top. It is one solid, so do not include the bottom of the hemisphere or the top of the cylinder.

$$\begin{aligned}
 SA &= LA_{\text{cylinder}} + LA_{\text{hemisphere}} + A_{\text{base circle}} \\
 &= \pi r h + \frac{1}{2}4\pi r^2 + \pi r^2 \\
 &= \pi(6)(13) + 2\pi 6^2 + \pi 6^2 \\
 &= 78\pi + 72\pi + 36\pi \\
 &= 186\pi \text{ in}^2 \qquad \text{"LA" stands for lateral area.}
 \end{aligned}$$

Volume of a Sphere

To see an animation of the volume of a sphere, see <http://www.rkm.com.au/ANIMATIONS/animation-Sphere-Volume.html> by Russell Knightley.

Volume of a Sphere: $V = \frac{4}{3}\pi r^3$.



Example 6: Find the volume of a sphere with a radius of 9 m.

Solution:

$$\begin{aligned} V &= \frac{4}{3}\pi 6^3 \\ &= \frac{4}{3}\pi(216) \\ &= 288\pi \text{ m}^3 \end{aligned}$$

Example 7: A sphere has a volume of 14137.167 ft^3 , what is the radius?

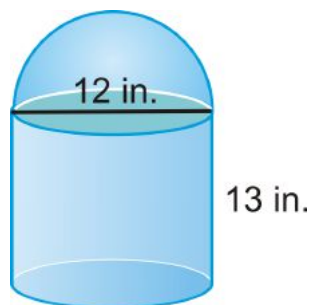
Solution:

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ 14137.167 &= \frac{4}{3}\pi r^3 \\ \frac{3}{4\pi} \cdot 14137.167 &= r^3 \\ 3375 &= r^3 \end{aligned}$$

At this point, you will need to take the **cubed root** of 3375. Ask your teacher how to do this on your calculator.

$$\sqrt[3]{3375} = 15 = r$$

Example 8: Find the volume of the following solid.



Solution:

$$\begin{aligned} V_{\text{cylinder}} &= \pi 6^2(13) = 78\pi \\ V_{\text{hemisphere}} &= \frac{1}{2} \left(\frac{4}{3}\pi 6^3 \right) = 36\pi \\ V_{\text{total}} &= 78\pi + 36\pi = 114\pi \text{ in}^3 \end{aligned}$$

Know What? Revisited The radius would be $27 = 2\pi r$, or $r = 4.30 \text{ inches}$. The surface area would be $4\pi 4.3^2 \approx 232.35 \text{ in}^2$, and the volume would be $\frac{4}{3}\pi 4.3^3 \approx 333.04 \text{ in}^3$.

Review Questions

- Questions 1-3 look at the definition of a sphere.

- Questions 4-17 are similar to Examples 1, 2, 4, 6 and 7.
- Questions 18-21 are similar to Example 3 and 5.
- Questions 22-25 are similar to Example 8.
- Question 26 is a challenge.
- Questions 27-29 are similar to Example 8.
- Question 30 analyzes the formula for the surface area of a sphere.

1. Are there any cross-sections of a sphere that are not a circle? Explain your answer.
2. List all the parts of a sphere that are the **same** as a circle.
3. List any parts of a sphere that a circle does not have.

Find the surface area and volume of a sphere with: (Leave your answer in terms of π)

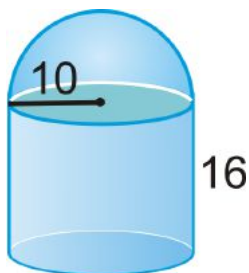
4. a radius of 8 in.
5. a diameter of 18 cm.
6. a radius of 20 ft.
7. a diameter of 4 m.
8. a radius of 15 ft.
9. a diameter of 32 in.
10. a circumference of 26π cm.
11. a circumference of 50π yds.
12. The surface area of a sphere is 121π in². What is the radius?
13. The volume of a sphere is 47916π m³. What is the radius?
14. The surface area of a sphere is 4π ft². What is the volume?
15. The volume of a sphere is 36π mi³. What is the surface area?
16. Find the radius of the sphere that has a volume of 335 cm³. Round your answer to the nearest hundredth.
17. Find the radius of the sphere that has a surface area 225π ft².

Find the surface area of the following shapes. Leave your answers in terms of π .

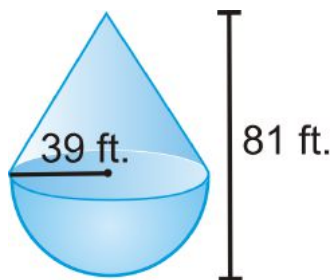
18.



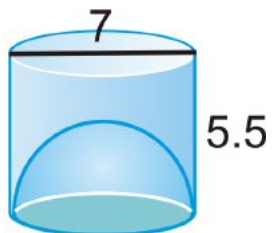
19.



20.

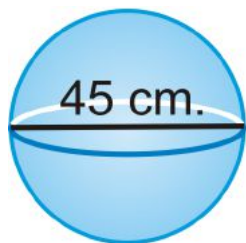


21. You may assume the bottom is *open*.

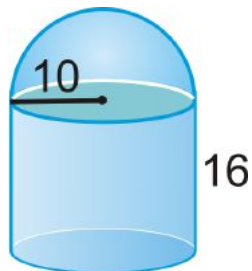


Find the volume of the following shapes. Round your answers to the nearest hundredth.

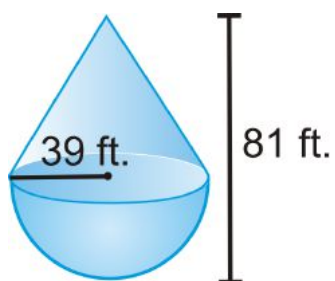
22.



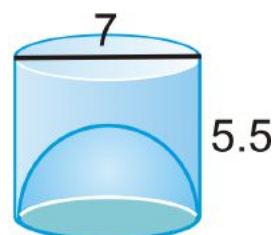
23.



24.



25.



26. A sphere has a radius of 5 cm. A right cylinder has the same radius and volume. Find the height of the cylinder.

Tennis balls with a 3 inch diameter are sold in cans of three. The can is a cylinder. Round your answers to the nearest hundredth.



27. What is the volume of one tennis ball?
28. What is the volume of the cylinder?
29. Assume the balls touch the can on the sides, top and bottom. What is the volume of the space *not* occupied by the tennis balls?
30. How does the formula of the surface area of a sphere relate to the area of a circle?

Review Queue Answers

1. *Answers will vary.* Possibilities are any type of ball, certain lights, or the 76/Unical orb.
2. 36π
3. 180π

7.7 Extension: Exploring Similar Solids

Learning Objectives

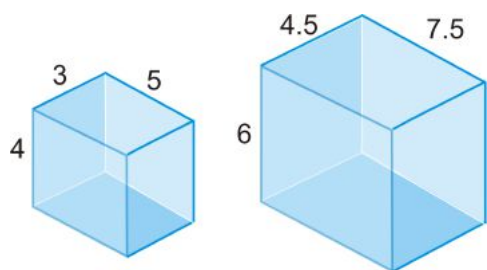
- Find the relationship between similar solids and their surface areas and volumes.

Similar Solids

Recall that two shapes are similar if all the corresponding angles are congruent and the corresponding sides are proportional.

Similar Solids: Two solids are similar if they are the same type of solid and their corresponding radii, heights, base lengths, widths, etc. are proportional.

Example 1: Are the two rectangular prisms similar? How do you know?

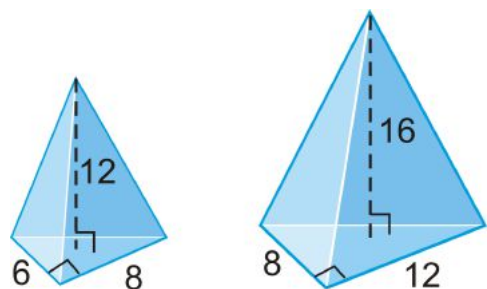


Solution: Match up the corresponding heights, widths, and lengths.

$$\frac{\text{small prism}}{\text{large prism}} = \frac{3}{4.5} = \frac{4}{6} = \frac{5}{7.5}$$

The congruent ratios tell us the two prisms are similar.

Example 2: Determine if the two triangular pyramids similar.



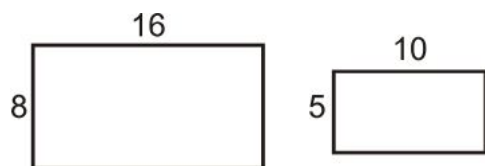
Solution: Just like Example 1, let's match up the corresponding parts.

$$\frac{6}{8} = \frac{3}{4} = \frac{12}{16} \text{ however, } \frac{8}{12} = \frac{2}{3}.$$

These triangle pyramids are not similar.

Surface Areas of Similar Solids

If two shapes are similar, then the ratio of the area is a square of the scale factor.

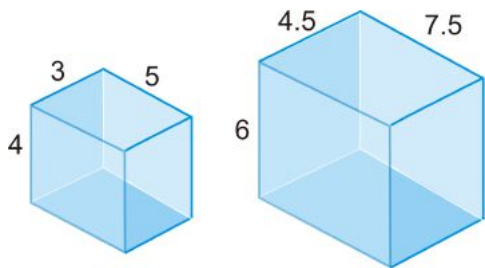


For example, the two rectangles are similar because their sides are in a ratio of 5:8. The area of the larger rectangle is $8(16) = 128 \text{ units}^2$. The area of the smaller rectangle is $5(10) = 50 \text{ units}^2$.

Comparing the areas in a ratio, it is $50 : 128 = 25 : 64 = 5^2 = 8^2$.

So, what happens with the surface areas of two similar solids?

Example 3: Find the surface area of the two similar rectangular prisms.



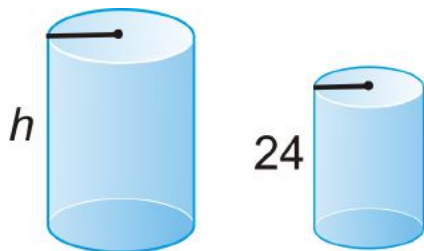
Solution:

$$\begin{aligned}
 SA_{\text{smaller}} &= 2(4 \cdot 3) + 2(4 \cdot 5) + 2(3 \cdot 5) \\
 &= 24 + 40 + 30 = 94 \text{ units}^2 \\
 SA_{\text{larger}} &= 2(6 \cdot 4.5) + 2(4.5 \cdot 7.5) + 2(6 \cdot 7.5) \\
 &= 54 + 67.5 + 90 = 211.5 \text{ units}^2
 \end{aligned}$$

Now, find the ratio of the areas. $\frac{94}{211.5} = \frac{4}{9} = \frac{2^2}{3^2}$. The sides are in a ratio of $\frac{4}{6} = \frac{2}{3}$, so the surface areas are in a ratio of $\frac{2^2}{3^2}$.

Surface Area Ratio: If two solids are similar with a scale factor of $\frac{a}{b}$, then the surface areas are in a ratio of $\left(\frac{a}{b}\right)^2$.

Example 4: Two similar cylinders are below. If the ratio of the areas is 16:25, what is the height of the taller cylinder?



Solution: First, we need to take the square root of the area ratio to find the scale factor, $\sqrt{\frac{16}{25}} = \frac{4}{5}$. Set up a proportion to find h .

$$\begin{aligned}
 \frac{4}{5} &= \frac{24}{h} \\
 4h &= 120 \\
 h &= 30
 \end{aligned}$$

Example 5: Using the cylinders from Example 4, if the area of the smaller cylinder is $1536\pi \text{ cm}^2$, what is the area of the larger cylinder?

Solution: Set up a proportion using the ratio of the areas, 16:25.

$$\begin{aligned}
 \frac{16}{25} &= \frac{1536\pi}{A} \\
 16A &= 38400\pi \\
 A &= 2400\pi \text{ cm}^2
 \end{aligned}$$

Volumes of Similar Solids

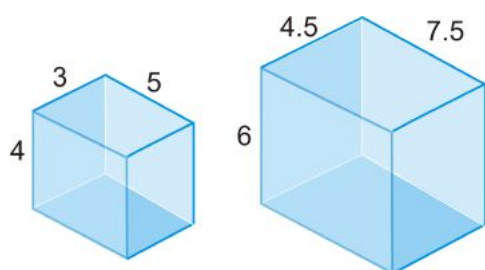
Let's look at what we know about similar solids so far.

Table 7.3:

	<i>Ratios</i>	<i>Units</i>
<i>Scale Factor</i>	$\frac{a}{b}$	in, ft, cm, m, etc.
<i>Ratio of the Surface Areas</i>	$\left(\frac{a}{b}\right)^2$	in^2, ft^2, cm^2, m^2 , etc.
<i>Ratio of the Volumes</i>	??	in^3, ft^3, cm^3, m^3 , etc.

If the ratio of the volumes follows the pattern from above, it should be the **cube** of the scale factor.

Example 6: Find the volume of the following rectangular prisms. Then, find the ratio of the volumes.



Solution:

$$V_{smaller} = 3(4)(5) = 60$$

$$V_{larger} = 4.5(6)(7.5) = 202.5$$

The ratio is $\frac{60}{202.5}$, which reduces to $\frac{8}{27} = \frac{2^3}{3^3}$.

Volume Ratio: If two solids are similar with a scale factor of $\frac{a}{b}$, then the volumes are in a ratio of $\left(\frac{a}{b}\right)^3$.

Example 7: Two spheres have radii in a ratio of 3:4. What is the ratio of their volumes?

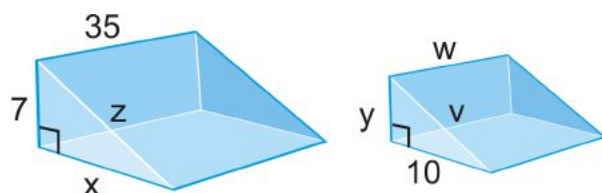
Solution: If we cube 3 and 4, we will have the ratio of the volumes. $3^3 : 4^3 = 27 : 64$.

Example 8: If the ratio of the volumes of two similar prisms is 125:8, what is the scale factor?

Solution: Take the **cubed root** of 125 and 8 to find the scale factor.

$$\sqrt[3]{125} : \sqrt[3]{8} = 5 : 2$$

Example 9: Two similar right triangle prisms are below. If the ratio of the volumes is 343:125, find the missing sides in both triangles.



Solution: The scale factor is 7:5, the cubed root. With the scale factor, we can now set up several proportions.

$$\frac{7}{5} = \frac{7}{y}$$

$$y = 5$$

$$\frac{7}{5} = \frac{x}{10}$$

$$x = 14$$

$$\frac{7}{5} = \frac{35}{w}$$

$$w = 25$$

$$7^2 + x^2 = z^2$$

$$7^2 + 14^2 = z^2$$

$$z = \sqrt{245} = 7\sqrt{5}$$

$$\frac{7}{5} = \frac{z}{v}$$

$$\frac{7}{5} = \frac{7\sqrt{5}}{v} \rightarrow v = 5\sqrt{5}$$

Example 10: The ratio of the surface areas of two similar cylinders is 16:81. What is the ratio of the volumes?

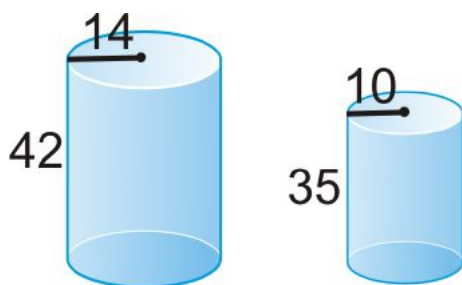
Solution: First, find the scale factor. If we take the square root of both numbers, the ratio is 4:9. Now, cube this to find the ratio of the volumes, $4^3 : 9^3 = 64 : 729$.

Review Questions

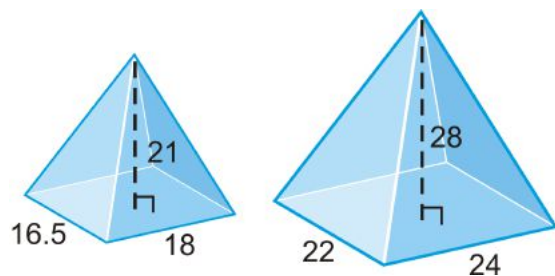
- Questions 1-4 are similar to Examples 1 and 2.
- Questions 5-14 are similar to Examples 3-8 and 10.
- Questions 15-18 are similar to Example 9.
- Questions 19 and 20 are similar to Example 1.

Determine if each pair of right solids are similar.

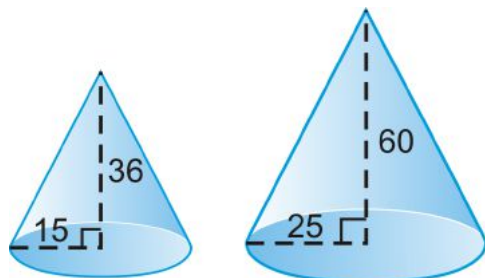
1.



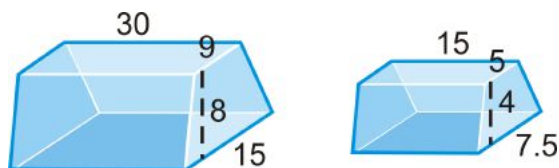
2.



3.

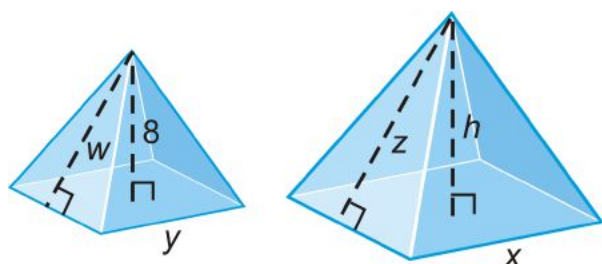


4.



5. Are all cubes similar? Why or why not?
6. Two prisms have a scale factor of 1:4. What is the ratio of their surface areas?
7. Two pyramids have a scale factor of 2:7. What is the ratio of their volumes?
8. Two spheres have radii of 5 and 9. What is the ratio of their volumes?
9. The surface area of two similar cones is in a ratio of 64:121. What is the scale factor?
10. The volume of two hemispheres is in a ratio of 125:1728. What is the scale factor?
11. A cone has a volume of 15π and is similar to another larger cone. If the scale factor is 5:9, what is the volume of the larger cone?
12. The ratio of the volumes of two similar pyramids is 8:27. What is the ratio of their total surface areas?
13. The ratio of the volumes of two tetrahedrons is 1000:1. The smaller tetrahedron has a side of length 6 cm. What is the side length of the larger tetrahedron?
14. The ratio of the surface areas of two cubes is 64:225. What is the ratio of the volumes?

Below are two similar square pyramids with a volume ratio of 8:27. The base lengths are equal to the heights. Use this to answer questions 15-18.



18. What is the scale factor?
19. What is the ratio of the surface areas?
20. Find h , x and y .
21. Find the volume of both pyramids.

Use the hemispheres below to answer questions 19-20.



1. Are the two hemispheres similar? How do you know?
2. Find the ratio of the surface areas and volumes.

7.8 Chapter 11 Review

Keywords, Theorems, & Formulas

Exploring Solids

- Polyhedron
- Face
- Edge
- Vertex
- Prism
- Pyramid
- Euler's Theorem
- Regular Polyhedron
- Regular Tetrahedron
- Cube
- Regular Octahedron
- Regular Dodecahedron
- Regular Icosahedron
- Cross-Section
- Net

Surface Area of Prisms & Cylinders

- Lateral Face
- Lateral Edge
- Base Edge
- Right Prism
- Oblique Prism
- Surface Area
- Lateral Area
- Surface Area of a Right Prism
- Cylinder
- Surface Area of a Right Cylinder

Surface Area of Pyramids & Cones

- Surface Area of a Regular Pyramid
- Cone
- Slant Height
- Surface Area of a Right Cone

Volume of Prisms & Cylinders

- Volume
- Volume of a Cube Postulate
- Volume Congruence Postulate
- Volume of a Rectangular Prism

- Volume of a Prism
- Cavalieri's Principle
- Volume of a Cylinder

Volume of Pyramids & Cones

- Volume of a Pyramid
- Volume of a Cone

Surface Area and Volume of Spheres

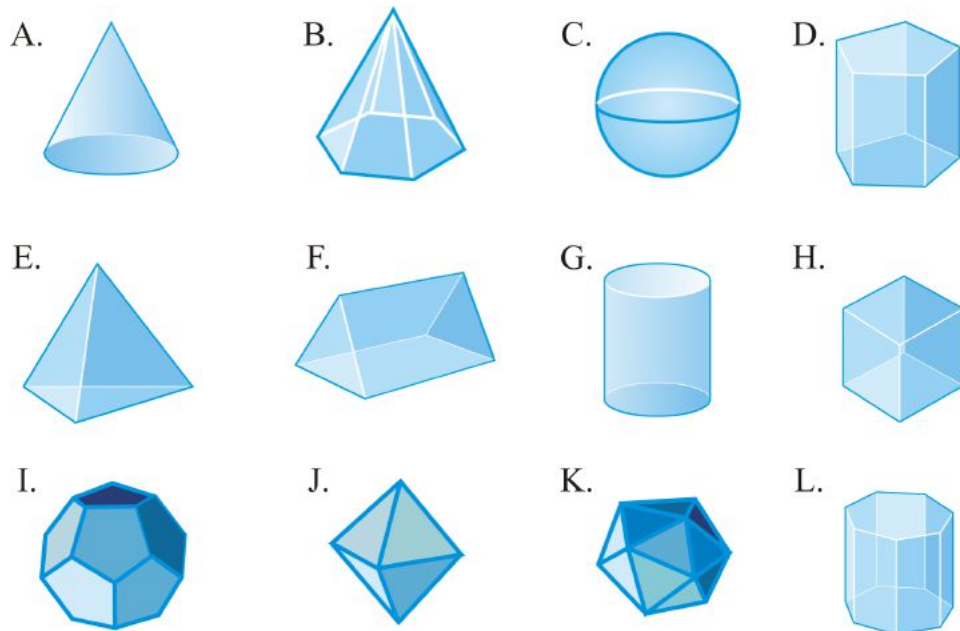
- Sphere
- Great Circle
- Surface Area of a Sphere
- Volume of a Sphere

Extension: Similar Solids

- Similar Solids
- Surface Area Ratio
- Volume Ratio

Review Questions

Match the shape with the correct name.



1. Triangular Prism
2. Icosahedron
3. Cylinder
4. Cone

5. Tetrahedron
6. Pentagonal Prism
7. Octahedron
8. Hexagonal Pyramid
9. Octagonal Prism
10. Sphere
11. Cube
12. Dodecahedron

Match the formula with its description.

13. Volume of a Prism - A. $\frac{1}{3}\pi r^2 h$
14. Volume of a Pyramid - B. $\pi r^2 h$
15. Volume of a Cone - C. $4\pi r^2$
16. Volume of a Cylinder - D. $\frac{4}{3}\pi r^3$
17. Volume of a Sphere - E. $\pi r^2 + \pi r l$
18. Surface Area of a Prism - F. $2\pi r^2 + 2\pi r h$
19. Surface Area of a Pyramid - G. $\frac{1}{3}Bh$
20. Surface Area of a Cone - H. Bh
21. Surface Area of a Cylinder - I. $B + \frac{1}{2}Pl$
22. Surface Area of a Sphere - J. The sum of the area of the bases and the area of each rectangular lateral face.

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9696>.

7.9 Study Guide

Keywords: Define, write theorems, and/or draw a diagram for each word below.

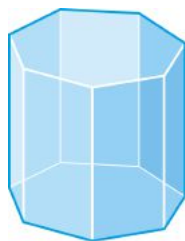
1st Section: Exploring Solids

Polyhedron

Face

Edge

Vertex

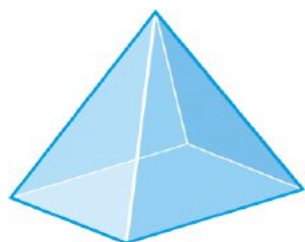


Prism

Pyramid

Euler's Theorem

Regular Polyhedron



Regular Tetrahedron

Cube

Regular Octahedron

Regular Dodecahedron

Regular Icosahedron

Cross-Section

Net

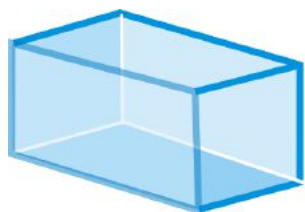
Homework:

2nd Section: Surface Area of Prisms & Cylinders

Lateral Face

Lateral Edge

Base Edge



Right Prism

Oblique Prism

Surface Area

Lateral Area

Surface Area of a Right Prism

Cylinder

Surface Area of a Right Cylinder

Homework:

3rd Section: Surface Area of Pyramids & Cones

Surface Area of a Regular Pyramid



Cone

Slant Height

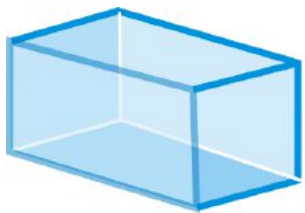
Surface Area of a Right Cone

Homework:

4th Section: Volume of Prisms & Cylinders

Volume

Volume of a Cube Postulate



Volume Congruence Postulate

Volume Addition Postulate

Volume of a Rectangular Prism

Volume of a Prism

Cavalieri's Principle

Volume of a Cylinder

Homework:

5th Section: Volume of Pyramids & Cones

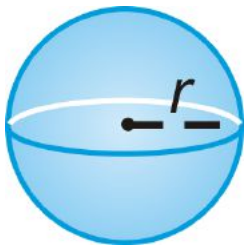
Volume of a Pyramid

Volume of a Cone

Homework:

6th Section: Surface Area and Volume of Spheres

Sphere



Great Circle

Surface Area of a Sphere

Volume of a Sphere

Homework:

Extension: Similar Solids

Similar Solids

Surface Area Ratio

Volume Ratio

Homework:

Chapter 8

Analyzing Conic Sections

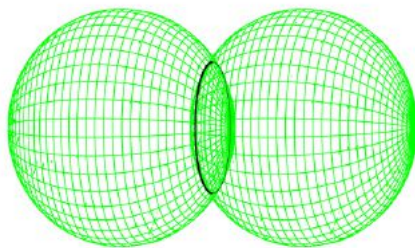
8.1 Introduction to Conic Sections

Learning Objectives

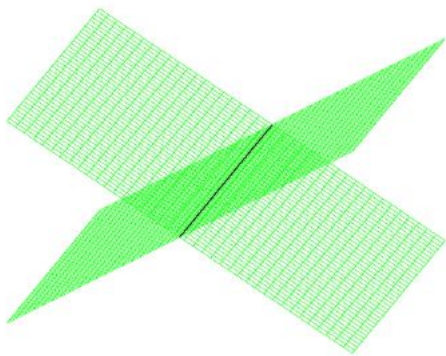
- Consider the results when two simple mathematical objects are intersected.
- Be comfortable working with an infinite two-sided cone.
- Know the basic types of figures that result from intersecting a plane and a cone.
- Know some of the history of the study of conic sections.

Introduction: Intersections of Figures

Some of the best mathematical shapes come from intersecting two other important shapes. Two spheres intersect to form a circle:

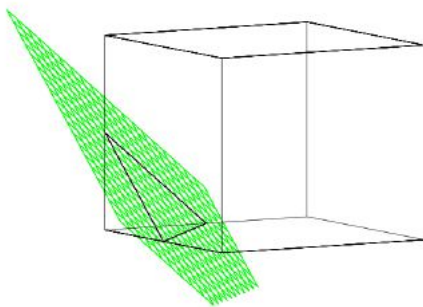


Two planes intersect to form a line:

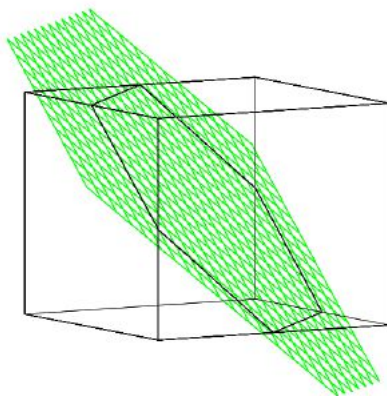


While simple and beautiful, for these two examples of intersections there isn't much else to investigate. For the spheres, no matter how we put them together, their intersection is always either nothing, a point (when they just touch), a circle of various sizes, or a sphere if they happen to be exactly the same size and coincide. Once we've exhausted this list, the inquiry is over. Planes are even simpler: the intersection of two distinct planes is either nothing (if the planes are parallel) or a line.

But some intersections yield more complex results. For instance a plane can intersect with a cube in numerous ways. Below a plane intersects a cube to form an equilateral triangle.



Here a plane intersects a cube and forms a regular hexagon.



Review Questions

1. Describe all the types of shapes that can be produced by the intersection of a plane and a cube.
2. What is the side-length of the regular hexagon that is produced in the above diagram when the cube

has side-length 1?

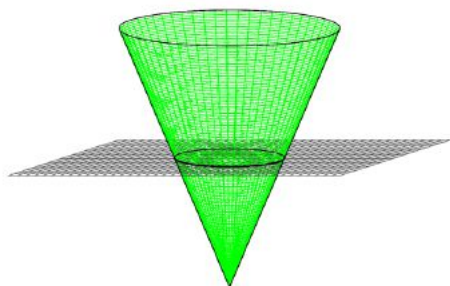
Review Answers

1. Square, rectangle, pentagon, others.
2. $length = \frac{\sqrt{2}}{2}$

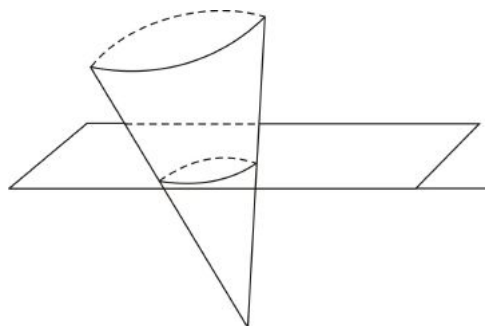
Intersections with Cones

One class of intersections is of particular interest: The intersection of a plane and a cone. These intersections are called **conic sections** and the first person known to have studied them extensively is the Ancient Greek mathematician Menaechmus in the 3rd Century B. C. E. Part of his interest in the conic sections came from his work on a classic Greek problem called “doubling the cube,” and we will describe this problem and Menaechmus’ approach that uses conic sections in section three. The intersections of the cone and the plane are so rich that the resulting shapes have continued to be of interest and generate new ideas from Menaechmus’ time until the present.

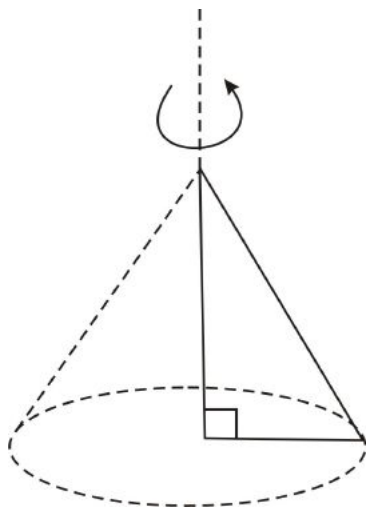
Before we really delve into what we mean by a plane and a cone, we can look at an intuitive example. Suppose by a cone we just mean an ice-cream cone. And by a plane we mean a piece of paper. Well, if you sliced through an upright ice-cream cone with a horizontal piece of paper you would find that the two objects intersect at a circle.



That’s very nice. But unlike the intersection of two spheres, which also resulted in a circle, that’s not *all* we get. If we tilt the paper (or the ice cream cone) things start to get tricky.

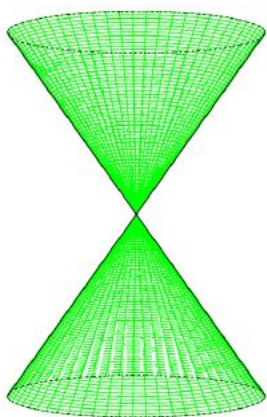


First things first, we better make sure we know what we mean by “plane” and “cone”. Let’s use the simplest definitions possible. So by “plane”, we mean the infinitely thin flat geometric object that extends forever in all directions. Even though infinity is a tricky concept, this plane is in some sense simpler than one that ends arbitrarily. There is no boundary to think about with the infinite plane. And what do we mean by a cone? An ice-cream cone is a good start. In fact, it’s very similar to how the ancient Greeks defined a cone, as a right triangle rotated about one of its legs.



Like we do with the infinite geometric plane, we want to idealize this object a bit too. As with the plane, to avoid having to deal with a boundary, let's suppose it continues infinitely in the direction of its open end.

But the Greek mathematician Apollonius noticed that it helps even more to have it go to infinity in the other direction. This way, a cone can be thought of as an infinite collection of lines, and since geometric lines go on forever in both directions, a cone also extends to infinity in both directions. Here is a picture of what we will call a **cone** in this chapter (remember it extends to infinity in both directions).



A cone can be formally defined as a three-dimensional collection of lines, all forming an equal angle with a central line or axis. In the above picture, the central line is vertical.

Review Questions

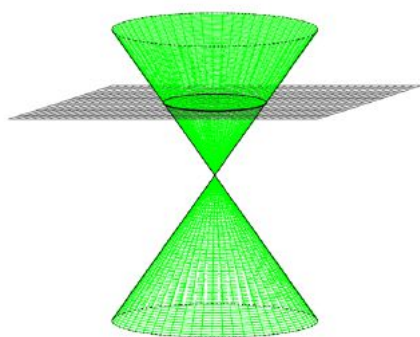
3. What other mathematical objects can be generated by a collection of lines?

Review Answers

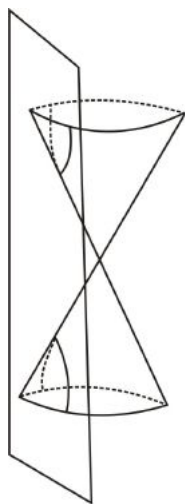
3. Cylinder (infinite), plane, three-sided infinite pyramids, others.

Intersections of an Infinite Cone and Plane

Like with the ice cream cone and the piece of paper, some intersections of this infinite cone with an infinite plane yield a circle.

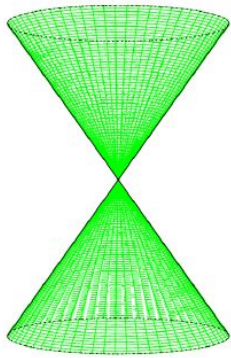


Other intersections yield something a bit less tidy. In the picture below, the plane is parallel to the axis of the cone.



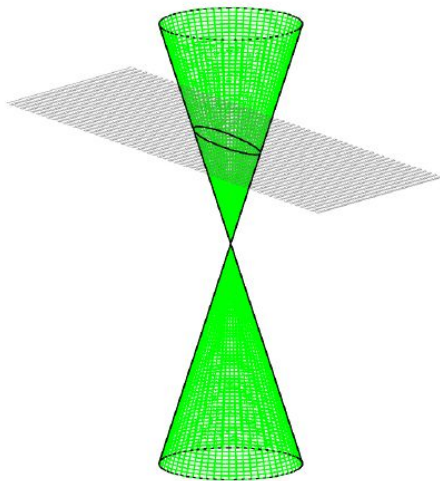
But it turns out that the set of intersections of a cone and a plane forms a beautiful, mathematically consistent, set of shapes that have interesting properties. So in this chapter we embark on a study of these intersections called *conic sections*.

Note: Although we will not prove it here, it doesn't matter if the cone is asymmetrical or tilted to one side (also called "oblique"). If you include "tilted cones" the same conic sections result. So, for simplicity's sake let's stick with non-tilted, or "right" cones and focus on what happens

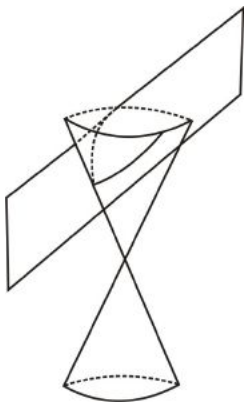


Let's begin by doing a rough tally of the kinds of shapes it seems we can generate by slicing planes through our cone. First of all, we have the **circle**, as we discovered with the ice cream cone above. A circle is formed when the plane is perpendicular to the line in the middle of the cone.

If we tilt the plane a little, but not so much that it intersects both cones, we get something more oval shaped. This is called an **ellipse**. Later you will learn many of the fascinating properties of the ellipse.



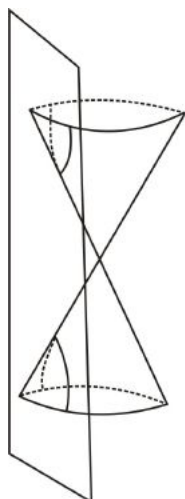
If really tilt the plane more so that it only intersects one side of the cone, but we get a big infinite “U”.



This shape is called a **parabola** and like the ellipse it has a number of surprising properties.

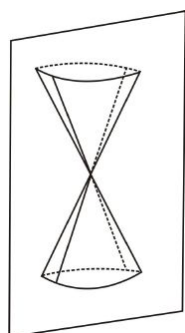
Then if we tilt it even further, we intersect both sides of the cone and get two big “U’s” going in opposite

directions.



This pair of objects is called a hyperbola, and, like the parabola and ellipse, the hyperbola has a number of interesting properties that we will discuss.

Finally, in a few cases we get what are called “degenerate” conics. For instance, if we line up a vertical plane with the vertex of the cone, we get two crossing lines.



Review Questions

4. Are there any other types of intersections between a plane and a cone besides the ones illustrated above?
5. Is it possible for a plane to miss a cone entirely?

Review Answers

4. Yes, the plane could meet the vertex, resulting in an intersection of a single point.
5. It is not possible for a plane to miss a cone entirely since both of the objects extend infinitely.

Applications and Importance

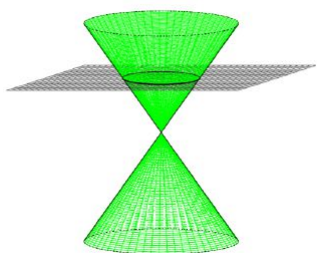
The intersections of cones and planes produce an interesting set of shapes, which we will study in the upcoming sections. We will also see numerous applications of these shapes to the physical world. Why is it that the intersection of a cone and a plane would produce so many applications? We don't seem to see

a lot of cones or planes in our daily life. But at closer inspection they are everywhere. Point sources of light, approximated by such sources as a flashlight or the sun, shine in cone-shaped array, so for instance the image of a flashlight against a slanted wall is an ellipse.

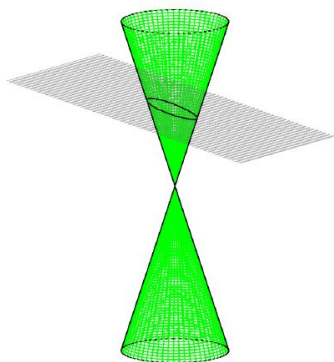
The geometric properties of conics, such as the focal property of the ellipse, turn out to have many physical ramifications, such as the design of telescopes. And, when we inspect the algebraic representations of conic sections, we will see that there are similarities with the law of gravity, which in turn has ramifications for planetary motion.

Lesson Summary

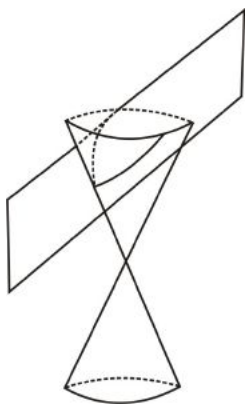
In summary, here are some of the ways that a plane can intersect a cone.



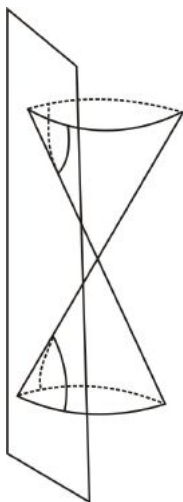
A circle



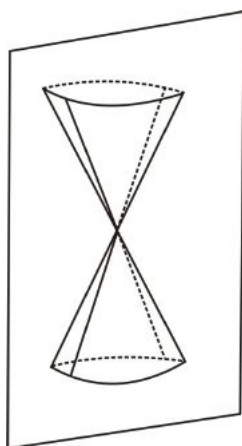
an “oval” called the ellipse



A big infinite “U” called a parabola.



Two big, infinite “U”s called a hyperbola.



Strange “degenerate” shapes like two crossing lines, as well as other examples you may have found in Review Question 1.

This array of shapes has a surprising amount of mathematical coherence, as well as a large number of interesting properties.

Vocabulary

Cone A three-dimensional collection of lines, all forming an equal angle with a central line or axis.

Conic section The points of intersection between a cone and a plane.

8.2 Circles and Ellipses

Learning Objectives

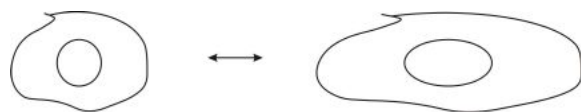
- Understand the difference between an “oval” and an ellipse.
- Recognize and work with equations for ellipses.
- Derive the focal property of ellipses.

- Understand the equivalence of different definitions of ellipses.
- Reconstruct Dandelin's sphere construction.
- Know some of the different ways people have approached ellipses throughout history.
- Understand some of the important applications of ellipses.

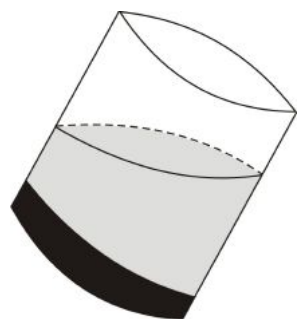
Introduction

Let's begin with the first class of shapes discussed in the last section. When the plane makes a finite intersection with one side of the cone, we get either a circle or the "oval-shaped" object illustrated in the previous section. It turns out that this is no ordinary oval, but something called an **ellipse**, a shape with special properties.

Like parallelograms, or any other shape with lots of interesting properties, ellipses can be defined by some of these properties, and then the other properties necessarily follow from the definition. For example, a parallelogram is typically defined as a quadrilateral with each pair of opposite pairs of sides parallel. Once you define it this way, it follows that the opposite sides must also be equal in length, and that the diagonals must bisect each other. Well, if you instead started by defining a parallelogram by one of these other properties, for instance opposite sides having equal lengths, then you would end up with the same class of shapes. The same thing happens for ellipses. One way to define an ellipse is as a "stretched out circle". It's the shape you would get if you sketched a circle on a deflated balloon and then stretched out the balloon evenly in two opposite directions:



It's also the shape of the surface of water that results when you tilt a round glass:



Or an ellipse could be thought of as the shape of a circle drawn on a piece of paper when it is viewed at an angle.

Equations of Ellipses

This "stretching" can be represented algebraically. For simplicity, take the circle of radius 1 centered at the origin (0,0). The distance formula tells us that this is the set of points (x,y) that is a distance 1 unit away from the origin.

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

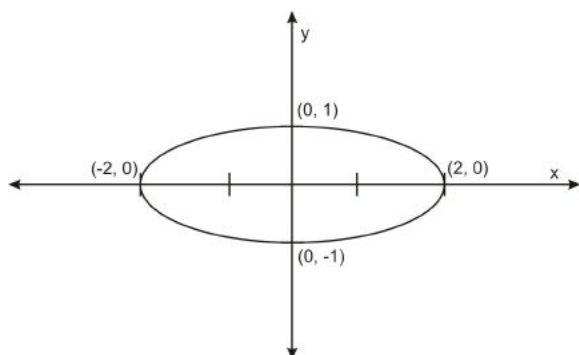
$$1 = \sqrt{(x - 0)^2 + (y - 0)^2}$$

$$1 = x^2 + y^2$$

This equation, $x^2 + y^2 = 1$, can be altered to stretch the circle in the horizontal (i.e. x -axis) direction by dividing the x variable by a constant $a > 1$,

$$\left(\frac{x}{a}\right)^2 + y^2 = 1$$

Why does this stretch the circle horizontally? Well, the effect of dividing x by a is that for each y -value in an ordered pair (x, y) that satisfies the original equation, the corresponding x value must be multiplied by a in order for the pair to make a solution to the altered equation. So solutions (x, y) of the circle are in one-to-one correspondence with solutions (ax, y) of the altered equation, hence stretching the corresponding graph to the left and right by a factor of a . For example, here is the graph of $\left(\frac{x}{2}\right)^2 + y^2 = 1$:



Generalizing the equation by allowing a stretch in the vertical direction, we get the following.

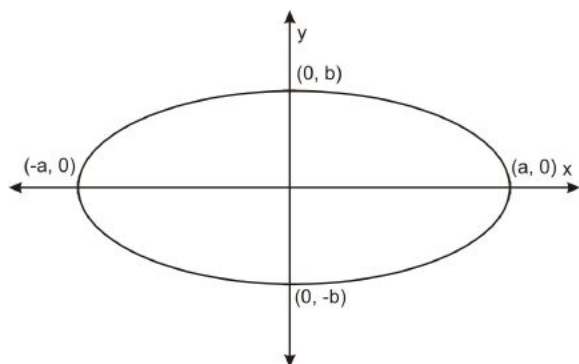
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

The factor a stretches the circle in the horizontal direction and the factor b stretches the circle in the vertical direction. If $a = b$, this is just a circle. When $a \neq b$, this equation represents an ellipse. The ellipse is stretched in the horizontal direction if $b < a$ and it is stretched in the vertical direction if $a < b$. Often the above equation is written as follows.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

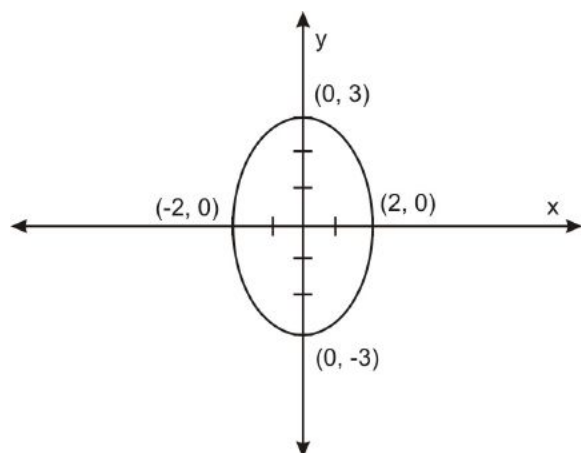
This is called the standard form of the equation of an ellipse, assuming that the ellipse is centered at $(0,0)$.

To sketch a graph of an ellipse with the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, start by plotting the four axes intercepts, which are easy to find by plugging in 0 for x and then for y . Then sketch the ellipse freehand, or with a graphing program or calculator.



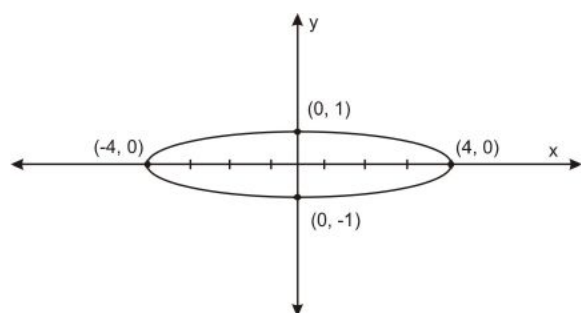
Example 1 Sketch the graph of $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Solution: This equation can be rewritten as $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$. After setting $x = 0$ and $y = 0$ to find the y - and x -intercepts, $(0,3)$, $(0,-3)$, $(2,0)$, and $(-2,0)$, we sketch the ellipse about these points:



Example 2 Sketch the graph of $\frac{x^2}{16} + y^2 = 1$.

Solution: This can be rewritten as $\frac{x^2}{4^2} + \frac{y^2}{1^2} = 1$. After finding the intercepts and sketching the graph, we have:

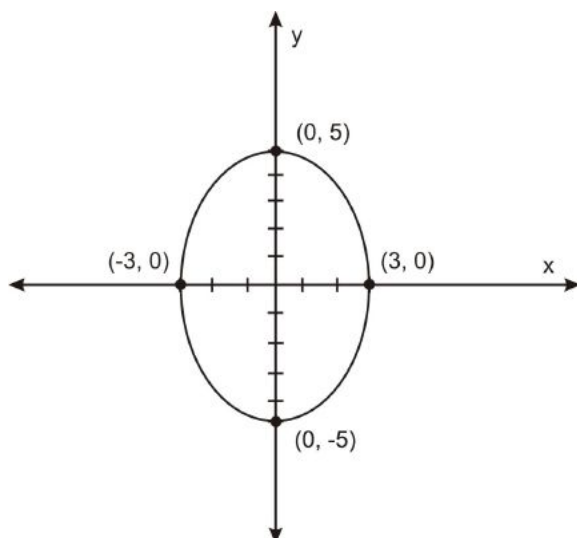


The segment spanning the long direction of the ellipse is called the **major axis**, and the segment spanning the short direction of the ellipse is called the **minor axis**. So in the last example the major axis is the segment from $(-4,0)$ to $(4,0)$ and the minor axis is the segment from $(0,-1)$ to $(0,1)$.

The major and minor axes are examples of what are sometimes called **reference lines**. Apollonius, the Ancient Greek mathematician who wrote an early treatise on conics, used these and other reference lines to orient conic sections. Though the Greeks did not use a coordinate plane to discuss geometry, these reference lines offer a framing perspective that is similar to the Cartesian plane that we use today. Apollonius' way of framing conics with reference lines was the closest mathematics came to the system of coordinate geometry that you know so well until Descartes' and Fermat's systematic work in the seventeenth century.

Example 3 Not all equations for ellipses start off in the standard form above. For example, $25x^2 + 9y^2 = 225$ is an ellipse. Put it in the proper form and graph it.

Solution: First, divide both sides by 225, to get: $\frac{x^2}{9} + \frac{y^2}{25} = 1$, or $\frac{x^2}{3^2} + \frac{y^2}{5^2} = 1$. Finding the intercepts and graphing, we have:



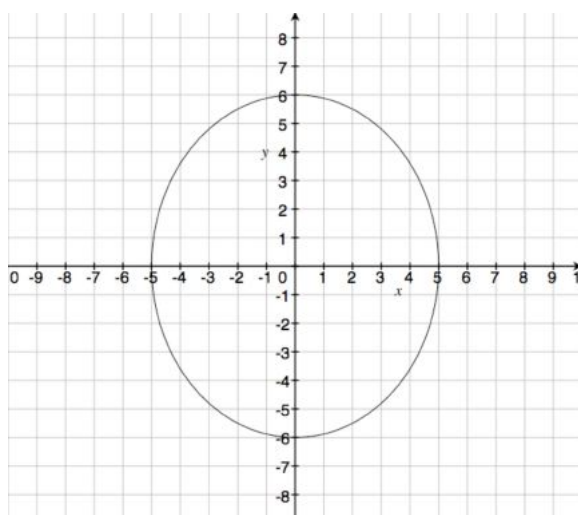
Review Questions

1. It was mentioned above that when a round glass of water is tilted, the surface of the water is an ellipse. Using our working definition of an ellipse as “stretched out circle”, explain why you think the water takes this shape.
2. Sketch the following ellipse: $36x^2 + 25y^2 = 900$
3. Now try sketching this ellipse where the numbers don't turn out to be so neat: $3x^2 + 4y^2 = 12$

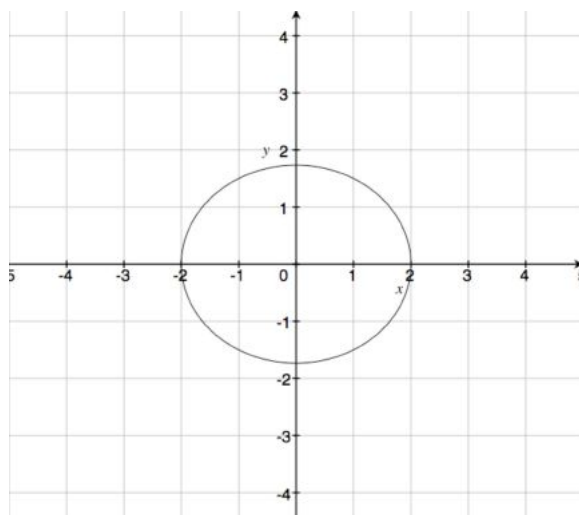
Review Answers

1. Answers may vary, but should explain why the shape that results stretches a circle in one direction because the width of the glass is constant.

2.

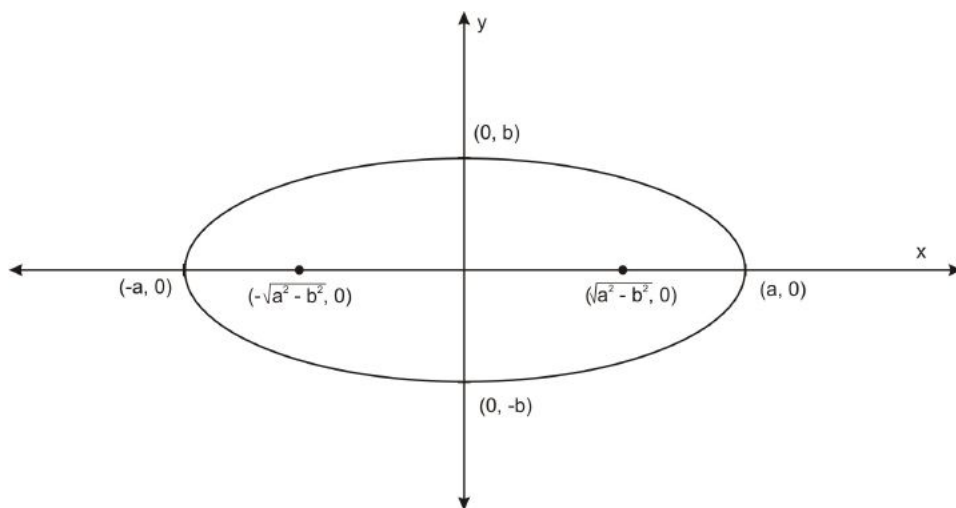


3.

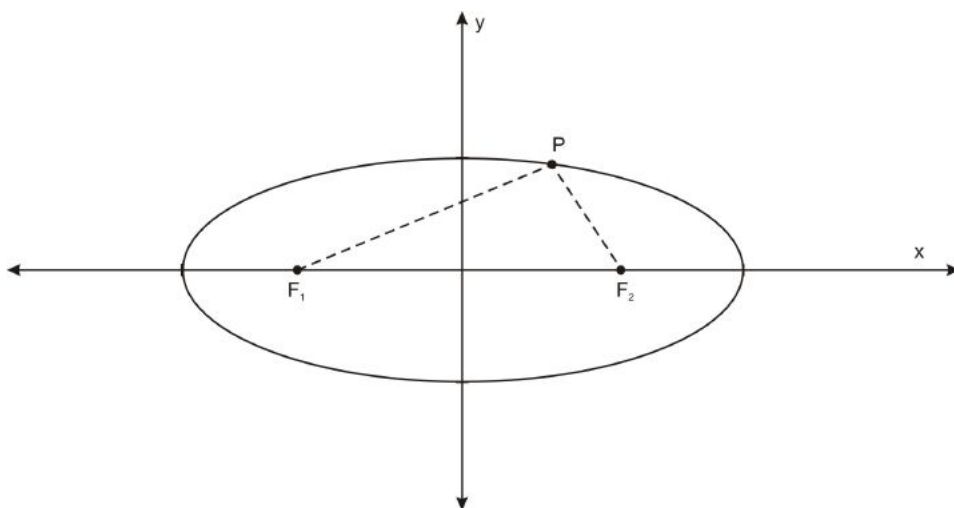


The Focal Property

In every ellipse there are two special points called the **foci** (foci is plural, **focus** is singular), which lie inside the ellipse and which can be used to define the shape. For an ellipse centered at $(0,0)$ that is wider than it is tall, its major axis is horizontal and its foci are at $(\sqrt{a^2 - b^2}, 0)$ and $(-\sqrt{a^2 - b^2}, 0)$.



What is the significance of these points? The ellipse has a geometric property relating to these points that is similar to a circle's relationship with its center. Remember a circle can be thought of as the set of points in a plane that are a certain distance from the center point. In fact, that is typically the definition of a circle. Well, the foci act like the center except that there are two of them. An ellipse is the set of points where the *sum* of the distance between each point on the ellipse and each of the two foci is a constant number. In the diagram below, for any point P on the ellipse, $F_1P + F_2P = d$, where F_1 and F_2 are the foci and d is a constant.



Technology

This definition using the sum of the focal distance gives us a great way to draw ellipses. Sure, you can just graph them on your calculator. But why not utilize a simpler technology that does the job just as well? As you know, a circle could be drawn by fixing a string to a piece of paper, tying the other end to a pencil, and then drawing the curve that keeps the string taut.

Similarly, an ellipse can be drawn by taking a string that is longer than the distance between two points, fixing the two ends of the string to the two points. Then drawing all points that can be drawn when the string is taut and the pencil is touching it. In the diagram above, the dotted line represents the string of fixed length, which is attached at the foci F_1 and F_2 . The string is looped around the pencil at P , and then the pencil is moved, keeping the string taut, drawing all the points whose sum of distances between F_1 and F_2 is the length of the string.

Review Questions

4. Use string and tacks to draw ellipses that
 - (a) are nearly circles
 - (b) are very different than circles

For each of these, what can you say about how the distance between the foci and the length of the string compare?

Review Answers

4. Drawings may vary. For ellipses that are nearly circles, the distance between the foci is small compared to the length of string.

The foci can also be used to measure how far an ellipse is “stretched” from a circle. The symbol ε stands for the **eccentricity** of an ellipse, and it is defined by the distance between the foci divided by the length of the major axis, or $\frac{\sqrt{a^2-b^2}}{a}$ for horizontally oriented ellipses and $\frac{\sqrt{b^2-a^2}}{b}$ for vertically oriented ellipses. Since a circle is an ellipse where $a = b$, circles have an eccentricity of 0.

Review Questions

5. What is the full range of the eccentricity of an ellipse? What does it look like near the extremes of this range?

Review Answers

5. The interval of possible values is $\varepsilon \in [0, 1)$. At $\varepsilon = 0$, the ellipse is a circle; as the eccentricity approaches 1 it becomes more and more elongated.

Turning the Definition of Ellipses on its Head

Often, this focal property is not thought of as a *property* of ellipses, but rather a *defining feature*. To see that these are equivalent, we have some work to do. Let's start by proving that stretched out circles actually have this focal property. We want to prove that for a "stretched out circle" defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are two positive numbers, the sum of the distance between every point on the stretched circle and the two foci $(\sqrt{a^2 - b^2}, 0)$ and $(-\sqrt{a^2 - b^2}, 0)$ is the same. So we need to prove that for every point on the shape defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the sum of the distances to $(\sqrt{a^2 - b^2}, 0)$ and $(-\sqrt{a^2 - b^2}, 0)$ is the same number.

Proof: Suppose a point (x, y) is on the curve defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then we can solve for y in the equation and express the point in terms of x :

$$\begin{aligned}\frac{y^2}{b^2} &= 1 - \frac{x^2}{a^2} \\ y^2 &= \frac{a^2b^2 - b^2x^2}{a^2} \\ y &= \sqrt{\frac{a^2b^2 - b^2x^2}{a^2}}\end{aligned}$$

Now, using the distance formula to compute the sum of the distance between the pairs of points: $(x, \sqrt{\frac{a^2b^2 - b^2x^2}{a^2}})$ and $(\sqrt{a^2 - b^2}, 0)$; and, $(x, \sqrt{\frac{a^2b^2 - b^2x^2}{a^2}})$ and $(-\sqrt{a^2 - b^2}, 0)$ We have:

$$\sqrt{(x - \sqrt{a^2 - b^2})^2 + \frac{a^2b^2 - b^2x^2}{a^2}} + \sqrt{(x + \sqrt{a^2 - b^2})^2 + \frac{a^2b^2 - b^2x^2}{a^2}}$$

This algebraic equation looks daunting, but simplifying the expressions inside these square roots results in a surprising result. This becomes

$$\begin{aligned}
& \sqrt{\frac{a^2(x - \sqrt{a^2 - b^2})^2 + a^2b^2 - b^2x^2}{a^2}} + \sqrt{\frac{a^2(x + \sqrt{a^2 - b^2})^2 + a^2b^2 - b^2x^2}{a^2}} \\
&= \frac{1}{a} \left(\sqrt{a^2(x - \sqrt{a^2 - b^2})^2 + a^2b^2 - b^2x^2} + \sqrt{a^2(x + \sqrt{a^2 - b^2})^2 + a^2b^2 - b^2x^2} \right) \\
&= \frac{1}{a} \left(\sqrt{a^2x^2 - 2a^2x\sqrt{a^2 - b^2} + a^4 - a^2b^2 + a^2b^2 - b^2x^2} + \sqrt{a^2x^2 - 2a^2x\sqrt{a^2 - b^2} + a^4 - a^2b^2 + a^2b^2 - b^2x^2} \right) \\
&= \frac{1}{a} \left(\sqrt{a^2x^2 - 2a^2x\sqrt{a^2 - b^2} + a^4 - b^2x^2} + \sqrt{a^2x^2 - 2a^2x\sqrt{a^2 - b^2} + a^4 - b^2x^2} \right) \\
&= \frac{1}{a} \left(\sqrt{(a^2 - x\sqrt{a^2 - b^2})^2} + \sqrt{(a^2 + x\sqrt{a^2 - b^2})^2} \right) \\
&= \frac{1}{a} (a^2 - x\sqrt{a^2 - b^2} + a^2 + x\sqrt{a^2 - b^2}) \\
&= \frac{1}{a} (2a^2) \\
&= 2a
\end{aligned}$$

What a miraculous collapse! One of the gnarliest algebraic expressions that I have ever encountered turned into the simple expression $2a$. Most importantly, this reduced expression is merely a constant—it doesn't depend on x or y . This is exactly what we were hoping for. *The sum of the distances between every point on the ellipse and the two foci is always $2a$.* If you ever find yourself needing to remember what this sum is, the number $2a$ can be remembered most easily by computing it from an easy point such as one of the shape's x -intercepts, say $(a, 0)$.

Review Questions

6. Compute the distance from the x -intercept $(a, 0)$ and the two foci and show that it is in fact $2a$.
7. What is the sum of the distances to the foci of the points on a vertically-oriented ellipse?

Review Answers

6. The distance between the x -intercept $(a, 0)$ and $(\sqrt{a^2 - b^2}, 0)$ is: $a - \sqrt{a^2 - b^2}$. The distance between the x -intercept $(a, 0)$ and $(-\sqrt{a^2 - b^2}, 0)$ is: $a + \sqrt{a^2 - b^2}$. Together these add to: $a - \sqrt{a^2 - b^2} + a + \sqrt{a^2 - b^2} = 2a$
7. $2b$

Defining an Ellipse by Focal Distance

So all this computing simply means that “stretched-out circles”—what we’ve been calling ellipses—satisfy the focal property. What would be great is if we could *define* ellipses by the focal property. This would be a nice generalization of the way we define circles.

Recall that circles are defined as the set of points in a plane that are a constant distance from a center point. Analogously, ellipses could be defined as a set of points in a plane for which the sum of the distances

to two focus points is a constant. In other words, this definition would yield the exact same set of shapes as the “stretched out circle” definition that we started with.

Before proceeding we need to make sure of one thing. We’ve already proved that stretched out circles satisfy the focal property, but how do we know that any shape satisfying the focal property is in turn a stretched out circle? Well, the calculation above can simply be read backwards. In other words, suppose you have a set of points that satisfy the focal property, that each point whose sum of the distances to the points $(f, 0)$ and $(-f, 0)$ is a fixed distance d . Now note that for any two positive numbers d and f with $2f < d$, there exist positive numbers a and b such that $d = 2a$ and $f = \sqrt{a^2 - b^2}$. (this fact is a bit subtle and is part of an exercise below.) So now we have a shape where every point (x, y) has a sum of distances to the points $(\sqrt{a^2 - b^2}, 0)$ and $(-\sqrt{a^2 - b^2}, 0)$ which equals $2a$. Using these expressions, the algebraic steps of the proof above can simply be read backwards.

Review Questions

8. Explain why for any two positive numbers d and f with $2f < d$, there exist positive numbers a and b such that $d = 2a$ and $f = \sqrt{a^2 - b^2}$.
9. We just told you that the above proof could be read backwards. But you need to be careful when following algebraic steps backwards, especially ones involving squares or square roots.

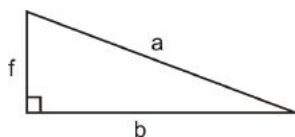
(a) For example, what happens when you follow this argument backwards?

$$x^2 = 4$$

(b) Write a convincing argument that it is okay to follow the steps backwards in the above proof that every stretched circle has the focal property.

Review Answers

8. Set $a = \frac{d}{2}$. Then we need to show that for f satisfying $2f < d$, there exists a number b such that $f = \sqrt{a^2 - b^2}$. Since $2f < d$, $2f < 2a$ by the definition of a (using the assumption $d > 0$). So $f < a$. We can find b geometrically. Since $f < a$, there is a right triangle with one leg having length f and hypotenuse a . Call the other leg of the triangle b .



Then the Pythagorean Theorem tells us that $f^2 + b^2 = a^2$, or equivalently $f = \sqrt{a^2 - b^2}$. So a and b satisfy the necessary requirements.

9. (a) Taking the square root of both sides of $x^2 = 4$ yields two solutions, $x = \pm 2$, instead of the one value we already know ($x = -2$). The problem is that the operation of squaring a number is not a one-to-one function. Both $(-2)^2$ and 2^2 yield the same number. So some information is lost during this step, and it cannot be perfectly “undone”, like other algebraic maneuvers.
- (b) The student’s reason should include the fact no information is lost (through squaring both sides or other operations) in any of these steps, so that each step is completely reversible.

Equation of an Ellipse Not Centered at the Origin

All the ellipses we’ve looked at so far are centered around the origin $(0, 0)$. To find an equation for ellipses centered around another point, say (h, k) , simply replace x with $x - h$ and y with $y - k$. This will shift all

the points of the ellipse to the right h units (or left if $h < 0$) and to up k units (or down if $k < 0$). So the general form for a horizontally- or vertically-oriented ellipse is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

It is centered about the point (h, k) . If $b < a$, the ellipse is horizontally oriented and has foci $(h + \sqrt{a^2 - b^2}, k)$ and $(h - \sqrt{a^2 - b^2}, k)$ on its horizontal major axis. If $a < b$, it is vertically oriented and has foci $(h, k + \sqrt{a^2 - b^2})$ and $(h, k - \sqrt{a^2 - b^2})$ on its vertical major axis.

Example 4

Graph the equation $4x^2 + 8x + 9y^2 - 36y + 4 = 0$.

Solution : We need to get the equation into the form of general equation above. The first step is to group all the x terms and y terms, factor out the leading coefficients of x^2 and y^2 , and move the constants to the other side of the equation:

$$4(x^2 + 2x) + 9(y^2 - 4y) = -4$$

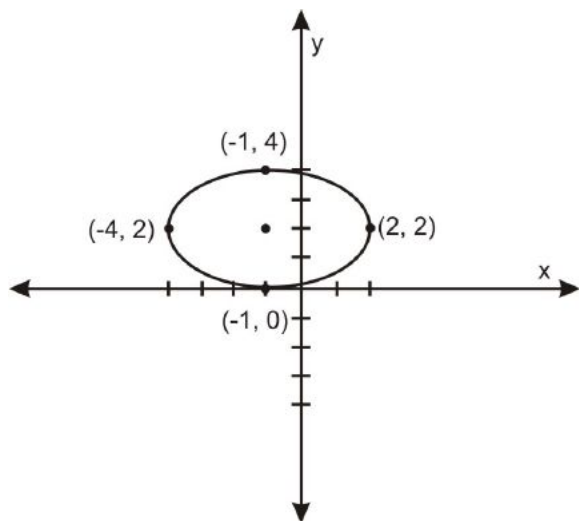
Now, we “complete the square” by adding the appropriate terms to the x expressions and the y expressions to make a perfect square. (See http://authors.ck12.org/wiki/index.php/Algebra_I-Chapter-10#Solving_Quadratic_Equations_by_Completing_the_Square for more on completing the square.)

$$4(x^2 + 2x + 1) + 9(y^2 - 4y + 4) = -4 + 4 + 36$$

Now we factor and divide by the coefficients to get:

$$\frac{(x+1)^2}{9} + \frac{(y-2)^2}{4} = 1$$

And there we have it. Once it's in this form, we see this is an ellipse is centered around the point $(-1, 2)$, it has a horizontal major axis of length 3 and a vertical minor axis of length 2, and from this we can make a sketch of the ellipse:



Review Questions

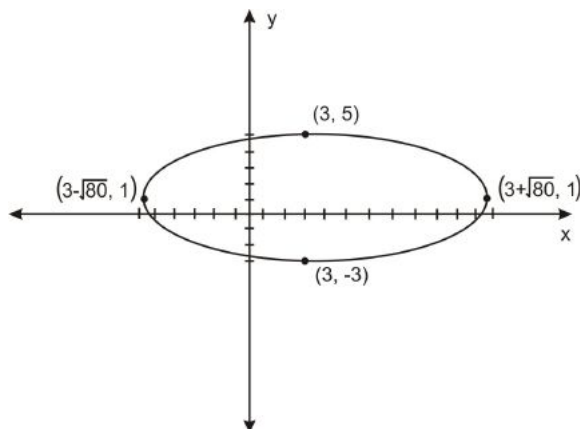
- Explain why subtracting h from the x -term and k from the y -term in the equation for an ellipse shifter the ellipse h horizontally and k vertically.
- Graph this ellipse. $x^2 - 6x + 5y^2 - 10y - 66 = 0$
- Now try this one that doesn't have such nice numbers!

$$16x^2 - 48x + 125y^2 + 150y + 61 = 0$$
- Now try this one. $3x^2 - 12x + 5y^2 + 10y - 3 = 0$. What goes wrong? Explain what you think the graph of this equation might look like.
- What about this one. $5x^2 - 15x - 2y^2 + 8y - 50 = 0$? What goes wrong here? Explain what you think the graph of this equation might look like.

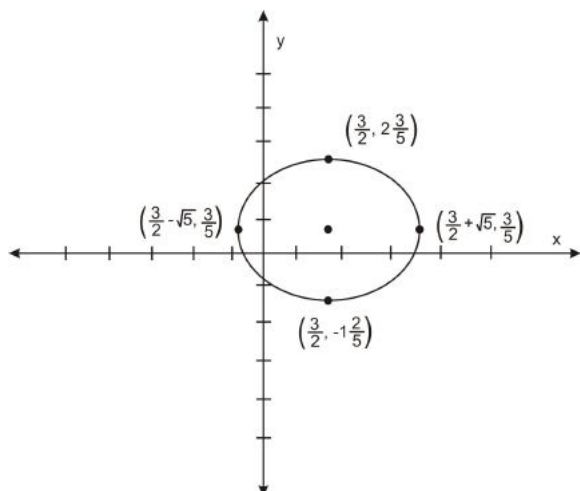
Review Answers

- If (x, y) is a solution to $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ then $(x+h, y+k)$ is a solution to $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. This produces a graph that is shifted horizontally by h and vertically by k .

11.



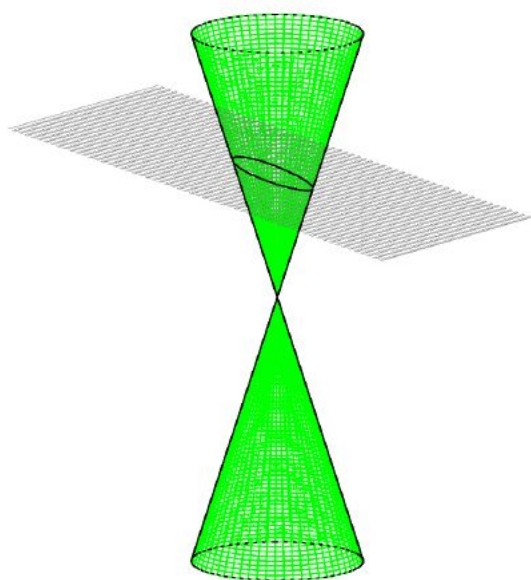
12.



- After completing the square, we have the sum of positive numbers equaling a negative number. This is an impossibility, so the equation has no solutions.
- After completing the square, the x term and the y term are opposite signs. If you plot some points you will see that the graph has two disconnected sections. This class of conic sections will be discussed in the next section.

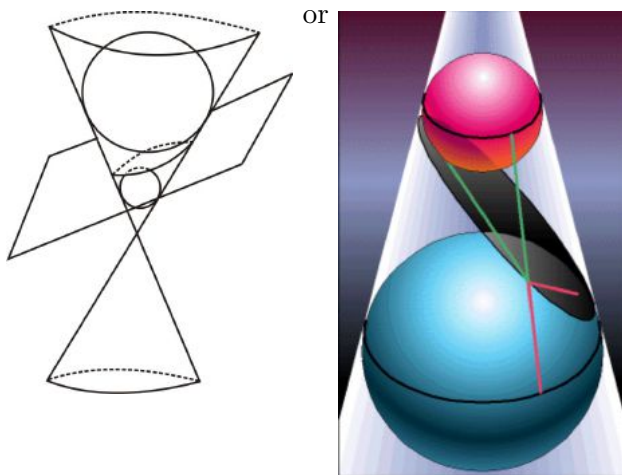
Difference Between an Ellipse and an Oval or Proving the Sliced Cone Definition of an Ellipse

There is still one critical step missing in our exploration of ellipses. We showed that “stretched-out circles” satisfy the focal property, and that any shape satisfying this property is in fact a “stretched-out circle”. So these are actually the same class of shapes, and they are called ellipses. But not any oval-shaped curve is an ellipse. Draw a random oval and you’re not likely to be able to find two points that satisfy the focal property. In particular, when we cut a cone with a tilted plane, how do we know that the oval-shaped curve that results is a “stretched out circle” satisfying the focal property?



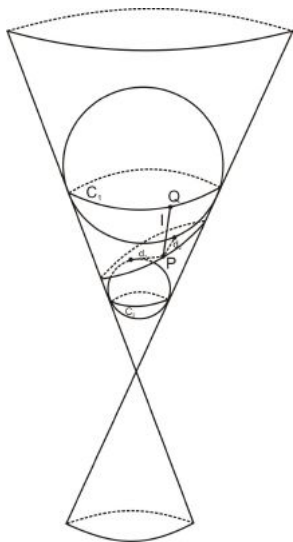
Amazingly, the Ancient Greeks had an argument for this fact over two millennia ago. While it is impressive that this problem was solved so long ago, the argument itself involves an intricate construction that isn’t as illuminating as a more modern the one I’m going to show you instead. Most mathematicians prefer to use a more modern argument that is simply stunning in its simplicity. This modern argument isn’t fancy—the Greeks had all the tools they need to understand it—they just didn’t happen to think of it. It wasn’t until 1822 that the French mathematician Germinal Dandelin thought of this very clever construction. Dandelin found a way to find the foci and prove the focal property in one fell swoop. Here’s what he said.

Take the conic section in question. Then choose a sphere that is just the right size so that when it’s dropped into the conic, it touches the intersecting plane, as well as being snug against the cone on all sides. If you prefer, you can think of the sphere as a perfectly round balloon that is blown up until it “just fits” inside the cone, still touching the plane. Then do the same on the *other* side of the plane. After we’ve drawn both of these spheres we have this picture:

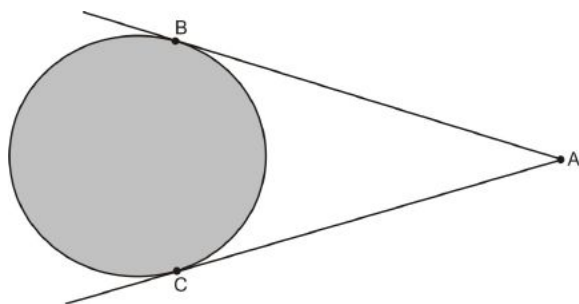


These spheres are often called “Dandelin spheres”, named after their discoverer. It turns out that not only is our shape an ellipse (which, like all ellipses satisfies the focal property). But these spheres touch the ellipse exactly at the two foci. To see this, consider this geometric argument.

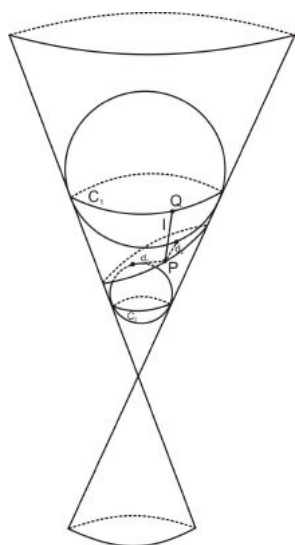
The first thing to notice is that the circles C_1 and C_2 shown on the diagram below, where each sphere lies snug against the cone, lie in parallel planes to one another. In particular, each line passing through these circles and the vertex of the cone, such as the line l drawn below, cuts off equal segments between the two circles. Let's call d the shortest distance along the cone between circles C_1 and C_2 . This can also be thought of as the shortest distance between C_1 and C_2 that passes through the vertex of the cone.



The next thing to remember is a property of tangents to spheres that you may have learned in geometry. If two segments are drawn between a point and a sphere, and if the line containing each segment is tangent to the sphere, then the two segments are equal. In the diagram below, $AB = AC$. (This follows from the fact that tangents are perpendicular to the radii of a sphere and that two congruent triangles are formed in this configuration. See [this description](#) for more about this property.)



Now consider the point P on the ellipse drawn below. Let \overline{QR} be the segment of length d between C_1 and C_2 that passes through P . The distances between the two foci are marked d_1 and d_2 . But $d_1 = RP$ and $d_2 = PQ$ by the property of tangents to spheres discussed above. So $d_1 + d_2 = RP + PQ = QR = d$. And this sum will always equal d , no matter what point P on the ellipse is chosen. So this proves the focal property of ellipses: that the sum of the distances between any point on the ellipse and the two foci is constant.



Review Questions

- What do the Dandelin spheres look like in the case of a circle?
- What is the area of an ellipse with the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$? (Hint: use a geometric argument starting with the area of a circle.)
- What is the perimeter of an ellipse with the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$?

Review Answers

- The Dandelin spheres for a circle lie directly above one another, and both touch the circle at the center point.
- The area of an ellipse is $ab\pi$. To see why this is true, start with a circle of radius 1, which has an area of π . Then imagine an approximation with rectangles of the circle. Then stretch the rectangles by a factor of a in the x -direction and by a factor of b in the y -direction to obtain an approximation of the ellipse. This makes the rectangles a times wider and b times taller, giving an area that is ab multiplied by the area of the approximation of the circle. Since this is true of any approximation of

the circle, the area of the ellipse must be $ab\pi$.

17. This is actually a much more difficult question than the previous one. You're on your own! Even the great Indian mathematician Ramanujan could only come up with an approximation: $p \approx \pi[3(a+b) - \sqrt{(3a+b)(a+3b)}]$.

Applications

The number of places ellipses appear in the natural world is immense. Consider, for instance, how often the simplest kind of ellipse, the circle, appears in your life. You see circles emanating as waves when you throw a rock into a pond. You see circular pupils when you look at a set of eyes. You see what is roughly a circle as the image of the sun or moon in the sky. The path of an object swung around on a string is a circle. When any of these circles are viewed at an angle you see an elongated circle, or an ellipse. So it is important to keep in mind that the discussion that follows covers only a few of the instances of ellipses in our daily lives.

Planetary Motion

When a planet orbits the sun (or when any object orbits any other), it takes an elliptical path and the sun lies at one of the two foci of the ellipse. Johannes Kepler first proposed this at the beginning of the seventeenth century as one of three laws of planetary motions, after analyzing observational data of Tycho Brahe. His law is accurate enough to produce modern computations which are still used to predict the motion of artificial satellites. A century later, Newton's law of gravity offers an explanation of why this law might be true.

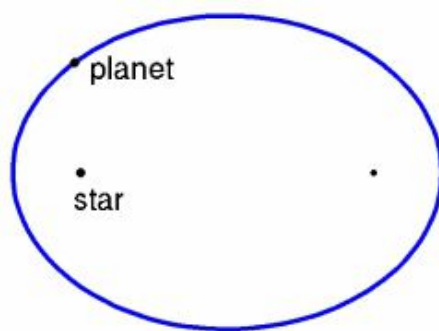


Figure 8.1

Review Questions

18. Though planets take an elliptical path around the sun, these ellipses often have a very low eccentricity, meaning they are close to being circles. The diagram above exaggerates the elliptical shape of a planet's orbit. The Earth's orbit has an eccentricity of 0.0167. Its minimum distance from the sun is 146 million km. What is its maximum distance from the sun? If the sun's diameter is 1.4 million kilometers. Do both foci of the Earth's orbit lie within the sun? Recall that the eccentricity of an ellipse is $\varepsilon = \frac{\sqrt{a^2 - b^2}}{a}$.
19. While the elliptical paths of planets are ellipses that are closely approximated by circles, comets and asteroids often have orbits that are ellipses with very high eccentricity. Halley's comet has an

eccentricity of 0.967, and comes within 54.6 million miles of the sun at its closest point, or “perihelion”. What is the furthest point it reaches from the sun?

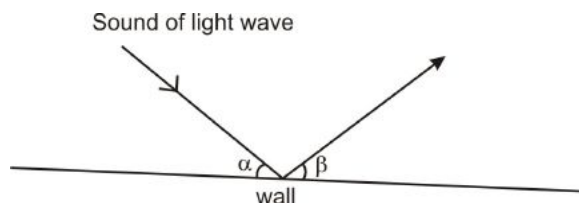
Review Answers

18. Assume that the orbit of the sun is an ellipse centered at $(0,0)$. Then we can use the distance from the origin to the focus $\sqrt{a^2 - b^2}$ to set up the equations $146 + 146 + 2\sqrt{a^2 - b^2} = 2a$ and $0.167 = \frac{\sqrt{a^2 - b^2}}{a}$. Solving we get $a = 175.270$, $b = 175.245$, and the distance from $(0,0)$ to the foci, $c = 2.927$ (all units are in millions of km). Finally the maximum distance from the earth to the sun is approximately 152 million km. From Kepler’s law, we know one of the foci of its orbit is at the center of the sun. The other foci is $2(2.927) = 5.854$ million kilometers away, so it is outside the sun (but not by very far!).
19. ~ 3.25 billion miles.

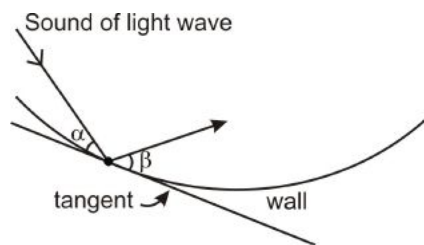
Echo Rooms

The National Statuary Hall in the United States Capital Building is an example of an ellipse-shaped room, sometimes called an “echo room”, which provide an interesting application to a property of ellipses. If a person whispers very quietly at one of the foci, the sound echoes in a way such that a person at the other focus can often hear them very clearly. Rumor has it that John Quincy Adams took advantage of this property to eavesdrop on conversations in this room.

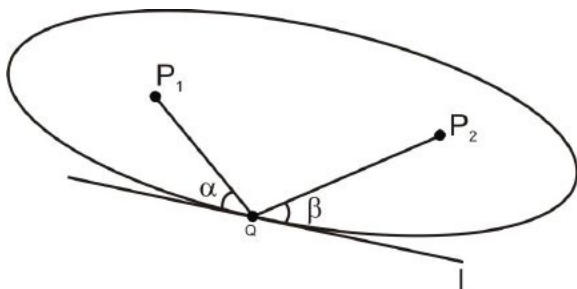
The property of ellipses that makes echo rooms work is called the “optical property.” So why echoes, if this is an optical property? Well, light rays and sound waves bounce around in similar ways. In particular, they both bounce off walls at equal angles. In the diagram below, $\alpha = \beta$.



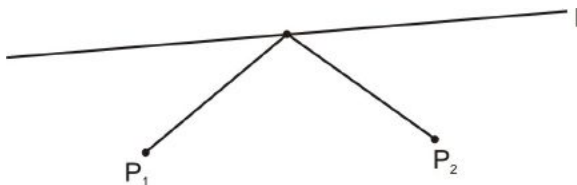
For a curved wall, they bounce at equal angles to the tangent line at that point:



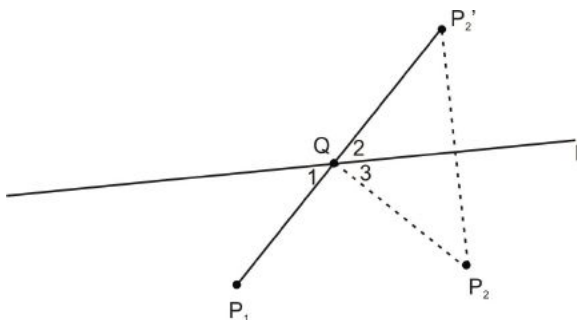
So the “optical property” of ellipses is that lines between a point on the ellipse and the two foci form equal angles to the tangent at that point, or in other words, whispers coming from one foci bounce directly to the other foci. In the diagram below, for each Q on the ellipse, $\angle\alpha \cong \angle\beta$.



This seems reasonable, given the symmetry of the ellipse, but how do we know it is true? First, let's prove an important property that is a bit more general. Suppose you have two points, P_1 and P_2 , and a line l , and you are interested in the shortest possible path between two P_1 and P_2 , that intersects with l .



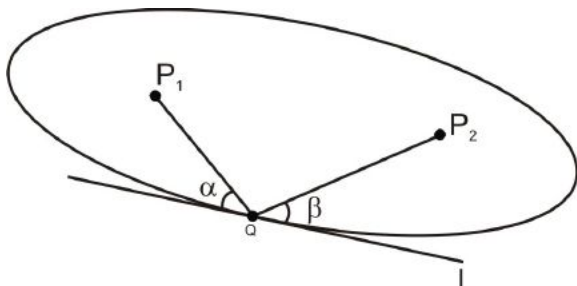
A nice way to find this is to reflect P_2 across l , obtaining the point P'_2 in the diagram below:



Since l lies between P_1 and P'_2 , it's much easier to find the shortest distance between P_1 and P'_2 that passes through l . It's simply the shortest path between P_1 and P'_2 : a straight line! But for every path between P_1 and P_2 , we can reflect the part from l to P_2 to get a path equal in length between P_1 and P'_2 . So the shortest path between P_1 and P_2 is simply the shortest path between P_1 and P'_2 (the straight line), with the part between l and P'_2 reflected to get a path between l and P_2 . So the shortest path from P_1 to P_2 that intersects with l is P_1Q followed by QP_2 .

The part of the above diagram that is going to help us prove the optical property is that since $\angle 1 \cong \angle 2$ (vertical angles) and $\angle 2 \cong \angle 3$ (reflected angles), then $\angle 1 \cong \angle 3$ (transitive property). Thus the shortest path from P_1 to P_2 that intersects with l consists of two segments that meet the line at equal angles.

Now to prove the optical property of the ellipse, apply the above situation to the ellipse. In the picture below, P_1 and P_2 are foci of the ellipse and Q is the intersection with tangent line l .



The optical property states that $\angle\alpha \cong \angle\beta$. This is true because all points other than Q on line l lie outside the ellipse. Points outside the ellipse have a combined distance to the two foci that is greater than points on the ellipse. So Q is the point on line l with the smallest combined distance to points P_1 to P_2 . Thus, like we showed in general above, $\angle\alpha \cong \angle\beta$. So the optical property has been proved.

Review Questions

- Design the largest possible echo room with the following constraints: You would like to spy on someone who will be 3 m from the tip of the ellipse. The room cannot be more than 100 m wide in any direction. How far from the person you're spying on will you be standing.

Review Answers

- The echo room has a major axis of 100 m and a minor axis of 34.12 m. Situating the room in the coordinate plane, the room can be represented by the equation: $\frac{x^2}{2500} + \frac{y^2}{291} = 1$. You will be 94 m from the person you are spying on.

Sundials:



Figure 8.2

Conic sections help us solve the problem of making a sundial. Depending on the season, the sun shines at a different angle. However, due to the elliptical nature of the Earth's orbit about the sun, a shadow-casting stick can be placed in such a way that the shadow always tells the correct time of day, no matter what the time of year, as long as the stick is lined up with the earth's pole of rotation.

Review Questions

21. No matter what the orientation of a stick, if you trace out the path that the shadow of the tip makes on a flat surface, you will find it is an ellipse. Describe why this is true. (HINT: for simplicity, you can assume that you are making the measurements throughout the course of one day and that with the exception of the earth rotating about a pole, the sun and the earth are fixed with respect to one another.)
22. It was mentioned earlier in the chapter that when a round glass of water is tilted, the surface of the water is an ellipse. Or, in other words, this statement is claiming that the cross section of a cylinder is an ellipse. Prove that this is true. (Hint: to prove that this is an ellipse all you need to do is show that for any cross section of a cylinder there exists a cone that has the same cross section.)
23. From the exercise above, it appears that there is some overlap between “conic sections” and “cylindrical sections”. Are any of the classes of conic sections we found in the last section not cylindrical sections? Are there any cylindrical sections that are not conic sections?

Review Answers

21. Answers may vary.
22. Answers may vary.
23. Answers may vary.

Vocabulary

Ellipse A conic section that can be equivalently defined as: 1) any finite conic section, 2) a circle which has been dilated (or “stretched”) in one direction, 3) the set of points in which the sum of distances to two special points called the foci is constant.

Major axis The segment spanning an ellipse in the longest direction.

Minor axis The segment spanning an ellipse in the shortest direction.

Focus One of two points that defines an ellipse in the above definition.

Eccentricity A measure of how “stretched out” an ellipse is. Formally, it is the distance between the two foci divided by the length of the major axis. The eccentricity ranges from 0 (a circle) to points close to 1, which are very elongated ellipses.

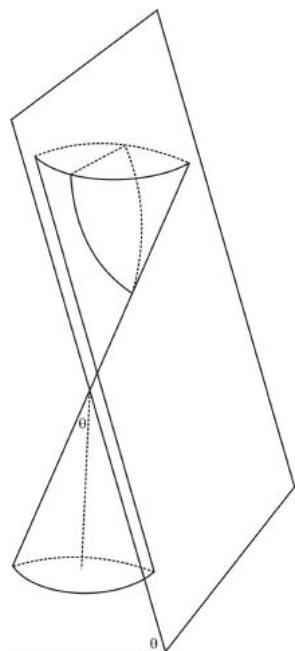
8.3 Parabolas

Learning Objectives

- Understand what results when a cone is intersected by a plane parallel to one side of the cone.
- Understand the focal-directrix property for parabolas.
- Recognize and work with equations for parabolas.
- Understand that all parabolas are self-similar.
- Understand the equivalence of different definitions of parabolas.

Introduction

We've examined ellipses and circles, the two cases when a plane intersects only one side of the cone and creates a finite cross-section. Is it possible for a plane to intersect only one side of the cone, but create an infinite cross-section? It turns out that this is possible if and only if the plane is parallel to one of the lines making up the cone. Or, in other words, the angle θ between the plane and the horizon, is equal to the angle formed by a side of the cone and the horizontal plane.



In the image above, if you tilt the plane a little bit to the left it will cut off a finite ellipse (possibly a very large one if you only tilt it a little.) Tilt the plane to the right and it will intersect both sides of the cone, making a two-part conic section called a hyperbola, which will be discussed in the next section.

When the plane is parallel to from the side of the cone, the infinite shape that results from the intersection of the plane and the cone is called a **parabola**. Like the ellipse, it has a number of interesting geometric properties.

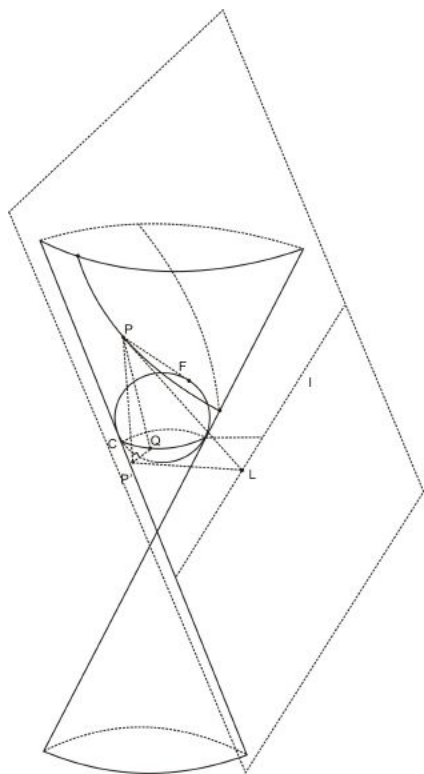
Focus-Directrix Property

Like the ellipse, the parabola has a focal property. And, also like the ellipse, a construction similar to Dandelin's with the spheres can show us what it is. Dandelin himself didn't prove the focal property for parabolas that we are about to discuss, but Pierce Morton used a sphere construction similar to Dandelin's to prove the focal property of parabolas in 1829. We'll look at Morton's argument here.

In contrast with the argument we made for the ellipse, for a parabola we can only fit one tangent sphere inside the cone. That is, only one sphere can be tangent to both the cone and the cutting plane. In the diagram below, the sphere fits underneath the cutting plane, but there is no room for a sphere to lie on top of the cutting plane and still be tangent to the cone.

As with the ellipse, the point where the sphere intersects the plane is called a **focus**. But because there is only one sphere in this construction, and this is related to the fact that a parabola has only one focus. The other geometric object of interest is called the **directrix**. This is the line that results from the intersection between the cutting plane and the plane that contains the circle of contact between the sphere and the cone. In the diagram below, the directrix is labeled l and is found by intersecting the plane defined by

circle C and the cutting plane (the planes are shown in dashed lines for clarity). Finally, we will call the angle between the planes θ .



In the above diagram, we have labeled the point where the sphere contacts the cutting plane with F , and we'll call that point the focus of the parabola. Suppose P is an arbitrarily chosen point on the parabola. Then, let Q be the point on circle c such that \overline{PQ} is tangent to the sphere. In other words Q is chosen so that \overline{PQ} lies on the cone itself. Let L be the point on the directrix l such that \overline{PL} is perpendicular to l . Then $PF = PQ$ since both segments are tangents to the sphere from the same point P . We can also show that $PQ = PL$. This follows from the fact that the cutting plane is parallel to one side of the cone. Consider the point P' that is the projection of P onto the plane containing circle C . Then $\square PP'Q$ and $\square PP'L$ are both right angles by the definition of a projection. $\square PQP'$ and $\square PLP'$ are both equal to the angle $90 - \theta$, where θ is the angle defined above, because the cutting plane and the cone both have an angle of θ with the horizon. Since they also share a side, triangles $\triangle PQP'$ and $\triangle PLP'$ are congruent by AAS. So the corresponding sides \overline{PQ} and \overline{PL} are congruent. By the transitive property we have $PF = PL$, so the distance between the point P on the parabola to the focus is the same as the distance between P and the directrix l . We have just proven the focus-directrix property of parabolas.

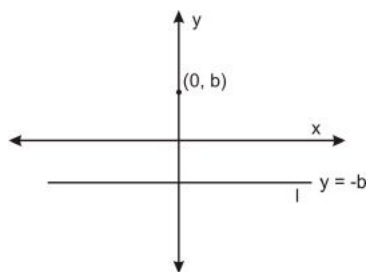
Equations and Graphs of Parabolas

The equation of a parabola is simpler than that of the ellipse. We will arrive at the equation for a parabola in two ways.

Method 1: Using the Distance formula

The first method arises directly from the focus-directrix property discussed in the previous section. Suppose we have a line and a point not on that line in a plane, and we want to find the equation of the set of points in the plane that is equidistant to these two objects. Without losing any generality, we can orient the line

horizontally and the point on the y -axis, with the origin halfway between them. Since the parabola is the set of points equidistant from the line and the point, The parabola passes through the origin, $(0,0)$. The picture below shows this configuration. The point directly between the directrix and the focus (the origin in this case) is called the **vertex** of the parabola. Suppose the focus is located at $(0, b)$. Then the directrix must be $y = -b$.



Thus, the parabola is the set of points (x, y) equidistant from the line $y = -b$ and the focus point $(0, b)$. The distance to the line is the vertical segment from (x, y) down to $(0, -b)$, which has length $y - (-b) = y + b$. The distance from (x, y) to the focus $(0, b)$ is distance $= \sqrt{(x - 0)^2 + (y - b)^2}$ by the distance formula. So the equation of the parabola is the set of points where these two distances equal.

$$y + b = \sqrt{(x - 0)^2 + (y - b)^2}$$

Since distances are always positive, we can square both sides without losing any information, obtaining the following.

$$\begin{aligned} y^2 + 2by + b^2 &= x^2 + y^2 - 2by + b^2 \\ 2by &= x^2 - 2by \\ 4by &= x^2 \\ y &= \frac{1}{4b}x^2 \end{aligned}$$

But b was chosen arbitrarily and could have been any positive number. And for any positive number, a , there always exists a number b such that $a = \frac{1}{4b}$, so we can rewrite this equation as:

$$y = ax^2$$

where a is any constant.

This is the general form of a parabola with a horizontal directrix, with a focus lying above it, and with a vertex at the origin. If a is negative, the parabola is reflected about the x -axis, resulting in a parabola with a horizontal directrix, with a focus lying below it, and with a vertex at the origin. The equation can be shifted horizontally or vertically by moving the vertex, resulting in the general form of a parabola with a horizontal directrix and passing through a vertex of (h, k) :

$$y - k = a(x - h)^2$$

Switching x and y , the equation for a parabola with a vertical directrix and with a vertex at (h, k) is:

$$x - h = a(y - k)^2$$

Example 1

Sketch a graph of the parabola $y = 3x^2 + 12x + 17$.

Solution: First, we need to factor out the coefficient of the x^2 term and complete the square:

$$y = 3(x^2 + 4x) + 17$$

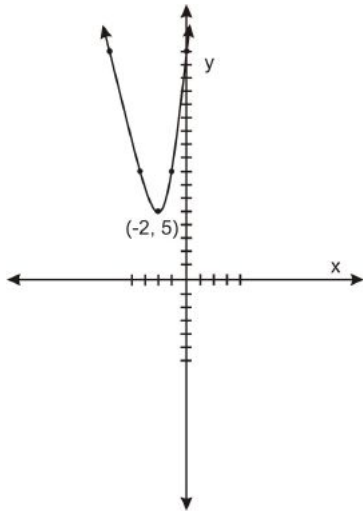
$$y = 3(x^2 + 4x + 4) + 17 - 12$$

$$y = 3(x + 2)^2 + 5$$

Now we write it in the form $x - h = a(y - k)^2$:

$$y - 5 = 3(x + 2)^2$$

So the vertex is at $(-2, 5)$ and plotting a few x -values on either side of $x = -2$, we can draw the following sketch of the parabola:

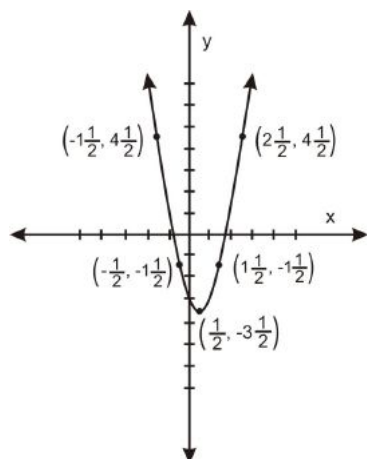


Review Questions

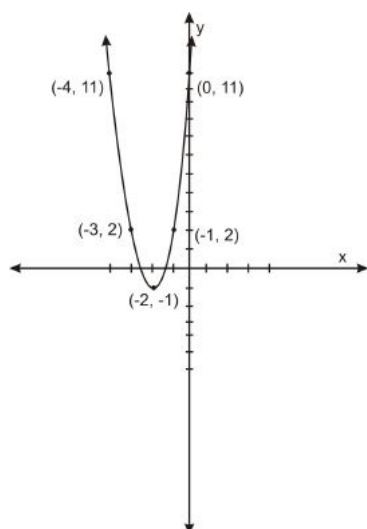
1. Sketch a graph of the following parabola: $y = 2x^2 - 2x - 3$
2. Sketch a graph of the following parabola: $3x^2 + 12x + 11 - y = 0$
3. Sketch a graph of the following parabola: $0 = x^2 - y + 3x + 5$
4. Identify which of the following equations are parabolas: $y - 5x + x^2 = 3$, $x^2 - 3y^2 + 3y - 2x + 15 = 0$, $x - 6y^2 + 20x - 100 = 0$
5. Draw a sketch of the following parabola. Also identify its directrix and focus. $3x^2 + 6x - y = 0$
6. Find the equation for a parabola with directrix $y = -2$ and focus $(3, 8)$.

Review Answers

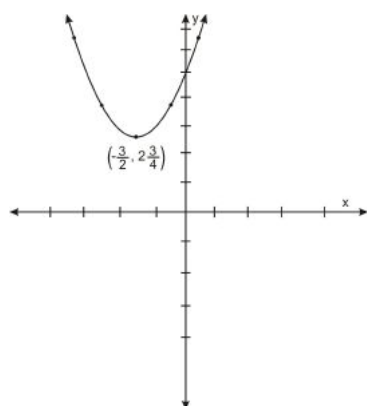
1.



2.

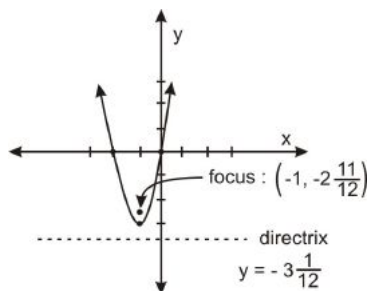


3.



4. $y - 5x + x^2 = 3$ and $x - 6y^2 + 20x - 100 = 0$

5.

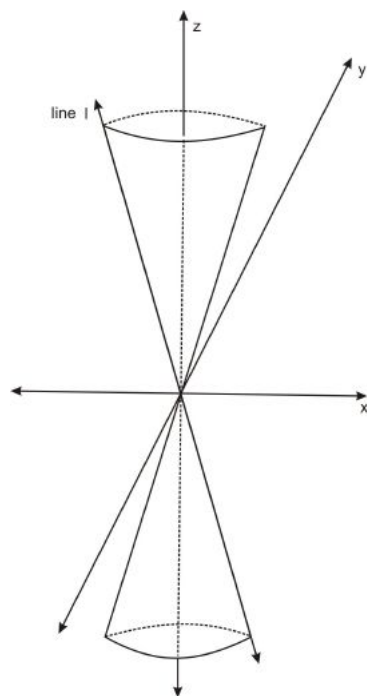


6. $y - 3 = \frac{1}{20}(x - 3)^2$

Method 2: Using Three Dimensional Analytical Geometry

Alternatively, we could have arrived at the above equation for a parabola without the argument of Pierce Morton or the use of Dandelin's spheres. The following argument is closer to Apollonius' approach to parabolas in Ancient Greece, though the closed equation forms and the coordinate grid that I will use are modern conventions.

Consider a cone oriented in space as pictured below:



If the cone opens at an angle such that at any point its radius to height ratio is a , then the cone could be defined as the set of points such that the distance from the z -axis is a times the z -coordinate. Or, in other words, the set of points (x, y, z) satisfying:

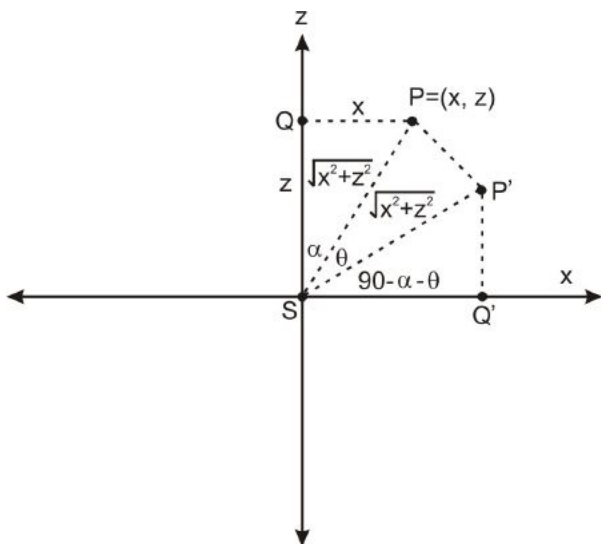
$$\sqrt{(x-0)^2 + (y-0)^2} = az$$

Or:

$$x^2 + y^2 = a^2 z^2$$

This equation works for negative values of x, y , and z , giving the general equation for a two-sided cone.

To consider the intersection of this cone with a plane that is parallel to the line marked l marked in the diagram above, it is most convenient to rotate the entire cone about the y axis until the left side of the cone is vertical, then intersect it with a vertical plane perpendicular to the x -axis. Such a rotation leaves the y -variable unchanged. To see what it does to the x and z variables, let's see what happens to the point (x, z) on the xz -plane when it is rotated by an angle of θ .



In the above diagram, $P(x, z)$ is rotated by an angle of θ to the point P' . We have marked the side lengths $QP = x$ and $SQ = z$. By the Pythagorean Theorem, $SP = \sqrt{x^2 + z^2}$. We also have $SP' = \sqrt{x^2 + z^2}$, since rotation leaves the distance from the origin unchanged. To find the x -coordinates of our rotated point P' , we can use the fact that $\cos(90 - \alpha - \theta) = \frac{SQ'}{\sqrt{x^2 + z^2}}$. But by properties of cosine we have:

$$\cos(90 - \alpha - \theta) = \sin(\alpha + \theta),$$

and substituting with the sine addition formula gives us:

$$\frac{SQ'}{\sqrt{x^2 + z^2}} = \sin(\alpha) \cos(\theta) + \cos(\alpha) \sin(\theta),$$

which we can use our diagram to change to:

$$\frac{SQ'}{\sqrt{x^2 + z^2}} = \frac{x}{\sqrt{x^2 + z^2}} \cos(\theta) + \frac{z}{\sqrt{x^2 + z^2}} \sin(\theta)$$

which simplifies to:

$$SQ' = x \cos(\theta) + z \sin(\theta)$$

To find the x -coordinates of our rotated point P' , we can use the fact that $\sin(90 - \alpha - \theta) = \frac{P'Q'}{\sqrt{x^2 + z^2}}$. But by properties of sine we have:

$$\sin(90 - \alpha - \theta) = \cos(\alpha + \theta)$$

and substituting with the cosine addition formula gives us:

$$\frac{P'Q'}{\sqrt{x^2 + z^2}} = \cos(\alpha) \cos(\theta) - \sin(\alpha) \sin(\theta),$$

which we can use our diagram to change to:

$$\frac{P'Q'}{\sqrt{x^2 + z^2}} = \frac{z}{\sqrt{x^2 + z^2}} \cos(\theta) - \frac{x}{\sqrt{x^2 + z^2}} \sin(\theta)$$

which simplifies to:

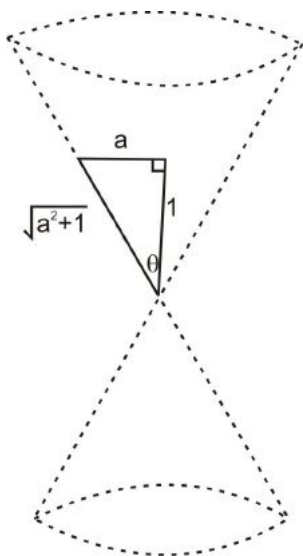
$$P'Q' = z \cos(\theta) - x \sin(\theta)$$

Looking back at the picture, this means that the coordinates of P' are $(x \cos(\theta) + z \sin(\theta), z \cos(\theta) - x \sin(\theta))$. In other words, in rotating from P to P' , the x -coordinate changes to $x \cos(\theta) + z \sin(\theta)$ and the z -coordinate changes to $z \cos(\theta) - x \sin(\theta)$.

If this rotation happens to every point on the cone, we can substitute $x \cos(\theta) + z \sin(\theta)$ for x and $z \cos(\theta) - x \sin(\theta)$ for z into our equation of the cone, resulting in a new equation for the cone after rotating by θ .

$$\begin{aligned} (x \cos(\theta) + z \sin(\theta))^2 + y^2 &= a^2(z \cos(\theta) - x \sin(\theta))^2 \\ x^2 \cos^2(\theta) + 2xz \cos(\theta) \sin(\theta) + z^2 \sin^2(\theta) + y^2 &= a^2(x^2 \sin^2(\theta) - 2xz \cos(\theta) \sin(\theta) + z^2 \cos^2(\theta)) \\ x^2 \cos^2(\theta) + 2xz \cos(\theta) \sin(\theta) + z^2 \sin^2(\theta) + y^2 &= a^2 x^2 \sin^2(\theta) - 2a^2 xz \cos(\theta) \sin(\theta) + a^2 z^2 \cos^2(\theta) \\ x^2 \cos^2(\theta) + 2xz \cos(\theta) \sin(\theta) + z^2 \sin^2(\theta) + y^2 &= a^2 x^2 \sin^2(\theta) - 2a^2 xz \cos(\theta) \sin(\theta) + a^2 z^2 \cos^2(\theta) \end{aligned}$$

Now in the case of the tilted cone, we want to tilt the cone such that the left side becomes vertical. Since the factor a determines how tilted the cone is, we can see from the triangle below that $\sin(\theta) = \frac{a}{\sqrt{1+a^2}}$ and $\cos(\theta) = \frac{1}{\sqrt{1+a^2}}$.

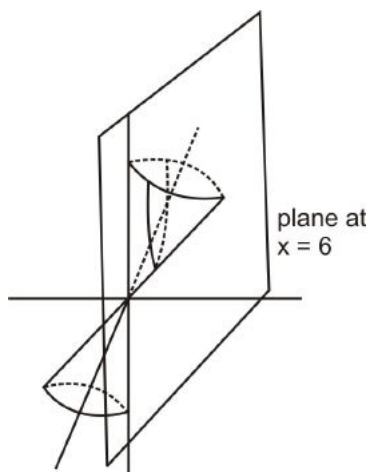


So the equation becomes:

$$\begin{aligned} x^2 \frac{1}{1+a^2} + 2xz \frac{a}{1+a^2} + z^2 \frac{a^2}{1+a^2} + y^2 &= a^2 x^2 \frac{a^2}{1+a^2} - 2a^2 xz \frac{a}{1+a^2} + a^2 z^2 \frac{1}{1+a^2} \\ x^2 + 2xza + z^2 a^2 + y^2(1+a^2) &= a^4 x^2 - 2a^3 xz + a^2 z^2 \\ x^2 + 2xza + y^2(1+a^2) &= a^4 x^2 - 2a^3 xz \end{aligned}$$

Now that we have tilted our cone, to take a cross section that is parallel to the left side of the cone, we can simply cut it with a vertical plane. The equation of a vertical plane going through $(b, 0, 0)$ and

perpendicular to the x -axis is $x = b$. Therefore, setting x equal to the constant in the equation above will give us the intersection of the tilted cone and a plane parallel to one side of the cone. b . Here is a picture of the rotation and the cross-section, which lies in an xz -plane.



Setting x equal to the constant b , we have:

$$\begin{aligned} b^2 + 2abz + y^2(1 + a^2) &= a^4b^2 - 2a^3bz \\ z(2ab + 2a^3b) &= -y^2(1 + a^2) + a^4b^2 - b^2 \\ z &= \left(\frac{-1 - a^2}{2ab + 2a^3b} \right) y^2 + (a^4b^2 - b^2) \end{aligned}$$

Although this coefficient and constant term seem complicated, a and b can be chosen so that the coefficient of the y^2 term can be equal to any number (you will explore this fact in an exercise). The constant term can be ignored since any parabola can be shifted vertically by any amount.

So the general form of a parabola is:

$$z = Ay^2$$

where A is any constant.

Or, using the more standard x - and y -coordinates the form of a parabola is

$$y = ax^2$$

As before, this equation can be adapted to produce the shifted and horizontally oriented forms.

Review Questions

- Explain why $\cos(90 - \alpha - \theta) = \sin(\alpha + \theta)$ in the above argument.
- Show that for any A there exist constants a and b such that $A = \frac{-1-a^2}{2ab+2a^3b}$.

Review Answers

- There are many arguments that work. One route is to use the fact that $\cos(\alpha) = \cos(-\alpha)$ for any α , and then the fact that $\cos(\alpha - 90) = \sin(\alpha)$ for any α .

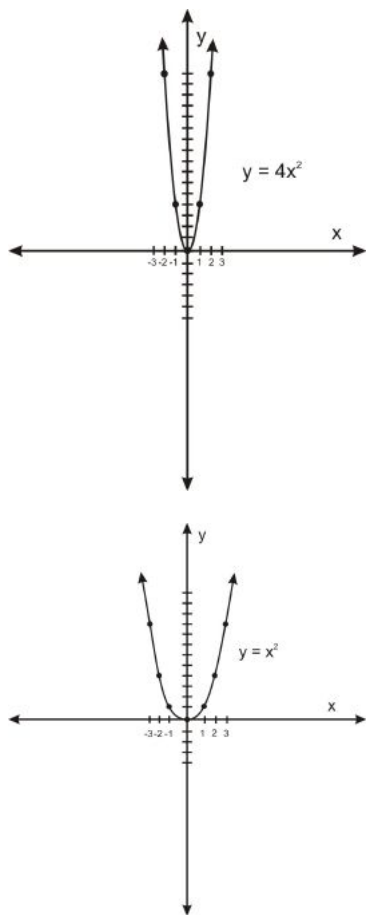
8. Solving for b in terms of A and a , we have:

$$\begin{aligned}A(2ab + 2a^3b) &= -1 - a^2 \\2Aab(1 + a^2) &= -(1 + a^2) \\2Aab &= -1 \\2Aab &= -1 \\b &= -\frac{1}{2Aa}\end{aligned}$$

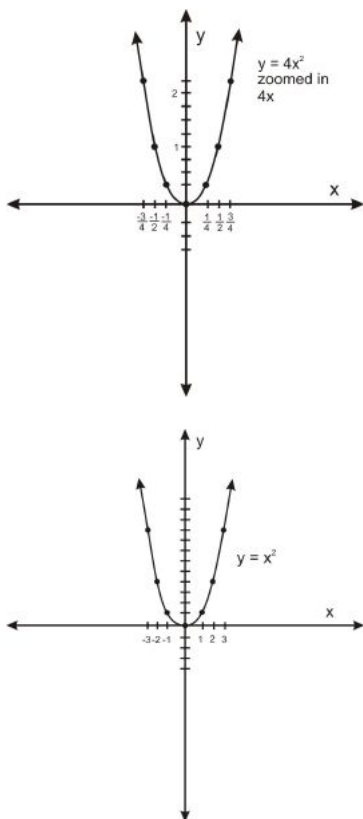
So we can set $b = 2Aa$ and the relationship will hold.

All Parabolas are the Same

There is a subtle point in the above two arguments that reveals a very interesting property of parabolas. This is the fact that *all parabolas have the same shape*. Or, in the language of geometry, any two parabolas are similar to one another. This means that one parabola can be scaled in or out to produce another parabola of exactly the same shape. This may come across as surprising, because parabolas where x^2 has a large coefficient certainly look much “steeper” than parabolas with a small coefficient when examined over the same domain, as shown in the graphs below.

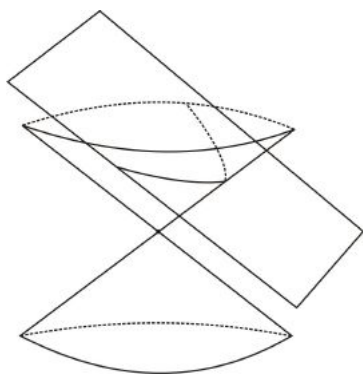


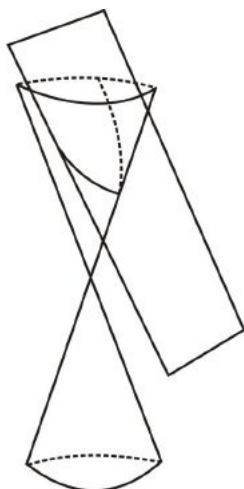
But when one of the parabolas is scaled appropriately, these parabolas are identical:



This fact about parabolas can be seen in the first argument simply from the fact that all parabolas are generated from a line and a point not on that line. This configuration of generating objects, a line and a point, is always the same shape. Any other line and point looks exactly the same—simply zoom in or out until the line and point are the same distance from one another. So the shapes that any two such configurations generate must also be the same shape.

In the second argument, there were two factors that might affect the shape of the parabola. The first is the distance between the cutting plane and the apex of the cone. But cones have the same proportions at any scale, so no matter what this distance, the picture can be reduced or enlarged, affecting this distance but not the shape of the cone or plane. So this parameter does not actually change the shape of the conic section that results. The other factor is the shape of the actual cone. This is its steepness, defined by the angle at the apex, or equivalently by the ratio between the radius and the height at any point. This is a bit trickier. It's not at all obvious that short, squat cones and tall, skinny cones would produce parabolas of the same shape.





According to what we found, any parabola produced by slicing any cone resulted in an equation of this form:

$$y = ax^2$$

We want to show that if we generate two such parabolas, that they actually have the same shape. So suppose we use two cone constructions and come up with these parabolas: $y = a_1x^2$ and $y = a_2x^2$. We want to show there is some scale factor, call it f , that shrinks or enlarges $y = a_1x^2$ into $y = a_2x^2$. To keep a shape the same, the scale factor needs to affect both the x - and y -variables. So we need to find an f such that $(fy) = a_1(fx)^2$ is equivalent to $y = a_2x^2$. The first equation can be written $y = (a_1f)x^2$, which is equivalent to the second equation when $a_1f = a_2$, or when $f = \frac{a_2}{a_1}$. Such an f always exists for non-negative numbers a_1 and a_2 . So the parabolas are indeed the same shape. If a is less than zero, then the parabola can be reflected vertically to produce a parabola of the same shape and with positive coefficient a .

Review Questions

9. What other classes of shapes have this property that all members of the class are similar to each other?
10. Explain why not all ellipses are similar. While enlarging or shrinking doesn't work to make two ellipses identical, how can you change the view of two ellipses that have different shapes so that they look the same?

Review Answers

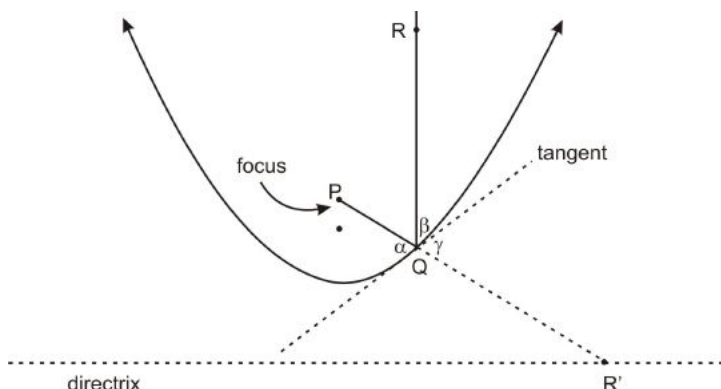
9. A few examples are: circles, squares, finite sections of one-sided cones of the same angle.
10. The eccentricity of ellipses defines the shape, so when the eccentricity is different for two ellipses, the ellipses are not similar to one another. Viewing one of the ellipses at an angle, however, changes the perceived eccentricity of that ellipse, and the angle can be chosen to match the perceived eccentricity to the eccentricity of the other ellipse, producing an image that is similar to the other ellipse.

Applications

“Burning Mirrors”

Diocles (~ 240 – 180 BCE) was a mathematician from Ancient Greece about whom we know very little. However, we know enough from a few scant documents that he thought about an important application of parabolas. It comes from what is sometimes called the “optical property” of parabolas. Remember the optical property of ellipses: lines from one focus “bounce off” the side of the ellipse to hit the other focus.

For parabolas, since parabolas have only one focus, the directrix plays a role. For the parabola, the optical property is that lines perpendicular to the directrix “bounce off” the parabola and converge at the focus. Or, alternatively, lines from the focus “bounce off” off the parabola and continue perpendicular to the directrix. As with the ellipse, “bouncing off” means that the two lines meet the parabola at equal angles to the tangent.



In the above diagram, the optical property states that $\angle \alpha \cong \angle \beta$. The proof is similar to the proof of the optical property of ellipses. In the above diagram P is the focus and Q is a point on the parabola. Let R' be the point on the directrix that is obtained by extending PQ . Then PR' , the straight line, is clearly the shortest distance between P and R' that passes through the tangent line. Let R be on the line lying directly above Q such that $QR = QR'$. The R can be thought of as R' , reflected across the tangent line. Then $\angle \alpha \cong \angle \gamma$ (vertical angles) and $\angle \gamma \cong \angle \beta$ (reflected angles), and so $\angle \alpha \cong \angle \beta$ (transitive property).

The optical property has some interesting applications. Diocles described one potential application in his document “On Burning Mirrors”. He envisioned a parabolic-shaped mirror (basically a parabola rotated about its line of symmetry) which would collect light from the sun and focus it on the focal point, creating enough of a concentration of light to start a fire at that point. Some claim that Archimedes attempted to make such a contraption with copper plates to fight the Romans in Syracuse.

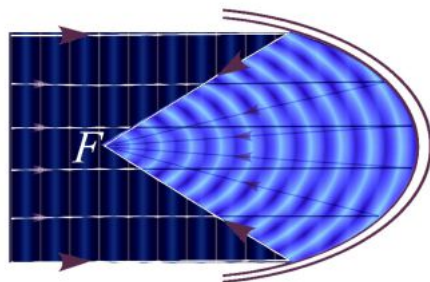


Figure 8.3

Review Questions

11. If nothing was used to deflect light before it entered the mirror, where would the sun have to be in relationship to you and the place you want to start a fire? Why is this a constraint? Design a way to circumvent this constraint.
12. You live in Ancient Greece and are defending your city against Roman invaders. Design a “burning mirror” that you plan to use to destroy an army that is approaching at a distance of three miles.

Review Answers

11. The fire-locale must lie on the segment between you and the sun. This is a problem because to start a ground fire, you would have to wait until evening when the sun is not as bright. A lens or mirror that changes the angle of the sun's rays could help you work around this constraint.
12. Answers will vary. The distance from the focus to the vertex should be 3 miles.

Headlights

The optical property is also responsible for parabola-shaped unidirectional lights, such as car headlights. If a bulb is placed at the focus of a parabolic mirror, the light rays reflect off the mirror parallel to each other, making a focused beam of light.

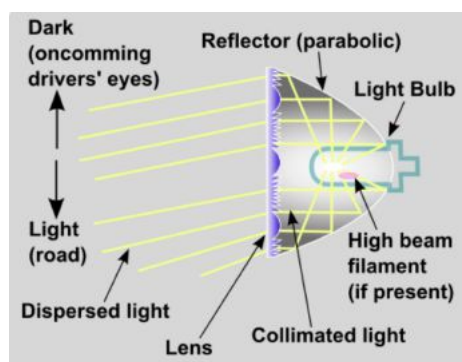


Figure 8.4

Review Questions

13. In the above diagram of a car headlight, the lens directs the beams of light downwards, to keep them out of the eyes of oncoming drivers. But if that was the only purpose of the lens, alternatively the lens could be omitted and the headlight could just be angled down slightly. But there is another purpose to the lens. What is it?

Review Answers

13. The lens also expands the array of light which is why it is called “dispersed light.” Without the lens, the headlight would only illuminate a strip the width of the headlight itself, which would not be very useful for driving.

Cassegrain Telescopes

Satellite telescopes take advantage of the optical property of parabolas to collect as much light from a distant star as possible. The dish of the satellite below is parabolic in shape and reflects light to the point in the middle.



Figure 8.5

This image at this point is then focused with a lens into the telescope as shown in the diagram below.

(from:http://commons.wikimedia.org/wiki/File:Telescope_cassegrain_principe.png) The

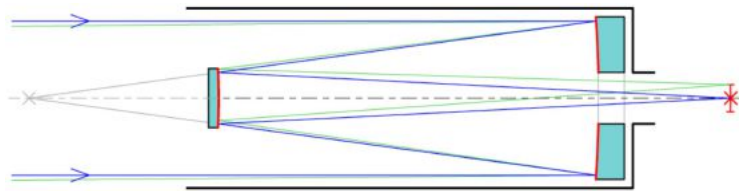


Figure 8.6

Technology

As with ellipses, you can examine the graphs of parabolas on a graphing calculator or computer program. But as with ellipses, there is a much more low-technology tool that can draw a parabola. It is a generalized version of the circle-drawing compass. Below is a model of a conic section producing compass drawn by Leonardo da Vinci. Like a compass for making circles, this tool assists a pencil in swiveling about a vertex. Unlike a compass, the pencil is held loosely in a shaft so that it can slide up and down, and the fixed side is held at a constant angle. As this angle changes, different conic sections result. This essentially turns the pencil shaft into the cone and the paper into the cutting plane.

Review Questions

14. Describe how the above compass should be set up to produce a parabola.

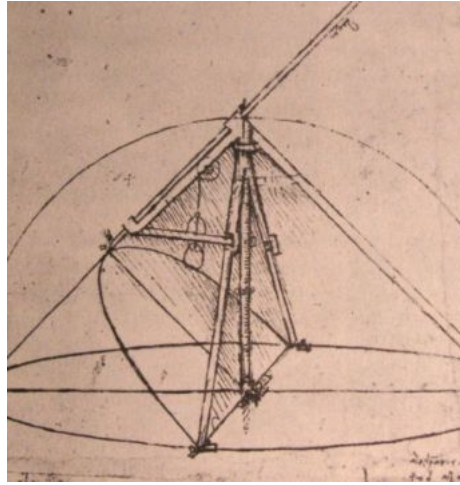


Figure 8.7

Review Answers

14. The pencil should be parallel to the paper at its most extended point.

Vocabulary

Parabola A conic section resulting from intersecting a cone with a plane that is parallel to one of the lines in the cone.

Focus The point to which all points on a parabola are the same distance as they are to a line, called the directrix.

Directrix A line to which all points on a parabola are the same distance as they are to a point called the focus.

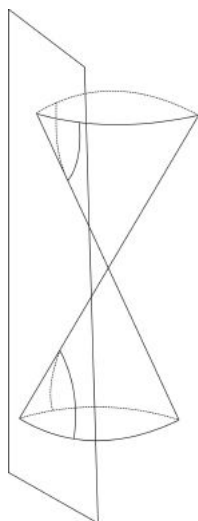
8.4 Hyperbolas

Learning Objectives

- Understand what results when a cone is intersected by a plane that intersects both sides of the cone.
- Understand the focal-directrix property for hyperbolas.
- Recognize and work with equations for hyperbolas.
- Understand how a hyperbola can be described in terms of its asymptotes and foci.
- Understand the equivalence of different definitions of hyperbolas.

Two-part Conic Sections

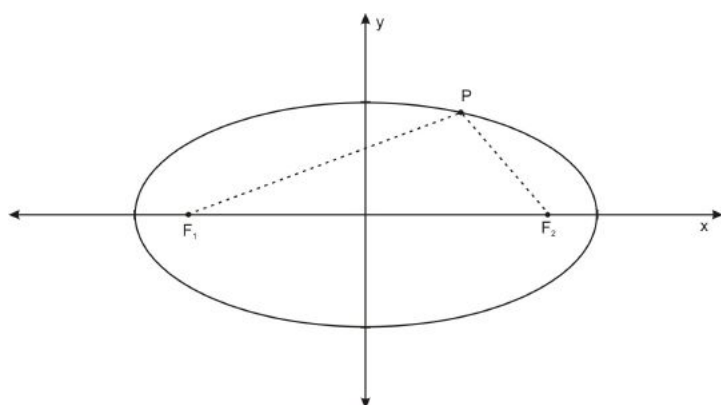
The final way of cutting a cone with a plane appears at first to be much messier. Not only does it result in an infinite shape, but there are two pieces that aren't even connected! When the plane slices through two parts of the cone, the two infinite "U"-shaped parts are together called a **hyperbola**.



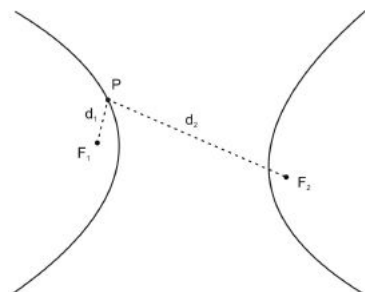
In this section we will see that this sprawling shape actually has some beautiful properties that make it as noble as its cousins.

The Focal Property

Even though this shape seems much harder to conceive of than an ellipse, the hyperbola has a defining focal property that is as simple as the ellipse's. Remember, an ellipse has two foci and the shape can be defined as the set of points in a plane whose distances to these two foci have a fixed sum.



Hyperbolas also have two foci, and they can be defined as the set of points in a plane whose distances to these two points have the same *difference*. So in the picture below, for every point P on the hyperbola, $|d_2 - d_1| = C$ for some constant C .



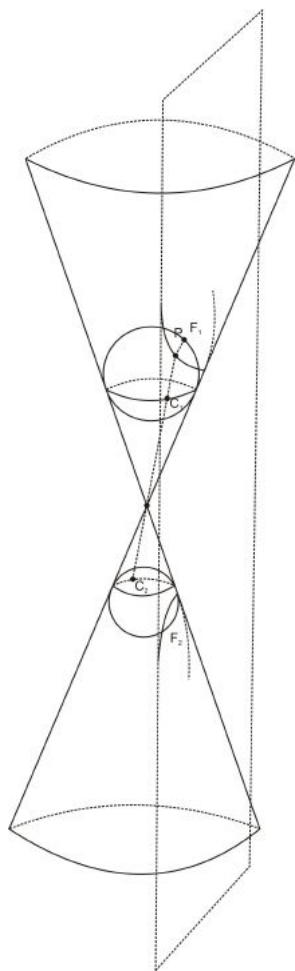
Review Questions

1. The hyperbola is infinite in size. In mathematics this is called **unbounded**, which means no circle, no matter how large, can enclose the shape. Explain why a focal property involving a *difference* results in an unbounded shape, while a focal property involving a sum results in a bounded shape.

Review Answers

1. Answer should include the following concept: In the case of an ellipse, we had two distances summing to a constant. Since the distances are both positive then there is a limit to the size of the numbers. In the case of hyperbolas, two very large positive numbers can have a much smaller difference.

To prove the focal property of hyperbolas, we examine Dandelin's sphere construction. Unlike the construction for ellipses, which used two spheres on one side of the cone, and the sphere construction for parabolas, which used one sphere on one side of the cone, this construction uses two spheres, one on each side of the cone. As with the ellipse construction, each sphere touches the plane at one of the foci of the hyperbola. And as with the argument for the elliptical focal property, the argument uses the fact that tangents from a common point to a sphere are equal.



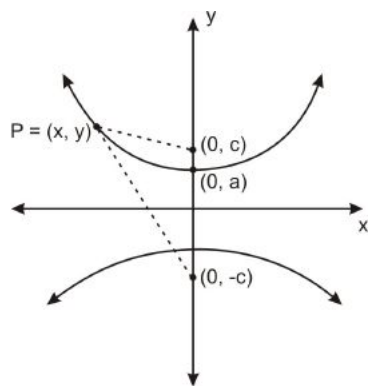
In the above diagram, suppose P is an arbitrary point on the hyperbola. We would like to examine the difference $PF_2 - PF_1$. Let C_1 be the point on the upper sphere that lies on the line between P and the vertex of the cone. Let C_2 be the point on the lower sphere when this line is extended (so P , C_1 , and

C_2 are all on the same line and $PC_1 + C_1C_2 = PC_2$ and the cone. By the common tangent property, $PF_1 = PC_1$ and $PF_2 = PC_2$ for some points C_1 and C_2 on the circles where the spheres meet the cone. So $PF_2 - PF_1 = PC_2 - PC_1 = (PC_1 + C_1C_2) - PC_1 = C_1C_2$. But C_1C_2 is the distance along the cone between the two circles of tangency and is constant regardless of the choice of C_1 and C_2 . So the difference $PF_2 - PF_1$ is constant.

Equations for Hyperbolas

To derive the equation for a hyperbola, we can't do exactly what we did for ellipses. Remember, when we first derived the equations for ellipses, we were defining them as "stretched out circles." Hyperbolas aren't as obviously a simple dilation of a shape as basic as a circle. Instead, we'll use the focal property to derive their equation. This was what we said could be done for ellipses when I said "simply read the proof of the focal property backwards." To derive the equation form of hyperbolas from the focal property, I'll actually show you the steps in the correct order.

Suppose we have a curve (actually a pair of curves), satisfying the focal property for hyperbolas. Let's orient the hyperbola so that the two foci are on the y -axis, and equidistant from the origin. In the diagram below the foci are labeled with the points $(0, c)$, and $(0, -c)$.



The focal property states that the difference in distances between an arbitrary point $P = (x, y)$ on the hyperbola and $(0, c)$, and $(0, -c)$ is a constant. In particular, we know that this constant can be computed from any point on the hyperbola, for instance the point on the y -axis marked $(0, a)$. The distance between $(0, a)$ and $(0, -c)$ is $a + c$ and the distance between $(0, a)$ and $(0, c)$ is $c - a$. The difference between these two quantities is $(c + a) - (c - a) = 2a$. So $2a$, the distance between the two y -intercepts of the hyperbola, is the constant in the focal property for the hyperbola. Using this distance formula and the focal property, we have:

$$\begin{aligned}
& \sqrt{(x-0)^2 + (y-(-c))^2} - \sqrt{(x-0)^2 + (y-c)^2} = 2a \\
& x^2 + (y+c)^2 - 2\sqrt{x^2 + (y+c)^2}\sqrt{x^2 + (y-c)^2} + x^2 + (y-c)^2 = 4a^2 \\
& 2x^2 + (y+c)^2 + (y-c)^2 - 4a^2 = 2\sqrt{x^2 + (y+c)^2}\sqrt{x^2 + (y-c)^2} \\
& (2x^2 + 2y^2 + 2c^2 - 4a^2)^2 = 4(x^2 + (y+c)^2)(x^2 + (y-c)^2) \\
& 4x^4 + 8x^2y^2 + 8c^2x^2 + 8c^2y^2 + 4y^4 - 16a^2x^2 + 4c^4 - 16a^2c^2 - 16a^2y^2 + 16a^4 = 4x^4 + 4x^2(y+c)^2 + 4x^2(y-c)^2 + 4(y+c)^2 + 4(y-c)^2 \\
& 4x^4 + 8x^2y^2 + 8c^2x^2 + 8c^2y^2 + 4y^4 - 16a^2x^2 + 4c^4 - 16a^2c^2 - 16a^2y^2 + 16a^4 = 4x^4 + 4x^2(y+c)^2 + 4x^2(y-c)^2 + 4(y^2 - c^2) + 4(y^2 - c^2) \\
& 4x^4 + 8x^2y^2 + 8c^2x^2 + 8c^2y^2 + 4y^4 - 16a^2x^2 + 4c^4 - 16a^2c^2 - 16a^2y^2 + 16a^4 = 4x^4 + 8x^2y^2 + 8c^2x^2 + 4y^4 - 8c^2y^2 + 4c^4 \\
& 16c^2y^2 - 16a^2x^2 - 16a^2c^2 - 16a^2y^2 + 16a^4 = 0 \\
& c^2y^2 - a^2x^2 - a^2c^2 - a^2y^2 + a^4 = 0 \\
& c^2y^2 - a^2y^2 - a^2x^2 = a^2c^2 - a^4 \\
& y^2(c^2 - a^2) - x^2(a^2) = a^2(c^2 - a^2) \\
& \frac{y^2}{a^2} - \frac{x^2}{c^2 - a^2} = 1
\end{aligned}$$

Like the collapse we witnessed when investigating the focal property for ellipses, this equation became pretty simple. We can even make it simpler. For any positive number b and a , there exists a c such that $b^2 = c^2 - a^2$. Thus the general form for a hyperbola that open upwards and downwards and whose foci lie on the y -axis is:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Switching x and y we have hyperbolas that open rightwards and leftwards and whose foci lie on the x -axis.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

These equations have so far been hyperbolas that are centered about the origin. For a hyperbola that is centered around the point (h, k) we have the shifted equations:

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

for a hyperbola opening up and down, and

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

For a hyperbola opening left and right.

Example 1

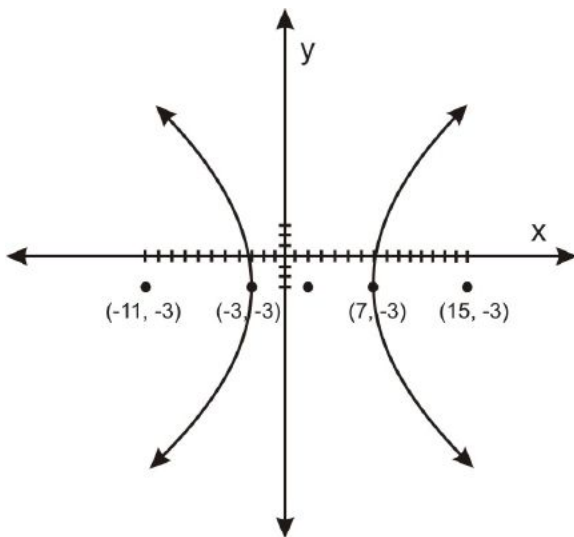
Show that the following equation is a hyperbola. Graph it, and show its foci.

$$144x^2 - 576x - 25y^2 - 150y - 3249 = 0$$

Solution: The positive leading coefficient for the x^2 term and the negative leading coefficient for the y^2 term indicate that this is a hyperbola that is horizontally oriented. Grouping and completing the square, we have:

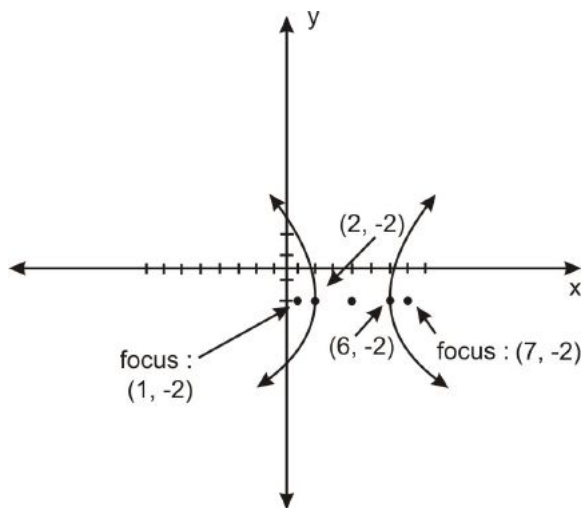
$$\begin{aligned}
 144(x^2 - 4x) - 25(y^2 + 6y) &= 3249 \\
 144(x^2 - 4x + 4) - 25(y^2 + 6y + 9) &= 3249 + 576 - 225 \\
 144(x - 2)^2 - 25(y + 3)^2 &= 3600 \\
 \frac{(x - 2)^2}{5^2} - \frac{(y + 3)^2}{12^2} &= 1
 \end{aligned}$$

So our hyperbola is centered at $(2, -3)$. Its vertices are 5 units to the right and left of $(2, -3)$, or at the points $(7, -3)$ and $(-3, -3)$. It opens to the right and left from these vertices. It's foci are c units to the left and right of $(2, -3)$, where $c = \sqrt{a^2 + b^2} = \sqrt{5^2 + 12^2} = 13$. So it's foci are at $(15, -3)$ and $(-11, -3)$. Plotting a few points near $(7, -3)$ and $(-3, -3)$, the graph looks like:



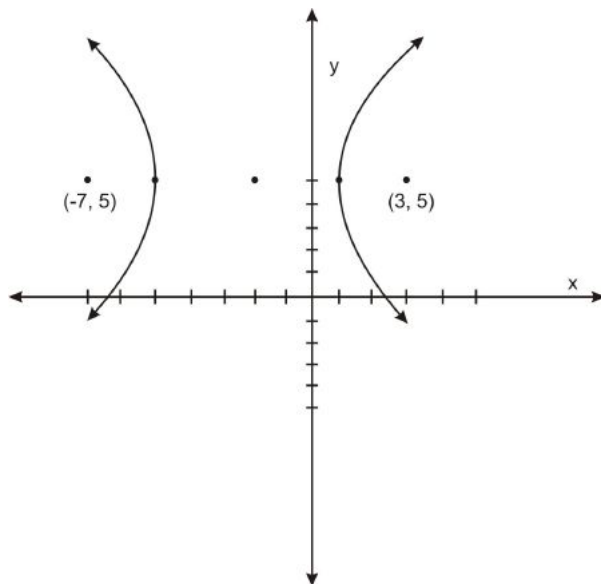
Review Questions

- Explain why for any positive number b and a , there exists a c such that $b^2 = c^2 - a^2$.
- Graph the following hyperbola and mark its foci: $16x^2 + 64x - 9y^2 + 90y - 305 = 0$
- Graph the following hyperbola and mark its foci: $9y^2 + 18y - x^2 + 4x - 4 = 0$
- Graph the following hyperbola and mark its foci: $25x^2 + 150x - 4y^2 + 24y + 89 = 0$
- Find the equation for the following hyperbola:

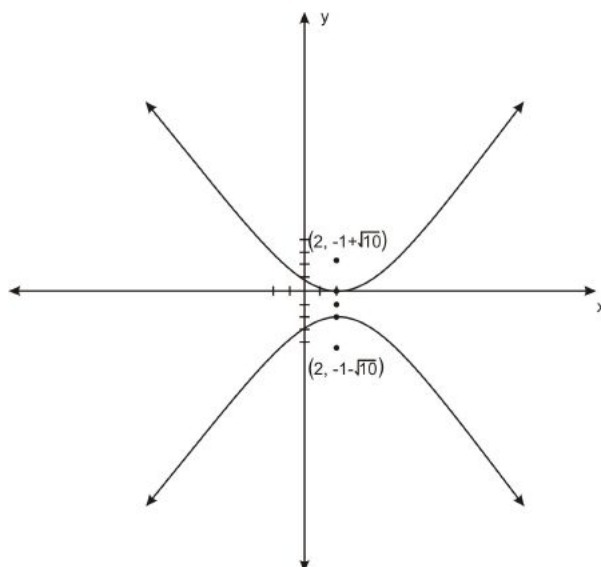


Review Answers

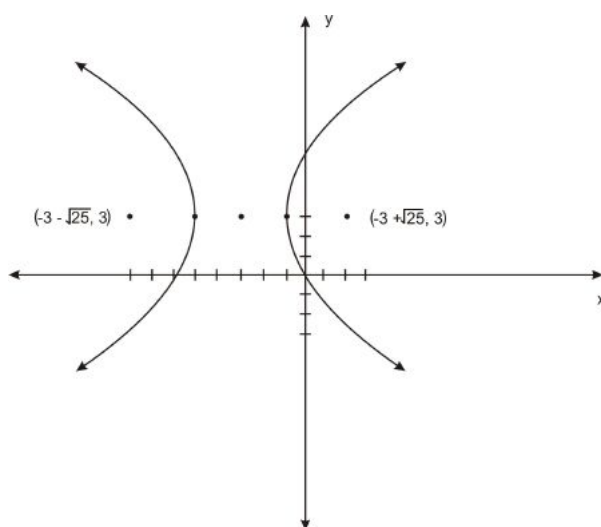
2. Let $c = \sqrt{a^2 + b^2}$. Since $a^2 + b^2$ is always positive for positive a and b , this number is always defined. Geometrically, let c be the hypotenuse of a right triangle with side lengths a and b .
- 3.



4.



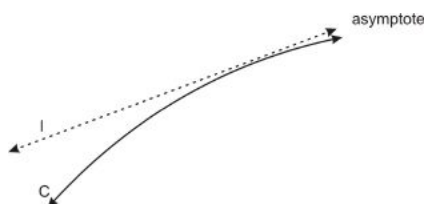
5.



6. $\frac{(x-4)^2}{4} - \frac{(y+2)^2}{45} = 1$

Asymptotes

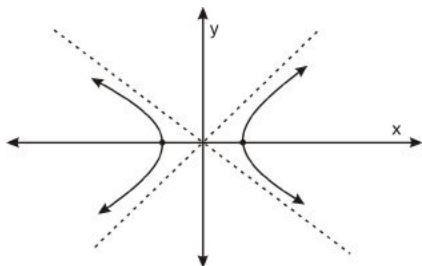
In addition to their focal property, hyperbolas also have another interesting geometric property. Unlike a parabola, a hyperbola becomes infinitesimally close to a certain line as the x - or y -coordinates approach infinity. Such a line is called an **asymptote**.



Before we try to prove this property of hyperbolas, we have to figure out what we mean by “infinitesimally close.” Here we mean two things: 1) The further you go along the curve, the closer you get to the asymptote, and 2) If you name a distance, no matter how small, eventually the curve will be that close to the asymptote. Or, using the language of limits, as we go further from the vertex of the hyperbola the limit of the distance

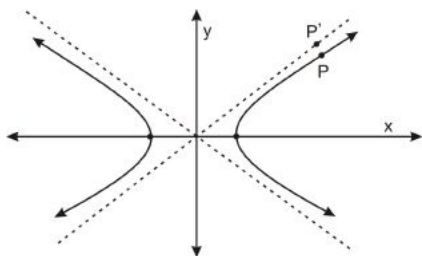
between the hyperbola and the asymptote is 0.

So now we need to prove such a line (or lines) exist for the hyperbola. It turns out there are two of them, and that they cross at the point at which the hyperbola is centered:



It also turns out that for a hyperbola of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the asymptotes are the lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$. (for a hyperbola of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ the asymptotes are the lines $y = \frac{a}{b}x$ and $y = -\frac{a}{b}x$. For a shifted hyperbola, the asymptotes shift accordingly.)

To prove that the hyperbola actually gets infinitesimally close to these lines as it goes to infinity, let's focus on the point P on the upper right leg of the hyperbola in the picture below.



Now consider the point P' , the point on the asymptote lying directly above P . If we can show that P and P' become infinitesimally close, then that's good enough (this subtlety will be explored in the next exercise.) Well, the point P is on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, so if it's x -value is x , it's y -value can be found by solving for y in the equation. So

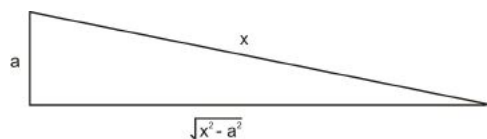
$$\begin{aligned}\frac{x^2}{a^2} - 1 &= \frac{y^2}{b^2} \\ y^2 &= \frac{b^2}{a^2}x^2 - b^2 \\ y &= \sqrt{\frac{b^2}{a^2}x^2 - b^2}\end{aligned}$$

This expression simplifies further, but we'll leave it for now and work on P' . Since P' lies directly above P , x is also the x -coordinate of P' . Similarly as with P , we can find the y -coordinate from the equation of the asymptote. Since $y = \frac{b}{a}x$ is the equation of the asymptote, we have $\frac{b}{a}x$ as the y -coordinate of P' . So $P = \left(x, \sqrt{\frac{b^2}{a^2}x^2 - b^2}\right)$, and $P' = \left(x, \frac{b}{a}x\right)$. Since these two points are vertically aligned, the distance between them points is simply the difference between the y -coordinate:

$$\text{Distance between } P \text{ and } P' = \frac{b}{a}x - \sqrt{\frac{b^2}{a^2}x^2 - b^2}$$

$$\begin{aligned}
&= \frac{b}{a}x - \sqrt{\frac{b^2}{a^2}x^2 - \frac{b^2}{a^2}a^2} \\
&= \frac{b}{a}x - \sqrt{\frac{b^2}{a^2}(x^2 - a^2)} \\
&= \frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2} \\
&= \frac{b}{a}\left(x - \sqrt{x^2 - a^2}\right)
\end{aligned}$$

So the distance is a constant multiplied by a quantity that depends on x . What happens to this quantity when x approaches infinity? This kind of question involves working with limits in a way you will study soon. For now, we can use a nice geometric picture to see what happens. In the picture below, we see that a , x , and $\sqrt{x^2 - a^2}$ can be thought of as side-lengths of a right triangle, where x is the hypotenuse. This can be verified with the Pythagorean Theorem.



In this picture, what happens to difference in lengths between the side of length x and the side of length $\sqrt{x^2 - a^2}$ when x approaches infinity and a remains fixed? The two side lengths become closer in length, so their difference approaches zero.

Going back to the previous diagram where we defined P and P' , this means that P and P' become infinitesimally close as x approaches infinity.

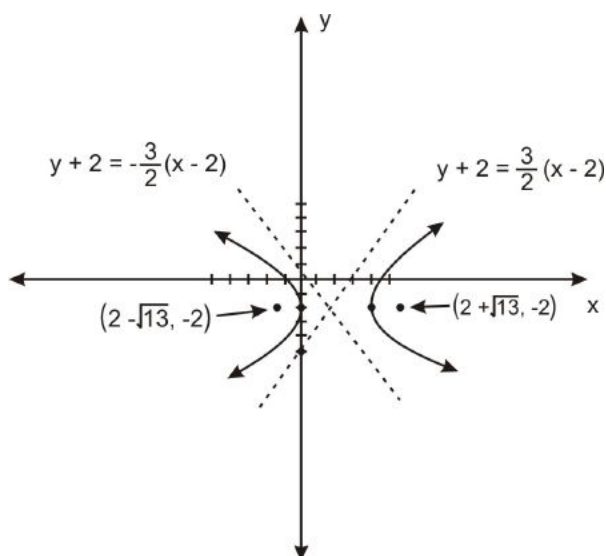
This same argument can be repeated to show that the other three legs of the hyperbola approach their respective asymptotes, and to show that the same holds in vertically-oriented and/or shifted parabolas.

Example: Graph the following hyperbola, drawing its foci and asymptotes and using them to create a better drawing: $9x^2 - 36x - 4y^2 - 16y - 16 = 0$.

Solution: First, we put the hyperbola into the standard form:

$$\begin{aligned}
9(x^2 - 4x) - 4(y^2 + 4y) &= 16 \\
9(x^2 - 4x + 4) - 4(y^2 + 4y + 4) &= 36 \\
\frac{(x - 2)^2}{4} - \frac{(y + 2)^2}{9} &= 1
\end{aligned}$$

So $a = 2$, $b = 3$ and $c = \sqrt{4 + 9} = \sqrt{13}$. The hyperbola is horizontally oriented, centered at the point $(2, -2)$, with foci at $(2 + \sqrt{13}, -2)$ and $(2 - \sqrt{13}, -2)$. After taking shifting into consideration, the asymptotes are the lines: $y + 2 = \frac{3}{2}(x - 2)$ and $y + 2 = -\frac{3}{2}(x - 2)$. So graphing the vertices and a few points on either side, we see the hyperbola looks something like this:



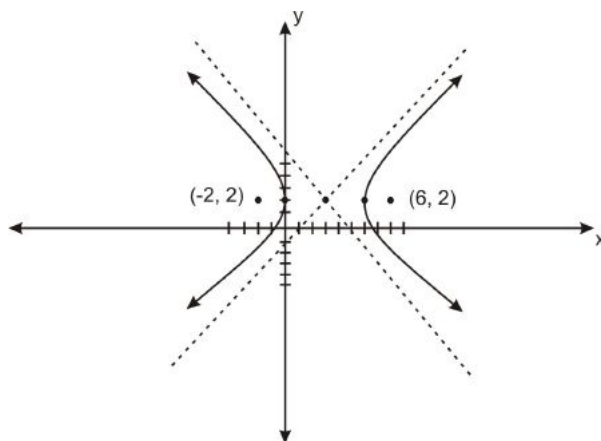
Review Questions

- In the diagram above, which was used in the proof that hyperbolas have asymptotes, why is it true that if P and P' become infinitesimally close, that this implies the hyperbola and the line become infinitesimally close. (HINT: what exactly do we mean by the “closeness” of a point on a hyperbola and a line?)
- Graph the following hyperbola, drawing its foci and asymptotes and using them to create a better drawing: $16x^2 - 96x - 9y^2 - 36y - 84 = 0$.
- Graph the following hyperbola, drawing its foci and asymptotes and using them to create a better drawing: $y^2 - 14y - 25x^2 - 200x - 376 = 0$.
- Find the equation for a hyperbola with asymptotes of slopes $\frac{5}{12}$ and $-\frac{5}{12}$, and foci at points $(2, 11)$ and $(2, 1)$.
- A hyperbolas with perpendicular asymptotes is called **perpendicular**. What does the equation of a perpendicular hyperbola look like?

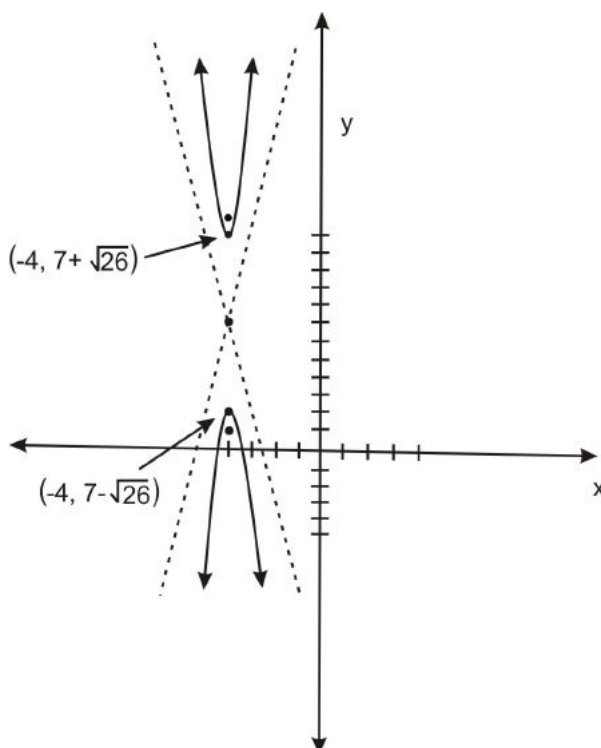
Review Answers

- The distance between a point and a line is the shortest segment between the point and a point on the line. We have shown that *some* distance—not necessarily the shortest—between P and a point on the asymptote becomes infinitesimally smaller. This means that the shortest distance between P and the asymptote must also become shorter.

8.



9.



10. $\frac{(y-6)^2}{25} - \frac{(x-2)^2}{144} = 1$

11. The slopes of perpendicular lines are negative reciprocals of each other. This means that $\frac{a}{b} = \frac{b}{a}$, which, for positive a and b means $a = b$.

Applications

Vexing Questions from Ancient Greece

This application of hyperbolas is the reason Menaechmus, one of the first serious thinkers about conic sections, got into them in the first place.

It is thought that conic sections first attracted the attention of the Ancient Greeks when Menaechmus used two conics to find a solution to another famous Greek problem. A famous class of Greek problems involves constructing a length with the simplest tools possible. One of these problems is often referred to as the problem of “doubling a cube.” The problem is to make a cube that is twice the volume of a given cube.



This, of course, does not simply mean making a cube whose sides are double in length. This would produce a cube with eight times the volume. If the first cube has side length one unit then it has volume $1^3 = 1$. In order for a cube to have double that volume, 2, it must have a side length of $\sqrt[3]{2}$. So the problem boils down to producing a segment whose length is $\sqrt[3]{2}$. Once this is done, the cube can be constructed out of segments of this length.

So Menaechmus was working on constructing a segment of length $\sqrt[3]{2}$. Now you are probably thinking, “Easy, just plug $\sqrt[3]{2}$ into your calculator.” But Menaechmus didn’t have a calculator into which he could type $2^{(1/3)}$. An even more important point is that *even if he did* have such a calculator, this wouldn’t solve the problem in a way Menaechmus would have been pleased with. He was trying to use the *simplest* technology possible, and a calculator—even if he had one—wouldn’t fit this bill. Plus, a calculator merely finds an estimate of this number, and the Greeks aimed to find a mathematical situation in which the *exact* number would be generated.

The very simplest tools the Greeks considered for making these kinds of constructions were an idealized compass and straightedge. Unknown to the Greeks at the time, a compass and straightedge don’t do the job for this length, though they didn’t know it at the time—it wasn’t discovered until 1837, when Pierre Laurent Wantzel demonstrated that this construction is impossible.

According to some historians (Heath [footnote]), Menaechmus did find a solution to the problem and he used conic sections to do so. Before his time, the problem had already been reduced to finding a specific set of “mean proportionals”, or a chain of numbers with equivalent subsequent ratios. In particular, to find numbers x and y such that 1 is to x as x is to y as y is to 2, or in fraction form:

$$\frac{1}{x} = \frac{x}{y} = \frac{y}{2}$$

How does this chain of proportions help find the number $\sqrt[3]{2}$? Well, looking at the first equality and cross-multiplying, we have $y = x^2$. Looking at the second equality, we have $y^2 = 2x$. Using the first equation to substitute x^2 for y in the second equation, we have $(x^2)^2 = 2x$, which reduces to $x^3 = 2$, the solution of which is $\sqrt[3]{2}$.

But solving the above chain of ratios is also equivalent to finding the intersection of the parabola $y = x^2$ and the hyperbola $xy = 2$. You’ll examine why this is true in an upcoming review questions.

But wait! Hold the presses! Did I say the hyperbola $xy = 2$? But that doesn’t look like the form of any of the hyperbolas we have looked at! This is because it is oriented with its foci on a diagonal, rather than horizontally or vertically. In the next section we will look at how rotating conic sections from their standard positions affects their forms. For now, you can check that $xy = 2$ indeed looks like a hyperbola by plugging in a few points near the origin—which you will do in the exercises below.

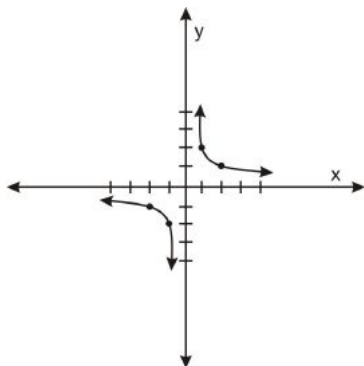
Review Questions

12. Draw a sketch of $xy = 2$ by plotting a few points near the origin.
13. What are the asymptotes of the hyperbola $xy = 2$? What are the foci?
14. Explain why solving the mean proportional $\frac{1}{x} = \frac{x}{y} = \frac{y}{2}$ is equivalent to finding the intersection of the parabola $y = x^2$ and the hyperbola $xy = 2$.
15. In the language of “compass and straightedge,” though Menaechmus’ construction can’t be done with a traditional compass, it can be done with the generalized compass discussed in the previous section.

Explain how this tool could be used to “double” an existing cube.

Review Answers

12.



13. The asymptotes are the x - and y -axes. The foci are $(2,2)$ and $(-2,-2)$ (these are relatively hard to find, but it is relatively easy to show they are the foci once they are found.)
14. These two equations are obtained by looking at the first equality and cross multiplying, as well as setting the first term equal to the third term and cross multiplying. These two equalities hold exactly the same amount of information as the chain of equalities.
15. Using a compass and straightedge, a coordinate grid with unit lengths can be drawn. See http://en.wikipedia.org/wiki/Compass_and_straightedge_constructions for more on using a compass and straightedge to make geometric constructions. Upon this, the two shapes can be drawn using the generalized compass as discussed in the last session. The distance between the y -axis and the intersection point is of length $\sqrt[3]{2}$.

Vocabulary

Hyperbola A conic section where the cutting plane intersects both sides of the cone, resulting in two infinite “U”-shapes curves.

Unbounded A shape which is so large that no circle, no matter how large, can enclose the shape.

Asymptote A line which a curve approaches as the curve and the line approach infinity, eventually becoming closer than any given positive number.

Perpendicular Hyperbola A hyperbola where the asymptotes are perpendicular.

8.5 General Algebraic Forms

Learning Objectives

- Understand how the cross sections of a cone relate to degree two polynomial equations.
- Understand what happens to the algebraic forms of conic sections when they are neither horizontally or vertically oriented.
- Recognize the algebraic form of different types of conic sections.

The Cross Sections of the Cone Are Degree Two Polynomial Equations

Let's examine all the equations of the conic sections we've studied in this chapter.

Ellipses: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are any positive numbers (the circle is the specific case when $a = b$).

Parabolas: $y = ax^2$ or $x = ay^2$ where a is any non-zero number.

Hyperbolas: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, where a and b are any positive numbers.

All these equations have in common that they are degree-2 polynomials, meaning the highest exponent of any variable—or sum of exponents of products of variables—is two. So for example, here are some degree two polynomial equations in a more general form:

$$\begin{aligned}2x^2 + 5y^2 - 3y + 4 &= 0 \\x^2 - 3y^2 + x - y + 3 &= 0 \\10x^2 - y - 5 &= 0 \\xy &= 2\end{aligned}$$

Some of these probably already look like conic sections to you. For example, in the first equation, we can complete the square to remove the $-3y$ term and we will see that we have an ellipse. In the second equation we can complete the square twice to remove both the x and $-y$ terms and we will have a hyperbola. This is a hyperbola, not an ellipse, because the coefficient of the x^2 and y^2 terms have opposite signs.

The third equation is a parabola since there is an x^2 term and y term but not a y^2 term. Do you see how you can solve for y , putting the equation in the standard form for a vertically oriented parabola?

But what about the fourth equation? Like the others, it is a degree-2 polynomial, since the exponents of the x and y term sum to 2. But the fourth equation looks nothing like any of the forms for conic sections that we've examined so far. Nonetheless, as we saw in the last section, $xy = 2$ appears to be a hyperbola with foci $(2,2)$ and $(-2,-2)$. The reason it doesn't fit either of the standard forms for hyperbolas is because it is *diagonally* oriented, rather than horizontally or vertically oriented (do you see how its two foci lie on a diagonal line, rather than a horizontal or vertical line?)

In order to see how such differently-oriented conic sections fit into our standard forms, we need to rotate them so that they are either horizontally or vertically oriented.

Rotation of Conics

Remember in the section on parabolas we discussed rotating objects in the plane. In that section we showed that if we take a point $P = (x, y)$ and rotate it θ degrees, it changes the x -coordinate to $x' \cos(\theta) - y' \sin(\theta)$ and the y -coordinate changes to $x' \sin(\theta) + y' \cos(\theta)$. (Note: in that section, because we were looking at a plane embedded in space, we happened to be looking at the xz -plane. We also we're using the "prime" symbols x' and y' . Also, in the particular case of that section we were rotating the cone clockwise. It is more standard to rotate counter-clockwise, so the signs on the sine functions have changed. So you can simply substitute x' for x and y' for z into the formulas in that section, switch the signs on the sine functions, and come up with these rotation rules for the xy -plane.)

Now suppose we have an equation of degree two polynomials, such as the $xy = 2$ example discussed above. In order to put it into the more recognizable form of a ellipse, parabola, or a hyperbola, we need to rotate it in such a way so that rotated version has no xy term. So we need to find an appropriate angle θ such that changing the x -coordinate to $x' \cos(\theta) - y' \sin(\theta)$ and the y -coordinate to $x' \sin(\theta) + y' \cos(\theta)$ results in

an equation with no xy term. We need to investigate what happens to a degree-two polynomial equation. Such an equation can be written in the form:

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

If $C = 0$, such as in the first three examples at the beginning of this section, we are done, as there is no xy term and we already know how to classify these into conics. If $C \neq 0$, we need to rotate the curve so that we have an equation with no xy term. When x is replaced by $x' \cos(\theta) - y' \sin(\theta)$ and y is replaced by $x' \sin(\theta) + y' \cos(\theta)$, only the first three terms of this equation are in danger of producing an xy term. To see if we can determine if an appropriate angle θ can always be found, let's substitute our new variables in for x and y into the first three terms of the equation:

$$A(x' \cos(\theta) - y' \sin(\theta))^2 + B(x' \sin(\theta) + y' \cos(\theta))^2 + C(x' \cos(\theta) - y' \sin(\theta))(x' \sin(\theta) + y' \cos(\theta))$$

Then, let's multiply this expression out, but only examine the terms that are a multiple of $x'y'$, since that is what we're trying to eliminate.

$$-2Ax'y' \cos(\theta) \sin(\theta) + 2Bx'y' \cos(\theta) \sin(\theta) + Cx'y'(\cos^2(\theta) - \sin^2(\theta))$$

This reduces to:

$$(-2A \cos(\theta) \sin(\theta) + 2B \cos(\theta) \sin(\theta)) + C(\cos^2(\theta) - \sin^2(\theta))x'y'$$

So we are interested in whether or not there is an angle θ such that the above coefficient of $x'y'$ is zero:

$$\begin{aligned} (-2A \cos(\theta) \sin(\theta) + 2B \cos(\theta) \sin(\theta)) + C(\cos^2(\theta) - \sin^2(\theta)) &= 0 \\ C(\cos^2(\theta) - \sin^2(\theta)) &= (2A \cos(\theta) \sin(\theta) - 2B \cos(\theta) \sin(\theta)) \\ C(\cos^2(\theta) - \sin^2(\theta)) &= 2(A - B)(\cos(\theta) \sin(\theta)) \end{aligned}$$

Separating the θ terms from the A , B , and C terms, we have:

$$\frac{2 \cos(\theta) \sin(\theta)}{\cos^2(\theta) - \sin^2(\theta)} = \frac{C}{A - B}$$

But remember the double angle formulas $\sin(2\theta) = 2 \cos(\theta) \sin(\theta)$ and $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ (see http://authors.ck12.org/wiki/index.php/Trigonometric_Identities#Double-Angle_Identities for more on the double angles). This means that we need to find an angle such that:

$$\frac{\sin(2\theta)}{\cos(2\theta)} = \frac{C}{A - B}$$

or:

$$\tan(2\theta) = \frac{C}{A - B}$$

Can we find such an angle? Well remember the graph of tangent (see http://authors.ck12.org/wiki/index.php/Circular_Functions#y_.3D_tan.28x.29.2C_The_Tangent_Graph for more on the tangent graph). It spans all values, so no matter what C , A , and B equal, we can find the appropriate tangent value. The only time this expression gives us trouble is if the denominator is zero, or $A = B$. But in this case there's an easy solution that you will find in the exercise below.

For the other cases, you can use the inverse tan function on your calculator or computer program to find out the value of 2θ , and then divide by 2 to find the value of θ .

Example: Rotate the following conic section so that it is oriented either horizontally or vertically, and then analyze the result:

$$79x^2 + 37y^2 + 42\sqrt{3}xy + (-200\sqrt{3} - 8)x + (8\sqrt{3} - 200)y + 4 = 0$$

Solution: To find how much to rotate this conic we need to solve for θ in the equation $\tan(2\theta) = \frac{C}{A-B}$. We have $C = 42\sqrt{3}$, $A = 79$, and $B = 37$, so we have:

$$\begin{aligned}\tan(2\theta) &= \frac{42\sqrt{3}}{79-37} \\ \tan(2\theta) &= \sqrt{3}\end{aligned}$$

The angle that solves this equation is $\theta = \frac{\pi}{6}$ (or 30°).

So we replace x with $x' \cos(30) - y' \sin(30) = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y'$, and y with $x' \sin(30) + y' \cos(30) = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$, giving us:

$$\begin{aligned}79\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)^2 + 37\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right)^2 + 42\sqrt{3}\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right) + \\ (-200\sqrt{3} - 8)\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right) + (8\sqrt{3} - 200)\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right) + 4 = 0\end{aligned}$$

Multiplying through by 4 we have:

$$\begin{aligned}79(\sqrt{3}x' - y')^2 + 37(x' + \sqrt{3}y')^2 + 42\sqrt{3}(\sqrt{3}x' - y')(x' + \sqrt{3}y') + \\ (-200\sqrt{3} - 8)(2\sqrt{3}x' - 2y') + (8\sqrt{3} - 200)(2x' + 2\sqrt{3}y') + 16 = 0\end{aligned}$$

Multiplying out we have:

$$\begin{aligned}237(x')^2 - 158\sqrt{3}x'y' + 79(y')^2 + 37(x')^2 + 74\sqrt{3}x'y' + 111(y')^2 + 126(x')^2 + 84\sqrt{3}x'y' - 126(y')^2 \\ - 1200x' - 16\sqrt{3}x' + 400\sqrt{3}y' + 16y' + 16\sqrt{3}x' - 400x' + 48y' - 400\sqrt{3}y' + 16 = 0\end{aligned}$$

Which simplifies to:

$$400(x')^2 + 64(y')^2 - 1600x' + 64y' + 16 = 0$$

Which, divided by 16 is:

$$25(x')^2 + 4(y')^2 - 100x' + 4y' + 1 = 0$$

Grouping the x' and y' terms and completing the squares, we have:

$$\begin{aligned}25((x')^2 - 4x' + 4) - 100 + 4\left((y')^2 + y' + \frac{1}{4}\right) - 1 + 1 = 0 \\ 25((x')^2 - 2)^2 + 4\left((y')^2 + \frac{1}{2}\right)^2 = 100 \\ \frac{((x')^2 - 2)^2}{4} + \frac{\left((y')^2 + \frac{1}{2}\right)^2}{25} = 1\end{aligned}$$

We recognize this as an ellipse, centered at the point $(2, -\frac{1}{2})$.

Review Questions

1. What's the easy solution to the above problem when $A = B$?
2. Use the above answer or a calculator or computer program to determine how much you would have to rotate the following conic to eliminate the xy term and orient it either horizontally or vertically:
 $x^2 + y^2 + 6x - 3y + 10xy + 100 = 0$.
3. Use the above answer or a calculator or computer program to determine how much you would have to rotate the following conic to eliminate the xy term and orient it either horizontally or vertically:
 $5x^2 - 3x + 4xy + 21 = 0$.
4. Rotate the following conic section so that it is oriented either horizontally or vertically, and then analyze the result: $3x^2 + 3y^2 + (4\sqrt{2} + 2)xy + 4\sqrt{2}x + 4\sqrt{2}y - 4 = 0$

Review Answers

1. $\theta = \frac{\pi}{2}$
2. $\theta = \frac{\pi}{2}$
3. $\theta \approx 31.72^\circ$
4. To figure out the angle of rotation, since $A = B$ we have $\theta = \frac{\pi}{4}$ (or 45°). After shifting the equation by this amount, we have a relation, $4x^2 + 8x - y^2 + 8y - 4 = 0$. Completing the square, this results in a shifted hyperbola: $\frac{(x+1)^2}{1} - \frac{(y-2)^2}{4} = 1$.

General Algebraic Forms

How can we look at a degree-2 polynomial equation and determine which conic section it depicts?

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

When $C = 0$ we have already discussed how to determine which conic section the equation refers to. In summary, if A and B are both positive, the conic section is an ellipse. This is also true if A and B are both negative, as the entire equation can be multiplied by -1 without changing the solution set. If A and B differ in sign, the equation is a hyperbola, and if A or B equals zero the equation is a parabola.

There are a few new, more general, rules I will show you that give more information about the case when $C \neq 0$ and hence the conic section needs to be rotated to achieve horizontal or vertical orientation.

If $C^2 < 4AB$, the equation is an ellipse (note when $C = 0$ this holds whenever A and B are the same sign, which is consistent with our simpler rule stated above.)

If $C^2 > 4AB$, the equation is a hyperbola (note when $C = 0$ this holds whenever A and B are the opposite sign, which is consistent with our simpler rule stated above.)

If $C^2 = 4AB$, the equation is a parabola (note when $C = 0$, either A or B equals zero, which is consistent with our simpler rule stated above.)

Review Questions

5. State what type of conic section is represented by the following equation: $5x^2 + 6y^2 + 2x - 5y + xy = 0$.
6. State what type of conic section is represented by the following equation: $x^2 + 3y - 20xy + 20 = 0$.
7. The rules above do not account for “degenerate” conic sections, that is the conic section that looks like an X made by the intersection of a plane containing the line at the center of the cone. Explain the conditions on the coefficients that lead to the degenerate conic sections.

Review Answers

5. Ellipse
6. Hyperbola
7. $C = 0$, and either $A = 0$ or $B = 0$ (or both).

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